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Existence and nonexistence of positive solutions to a discrete boundary value problem

JOHNNY HENDERSON[1], RODICA LUCA[2] and ALEXANDRU TUDORACHE[3]

ABSTRACT. We study the existence and nonexistence of positive solutions for a system of nonlinear second-order difference equations subject to coupled multi-point boundary conditions which contain some positive constants.

1. INTRODUCTION

The mathematical modeling of many nonlinear problems from computer science, economics, mechanical engineering, control systems, biological neural networks and others leads to the consideration of nonlinear difference equations (see [11], [12]). In the last decades, many authors have investigated such problems by using various methods, such as fixed point theorems, the critical point theory, upper and lower solutions, the fixed point index theory and the topological degree theory (see for example [1], [2], [3], [4], [5], [7], [10], [13], [14], [15]).

In this paper, we consider the system of nonlinear second-order difference equations

$$(S) \qquad \begin{cases} \Delta^2 u_{n-1} + s_n f(v_n) = 0, \quad n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + t_n g(u_n) = 0, \quad n = \overline{1, N-1}, \end{cases}$$

with the coupled multi-point boundary conditions

$$(BC) \qquad u_0 = 0, \;\; u_N = \sum_{i=1}^{p} a_i v_{\xi_i} + a_0, \;\; v_0 = 0, \;\; v_N = \sum_{i=1}^{q} b_i u_{\eta_i} + b_0,$$

where $N \in \mathbb{N}$, $N \geq 2$, p, $q \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for k, $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, a_0 and b_0 are positive constants.

Under some assumptions on the functions f and g, we shall prove the existence of positive solutions of problem $(S)-(BC)$. By a positive solution of $(S)-(BC)$ we mean a pair of sequences $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ satisfying (S) and (BC) with $u_n > 0$, $v_n > 0$ for all $n = \overline{1, N}$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem. The system (S) with the uncoupled multi-point boundary conditions

$$(BC_1) \qquad u_0 = \sum_{i=1}^{p} a_i u_{\xi_i} + a_0, \;\; u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \;\; v_0 = \sum_{i=1}^{r} c_i v_{\zeta_i}, \;\; v_N = \sum_{i=1}^{l} d_i v_{\rho_i} + b_0,$$

(a_0, $b_0 > 0$) has been investigated in [6]. Some systems of difference equations with parameters subject to multi-point boundary conditions were studied in [7] and [8] by using

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Corresponding author: Rodica Luca; rluca@math.tuiasi.ro

the Guo-Krasnosel'skii fixed point theorem. We also mention the paper [9], where we investigated the existence and multiplicity of positive solutions for the system $\Delta^2 u_{n-1} + f(n, v_n) = 0$, $\Delta^2 v_{n-1} + g(n, u_n) = 0$, $n = \overline{1, N-1}$, with the multi-point boundary conditions (BC_1) with $a_0 = b_0 = 0$, by using some theorems from the fixed point index theory.

In Section 2, we present some auxiliary results from [8] which investigate a system of second-order difference equations subject to the coupled boundary conditions (BC) with $a_0 = b_0 = 0$. In Section 3, we shall prove our main results, and in Section 4, we shall present an example which illustrates the obtained theorems.

2. Auxiliary results

In this section, we present some auxiliary results from [8] related to the following system of second-order difference equations

$$\text{(2.1)} \qquad \begin{cases} \Delta^2 u_{n-1} + x_n = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + y_n = 0, & n = \overline{1, N-1}, \end{cases}$$

with the coupled multi-point boundary conditions

$$\text{(2.2)} \qquad u_0 = 0, \quad u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \quad v_0 = 0, \quad v_N = \sum_{i=1}^{q} b_i u_{\eta_i},$$

where $N \in \mathbb{N}$, $N \geq 2$, $p,\ q \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $x_i,\ y_i \in \mathbb{R}$ for all $i = \overline{1, N-1}$.

Lemma 2.1. ([8]) *If $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \ldots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $\Delta_0 = N^2 - \left(\sum_{i=1}^{p} a_i \xi_i\right) \left(\sum_{i=1}^{q} b_i \eta_i\right) \neq 0$, and $x_i,\ y_i \in \mathbb{R}$ for all $i = \overline{1, N-1}$, then the unique solution of (2.1)-(2.2) is given by*

$$\text{(2.3)} \qquad \begin{aligned} u_n &= \sum_{j=1}^{N-1} G_1(n, j) x_j + \sum_{j=1}^{N-1} G_2(n, j) y_j, \quad n = \overline{0, N}, \\ v_n &= \sum_{j=1}^{N-1} G_3(n, j) y_j + \sum_{j=1}^{N-1} G_4(n, j) x_j, \quad n = \overline{0, N}, \end{aligned}$$

where

$$\text{(2.4)} \qquad \begin{aligned} G_1(n, j) &= g_0(n, j) + \frac{n}{\Delta_0} \left(\sum_{i=1}^{p} a_i \xi_i \right) \left(\sum_{i=1}^{q} b_i g_0(\eta_i, j) \right), \\ G_2(n, j) &= \frac{nN}{\Delta_0} \sum_{i=1}^{p} a_i g_0(\xi_i, j), \\ G_3(n, j) &= g_0(n, j) + \frac{n}{\Delta_0} \left(\sum_{i=1}^{q} b_i \eta_i \right) \left(\sum_{i=1}^{p} a_i g_0(\xi_i, j) \right), \\ G_4(n, j) &= \frac{nN}{\Delta_0} \sum_{i=1}^{q} b_i g_0(\eta_i, j), \end{aligned}$$

and

$$g_0(n, j) = \frac{1}{N} \begin{cases} j(N-n), & 1 \leq j \leq n \leq N, \\ n(N-j), & 0 \leq n \leq j \leq N-1, \end{cases}$$

for all $n = \overline{0, N}$ and $j = \overline{1, N-1}$.

Lemma 2.2. ([8]) *If* $a_i \geq 0$, $\xi_i \in \mathbb{N}$ *for all* $i = \overline{1,p}$, $b_i \geq 0$, $\eta_i \in \mathbb{N}$ *for all* $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, *and* $\Delta_0 = N^2 - (\sum_{i=1}^p a_i\xi_i)(\sum_{i=1}^q b_i\eta_i) > 0$, *then the functions* G_i, $i = \overline{1,4}$, *given by (2.4), satisfy* $G_i(n,j) \geq 0$ *for all* $n = \overline{0,N}$, $j = \overline{1,N-1}$, $i = \overline{1,4}$. *Moreover, if* $x_n \geq 0$, $y_n \geq 0$ *for all* $n = \overline{1,N-1}$, *then the solution* $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ *of problem (2.1)-(2.2) (given by (2.3)) satisfies* $u_n \geq 0$, $v_n \geq 0$ *for all* $n = \overline{0,N}$.

Lemma 2.3. ([8]) *Assume that* $a_i \geq 0$, $\xi_i \in \mathbb{N}$ *for all* $i = \overline{1,p}$, $b_i \geq 0$, $\eta_i \in \mathbb{N}$ *for all* $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, *and* $\Delta_0 > 0$. *Then the functions* G_i, $i = \overline{1,4}$ *satisfy the inequalities*

$a_1)$ $G_1(n,j) \leq I_1(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where*

$$I_1(j) = g_0(j,j) + \frac{N}{\Delta_0}\left(\sum_{i=1}^p a_i\xi_i\right)\left(\sum_{i=1}^q b_i g_0(\eta_i,j)\right);$$

$a_2)$ *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min_{n=\overline{c,N-c}} G_1(n,j) \geq \frac{c}{N} I_1(j)$, $\forall\, j = \overline{1,N-1}$;

$b_1)$ $G_2(n,j) \leq I_2(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where* $I_2(j) = \frac{N^2}{\Delta_0}\sum_{i=1}^p a_i g_0(\xi_i,j)$;

$b_2)$ *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min_{n=\overline{c,N-c}} G_2(n,j) \geq \frac{c}{N} I_2(j)$, $\forall\, j = \overline{1,N-1}$;

$c_1)$ $G_3(n,j) \leq I_3(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where*

$$I_3(j) = g_0(j,j) + \frac{N}{\Delta_0}\left(\sum_{i=1}^q b_i\eta_i\right)\left(\sum_{i=1}^p a_i g_0(\xi_i,j)\right);$$

$c_2)$ *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min_{n=\overline{c,N-c}} G_3(n,j) \geq \frac{c}{N} I_3(j)$, $\forall\, j = \overline{1,N-1}$;

$d_1)$ $G_4(n,j) \leq I_4(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where* $I_4(j) = \frac{N^2}{\Delta_0}\sum_{i=1}^q b_i g_0(\eta_i,j)$;

$d_2)$ *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min_{n=\overline{c,N-c}} G_4(n,j) \geq \frac{c}{N} I_4(j)$, $\forall\, j = \overline{1,N-1}$,

where $[\![N/2]\!]$ *is the largest integer not greater than* $N/2$.

Lemma 2.4. ([8]) *Assume that* $a_i \geq 0$, $\xi_i \in \mathbb{N}$ *for all* $i = \overline{1,p}$, $b_i \geq 0$, $\eta_i \in \mathbb{N}$ *for all* $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $\Delta_0 > 0$, $c \in \{1,\ldots,[\![N/2]\!]\}$, *and* x_n, $y_n \geq 0$ *for all* $n = \overline{1,N-1}$. *Then the solution of problem (2.1)-(2.2) satisfies the inequalities*

$$\min_{n=\overline{c,N-c}} u_n \geq \frac{c}{N}\max_{m=\overline{0,N}} u_m, \quad \min_{n=\overline{c,N-c}} v_n \geq \frac{c}{N}\max_{m=\overline{0,N}} v_m.$$

Our main existence result is based on the Schauder fixed point theorem which we present now.

Theorem 2.1. *Let* X *be a Banach space and* $Y \subset X$ *a nonempty, bounded, convex and closed subset. If the operator* $A : Y \to Y$ *is completely continuous (continuous, and compact, that is, it maps bounded sets into relatively compact sets), then* A *has at least one fixed point.*

3. Main results

We present first the assumptions that we shall use in the sequel.

$(H1)$ $a_i \geq 0$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $b_i \geq 0$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$ and $\Delta_0 = N^2 - \left(\sum_{i=1}^p a_i\xi_i\right)\left(\sum_{i=1}^q b_i\eta_i\right) > 0$.

$(H2)$ The constants s_n, $t_n \geq 0$ for all $n = \overline{1,N-1}$, and there exist i_0, $j_0 \in \{1,\ldots,N-1\}$ such that $s_{i_0} > 0$, $t_{j_0} > 0$.

$(H3)$ $f,\ g\ :\ [0,\infty) \to [0,\infty)$ are continuous functions and there exists $c_0 > 0$ such that $f(u) < \frac{c_0}{L}$, $g(u) < \frac{c_0}{L}$ for all $u \in [0, c_0]$, where $L = \max\{\sum_{i=1}^{N-1} s_i I_1(i) + \sum_{i=1}^{N-1} t_i I_2(i),\ \sum_{i=1}^{N-1} t_i I_3(i) + \sum_{i=1}^{N-1} s_i I_4(i)\}$ and $I_i,\ i = \overline{1,4}$ are defined in Lemma 2.3.

$(H4)$ $f,\ g : [0,\infty) \to [0,\infty)$ are continuous functions and satisfy the conditions
$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty, \quad \lim_{u\to\infty} \frac{g(u)}{u} = \infty.$$

Our first theorem is the following existence result for problem $(S) - (BC)$.

Theorem 3.2. *Assume that the assumptions $(H1) - (H3)$ hold. Then problem $(S) - (BC)$ has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Proof. We consider the system of second-order difference equations

$$\Delta^2 h_{n-1} = 0, \quad \Delta^2 k_{n-1} = 0, \quad n = \overline{1, N-1}, \tag{3.5}$$

with the coupled boundary conditions

$$h_0 = 0, \quad h_N = \sum_{i=1}^{p} a_i k_{\xi_i} + a_0, \quad k_0 = 0, \quad k_N = \sum_{i=1}^{q} b_i h_{\eta_i} + b_0, \tag{3.6}$$

with $a_0 > 0$ and $b_0 > 0$.

The above problem (3.5)-(3.6) has the solution

$$h_n = \frac{n}{\Delta_0}\left(a_0 N + b_0 \sum_{i=1}^{p} a_i \xi_i\right), \quad k_n = \frac{n}{\Delta_0}\left(b_0 N + a_0 \sum_{i=1}^{q} b_i \eta_i\right), \quad n = \overline{0, N}, \tag{3.7}$$

where $\Delta_0 = N^2 - \left(\sum_{i=1}^{p} a_i \xi_i\right)\left(\sum_{i=1}^{q} b_i \eta_i\right)$. By assumption $(H1)$, we obtain $h_n > 0$ and $k_n > 0$ for all $n = \overline{1, N}$.

We define the sequences $(x_n)_{n=\overline{0,N}}$ and $(y_n)_{n=\overline{0,N}}$ by $x_n = u_n - h_n$, $y_n = v_n - k_n$, $n = \overline{0, N}$, where $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ is a solution of $(S) - (BC)$. Then $(S) - (BC)$ can be equivalently written as

$$\begin{cases} \Delta^2 x_{n-1} + s_n f(y_n + k_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 y_{n-1} + t_n g(x_n + h_n) = 0, & n = \overline{1, N-1}, \end{cases} \tag{3.8}$$

with the boundary conditions

$$x_0 = 0, \quad x_N = \sum_{i=1}^{p} a_i y_{\xi_i}, \quad y_0 = 0, \quad y_N = \sum_{i=1}^{q} b_i x_{\eta_i}. \tag{3.9}$$

Using the Green's functions $G_i,\ i = \overline{1,4}$ from Section 2, a pair $((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}})$ is a solution of problem (3.8)-(3.9) if and only if it is a solution for the problem

$$\begin{cases} x_n = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) s_i f(y_i + k_i) + \sum_{i=1}^{N-1} G_2(n,i) t_i g(x_i + h_i), & n = \overline{0, N}, \\ y_n = \displaystyle\sum_{i=1}^{N-1} G_3(n,i) t_i g(x_i + h_i) + \sum_{i=1}^{N-1} G_4(n,i) s_i f(y_i + k_i), & n = \overline{0, N}, \end{cases} \tag{3.10}$$

where $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ are given in (3.7).

We consider the Banach space $X = \mathbb{R}^{N+1}$ with the maximum norm $\|u\| = \max\limits_{n=\overline{0,N}} |u_n|$, $u = (u_n)_{n=\overline{0,N}}$, and the space $Y = X \times X$ with the norm $\|(x,y)\|_Y = \|x\| + \|y\|$. We define the set

$$E = \{(x_n)_{n=\overline{0,N}},\ 0 \le x_n \le c_0,\ \forall\, n = \overline{0, N}\} \subset X.$$

We also define the operators $S_1,\ S_2 : E \times E \to X$ and $S : E \times E \to Y$ by

$$S_1(x,y) = \left(\sum_{i=1}^{N-1} G_1(n,i) s_i f(y_i + k_i) + \sum_{i=1}^{N-1} G_2(n,i) t_i g(x_i + h_i) \right)_{n=\overline{0,N}},$$
$$S_2(x,y) = \left(\sum_{i=1}^{N-1} G_3(n,i) t_i g(x_i + h_i) + \sum_{i=1}^{N-1} G_4(n,i) s_i f(y_i + k_i) \right)_{n=\overline{0,N}},$$

and $S(x,y) = (S_1(x,y), S_2(x,y))$ for $(x,y) = ((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}}) \in E \times E$.

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(H3)$ we deduce

$$f(y_n + k_n) \le \frac{c_0}{L},\ \ g(x_n + h_n) \le \frac{c_0}{L},\ \ \forall\, n = \overline{0,N},\ \ \forall\, (x_n)_n,\ (y_n)_n \in E. \tag{3.11}$$

Then, by using Lemma 2.2, we obtain $(S_1(x,y))_n \ge 0$, $(S_2(x,y))_n \ge 0$ for all $n = \overline{0,N}$ and $x = (x_n)_{n=\overline{0,N}}$, $y = (y_n)_{n=\overline{0,N}} \in E$. By Lemma 2.3, for all $(x,y) \in E \times E$, we have

$$\begin{aligned}
(S_1(x,y))_n &\le \sum_{i=1}^{N-1} I_1(i) s_i f(y_i + k_i) + \sum_{i=1}^{N-1} I_2(i) t_i g(x_i + h_i) \\
&\le \frac{c_0}{L} \left(\sum_{i=1}^{N-1} s_i I_1(i) + \sum_{i=1}^{N-1} t_i I_2(i) \right) \le c_0,\ \ \forall\, n = \overline{0,N}, \\
(S_2(x,y))_n &\le \sum_{i=1}^{N-1} I_3(i) t_i g(x_i + h_i) + \sum_{i=1}^{N-1} I_4(i) s_i f(y_i + k_i) \\
&\le \frac{c_0}{L} \left(\sum_{i=1}^{N-1} t_i I_3(i) + \sum_{i=1}^{N-1} s_i I_4(i) \right) \le c_0,\ \ \forall\, n = \overline{0,N}.
\end{aligned}$$

Therefore $S(E \times E) \subset E \times E$.

Using standard arguments, we deduce that S is completely continuous. By Theorem 2.1, we conclude that S has a fixed point $(x,y) = ((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}} \in E \times E$, which represents a solution for (3.8)-(3.9). This shows that our problem $(S)-(BC)$ has a positive solution $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ with $u_n = x_n + h_n,\ v_n = y_n + k_n,\ n = \overline{0,N}$, ($u_n > 0$ and $v_n > 0$ for all $n = \overline{1,N}$) for sufficiently small $a_0 > 0$ and $b_0 > 0$. □

In what follows, we present sufficient conditions for the nonexistence of positive solutions of $(S)-(BC)$.

Theorem 3.3. *Assume that assumptions $(H1)$, $(H2)$ and $(H4)$ hold. Then problem $(S)-(BC)$ has no positive solution for a_0 and b_0 sufficiently large.*

Proof. We suppose that $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ is a positive solution of $(S)-(BC)$. Then $(x,y) = ((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}})$ with $x_n = u_n - h_n,\ y_n = v_n - k_n,\ n = \overline{0,N}$, is a solution for problem (3.8)-(3.9), where $(h,k) = ((h_n)_{n=\overline{0,N}}, (k_n)_{n=\overline{0,N}})$ is the solution of problem (3.5)-(3.6) (given by (3.7)). By $(H2)$ there exists $c \in \{1, 2, \ldots, [\![N/2]\!]\}$ such that $i_0,\ j_0 \in \{c, \ldots, N-c\}$ and then $\sum_{i=c}^{N-c} s_i I_1(i) > 0$, $\sum_{i=c}^{N-c} t_i I_2(i) > 0$, $\sum_{i=c}^{N-c} t_i I_3(i) > 0$ and $\sum_{i=c}^{N-c} s_i I_4(i) > 0$. Now by using Lemma 2.2, we have $x_n \ge 0,\ y_n \ge 0$ for all $n = \overline{0,N}$, and by Lemma 2.4, we obtain $\min_{n=\overline{c,N-c}} x_n \ge \frac{c}{N}\|x\|$ and $\min_{n=\overline{c,N-c}} y_n \ge \frac{c}{N}\|y\|$.

Using now (3.7), we deduce that

$$\min_{n=\overline{c,N-c}} h_n = h_c = \frac{h_c}{h_N}\|h\| = \frac{c}{N}\|h\|, \quad \min_{n=\overline{c,N-c}} k_n = k_c = \frac{k_c}{k_N}\|k\| = \frac{c}{N}\|k\|.$$

Therefore, we obtain

$$\min_{n=\overline{c,N-c}}(x_n+h_n) \geq \frac{c}{N}\|x\| + \frac{c}{N}\|h\| = \frac{c}{N}(\|x\|+\|h\|) \geq \frac{c}{N}\|x+h\|,$$
$$\min_{n=\overline{c,N-c}}(y_n+k_n) \geq \frac{c}{N}\|y\| + \frac{c}{N}\|k\| = \frac{c}{N}(\|y\|+\|k\|) \geq \frac{c}{N}\|y+k\|.$$

We now consider

$$R = \left(\frac{c^2}{N^2} \sum_{i=c}^{N-c} t_i I_2(i) \right)^{-1} > 0. \tag{3.12}$$

By using $(H4)$, for R defined above, we conclude that there exists $M_0 > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for all $u \geq M_0$. We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that

$$\min_{n=\overline{c,N-c}}(x_n+h_n) \geq M_0, \quad \min_{n=\overline{c,N-c}}(y_n+k_n) \geq M_0. \tag{3.13}$$

By $(H2)$, (3.8), (3.9) and the above inequalities, we deduce that $\|x\| > 0$ and $\|y\| > 0$.

Now, by using Lemma 2.3 and the above considerations, we have

$$\begin{aligned}
x_c &= \sum_{i=1}^{N-1} G_1(c,i) s_i f(y_i+k_i) + \sum_{i=1}^{N-1} G_2(c,i) t_i g(x_i+h_i) \\
&\geq \frac{c}{N} \sum_{i=1}^{N-1} I_2(i) t_i g(x_i+h_i) \geq \frac{c}{N} \sum_{i=c}^{N-c} I_2(i) t_i g(x_i+h_i) \\
&\geq \frac{2Rc}{N} \sum_{i=c}^{N-c} I_2(i) t_i (x_i+h_i) \geq \frac{2Rc}{N} \sum_{i=c}^{N-c} I_2(i) t_i \min_{j=\overline{c,N-c}}(x_j+h_j) \\
&\geq \frac{2Rc^2}{N^2} \sum_{i=c}^{N-c} I_2(i) t_i \|x+h\| = 2\|x+h\| \geq 2\|x\|.
\end{aligned}$$

Therefore, we obtain $\|x\| \leq \frac{1}{2} x_c \leq \frac{1}{2}\|x\|$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution. $\square$

Remark 3.1. In the proof of Theorem 3.3, instead of the constant R from (3.12), we can also consider

$$\widetilde{R} = \left(\frac{c^2}{N^2} \sum_{i=c}^{N-c} s_i I_4(i) \right)^{-1}, \tag{3.14}$$

and in a similar manner as above we prove that $y_c \geq 2\|y+k\| \geq 2\|y\|$, and then $\|y\| \leq \frac{1}{2} y_c \leq \frac{1}{2}\|y\|$, which is a contradiction.

Similar results as Theorems 3.2 and 3.3 can be obtained if instead of boundary conditions (BC) we have

$$(BC_1) \qquad u_0 = a_0, \ \ u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \ \ v_0 = 0, \ \ v_N = \sum_{i=1}^{q} b_i u_{\eta_i} + b_0, \ \text{ or}$$

$$(BC_2) \qquad u_0 = 0, \ \ u_N = \sum_{i=1}^{p} a_i v_{\xi_i} + a_0, \ \ v_0 = b_0, \ \ v_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \ \text{ or}$$

$$(BC_3) \qquad u_0 = a_0, \ \ u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \ \ v_0 = b_0, \ \ v_N = \sum_{i=1}^{q} b_i u_{\eta_i},$$

where a_0 and b_0 are positive constants.

For problem $(S)-(BC_1)$, instead of the pair of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ from the proof of Theorem 3.2, the solution of system

$$\Delta^2\widetilde{h}_{n-1}=0,\quad \Delta^2\widetilde{k}_{n-1}=0,\quad n=\overline{1,N-1},$$

with the coupled boundary conditions

$$\widetilde{h}_0=a_0,\quad \widetilde{h}_N=\sum_{i=1}^{p}a_i\widetilde{k}_{\xi_i},\quad \widetilde{k}_0=0,\quad \widetilde{k}_N=\sum_{i=1}^{q}b_i\widetilde{h}_{\eta_i}+b_0,$$

is given by

$$\begin{aligned}\widetilde{h}_n&=\frac{1}{\Delta_0}\left\{n\left[b_0\left(\sum_{i=1}^{p}a_i\xi_i\right)+a_0\left[\left(\sum_{i=1}^{p}a_i\xi_i\right)\left(\sum_{i=1}^{q}b_i\right)-N\right]\right]\right.\\&\quad\left.+a_0\left[N^2-\left(\sum_{i=1}^{p}a_i\xi_i\right)\left(\sum_{i=1}^{q}b_i\eta_i\right)\right]\right\},\quad n=\overline{0,N},\\\widetilde{k}_n&=\frac{n}{\Delta_0}\left[a_0\sum_{i=1}^{q}b_i(N-\eta_i)+b_0N\right],\quad n=\overline{0,N}.\end{aligned}$$

By assumption (H1) we obtain $\widetilde{h}_n>0$ for all $n=\overline{0,N-1}$, and $\widetilde{k}_n>0$ for all $n=\overline{1,N}$. For the nonexistence of the positive solutions, we take here $\widetilde{R}$ given in (3.14), and we show that $y_c\geq 2\|y\|$, which leads us to a contradiction.

For problem $(S)-(BC_2)$, instead of the pair of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ from the proof of Theorem 3.2, the solution of system

$$\Delta^2\check{h}_{n-1}=0,\quad \Delta^2\check{k}_{n-1}=0,\quad n=\overline{1,N-1},$$

with the coupled boundary conditions

$$\check{h}_0=0,\quad \check{h}_N=\sum_{i=1}^{p}a_i\check{k}_{\xi_i}+a_0,\quad \check{k}_0=b_0,\quad \check{k}_N=\sum_{i=1}^{q}b_i\check{h}_{\eta_i},$$

is given by

$$\begin{aligned}\check{h}_n&=\frac{n}{\Delta_0}\left[a_0N+b_0\sum_{i=1}^{p}a_i(N-\xi_i)\right],\quad n=\overline{0,N},\\\check{k}_n&=\frac{1}{\Delta_0}\left\{n\left[a_0\left(\sum_{i=1}^{q}b_i\eta_i\right)+b_0\left[\left(\sum_{i=1}^{q}b_i\eta_i\right)\left(\sum_{i=1}^{p}a_i\right)-N\right]\right]\right.\\&\quad\left.+b_0\left[N^2-\left(\sum_{i=1}^{p}a_i\xi_i\right)\left(\sum_{i=1}^{q}b_i\eta_i\right)\right]\right\},\quad n=\overline{0,N}.\end{aligned}$$

By assumption (H1) we obtain $\check{h}_n>0$ for all $n=\overline{1,N}$, and $\check{k}_n>0$ for all $n=\overline{0,N-1}$. For the nonexistence of the positive solutions, we take here R given in (3.12), and we prove that $x_c\geq 2\|x\|$, which leads us to a contradiction.

For problem $(S)-(BC_3)$, instead of the pair of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ from the proof of Theorem 3.2, the solution of problem

$$\Delta^2\hat{h}_{n-1}=0,\quad \Delta^2\hat{k}_{n-1}=0,\quad n=\overline{1,N-1},$$

with the coupled boundary conditions

$$\hat{h}_0=a_0,\quad \hat{h}_N=\sum_{i=1}^{p}a_i\hat{k}_{\xi_i},\quad \hat{k}_0=b_0,\quad \hat{k}_N=\sum_{i=1}^{q}b_i\hat{h}_{\eta_i},$$

is given by

$$\hat{h}_n = \frac{1}{\Delta_0}\left\{ n\left[-a_0N + a_0\left(\sum_{i=1}^{q} b_i\right)\left(\sum_{i=1}^{p} a_i\xi_i\right) + b_0\sum_{i=1}^{p} a_i(N-\xi_i)\right] \right.$$
$$\left. + a_0\left[N^2 - \left(\sum_{i=1}^{p} a_i\xi_i\right)\left(\sum_{i=1}^{q} b_i\eta_i\right)\right]\right\}, \quad n = \overline{0,N},$$
$$\hat{k}_n = \frac{1}{\Delta_0}\left\{ n\left[-b_0N + b_0\left(\sum_{i=1}^{p} a_i\right)\left(\sum_{i=1}^{q} b_i\eta_i\right) + a_0\sum_{i=1}^{q} b_i(N-\eta_i)\right] \right.$$
$$\left. + b_0\left[N^2 - \left(\sum_{i=1}^{p} a_i\xi_i\right)\left(\sum_{i=1}^{q} b_i\eta_i\right)\right]\right\}, \quad n = \overline{0,N}.$$

By assumption (H1) we obtain $\hat{h}_n > 0$ and $\hat{k}_n > 0$ for all $n = \overline{0, N-1}$.

Therefore we also obtain the following results.

Theorem 3.4. *Assume that assumptions $(H1)-(H3)$ hold. Then problem $(S)-(BC_1)$ has at least one positive solution ($u_n > 0$ for all $n = \overline{0,N-1}$, and $v_n > 0$ for all $n = \overline{1,N}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Theorem 3.5. *Assume that assumptions $(H1)$, $(H2)$ and $(H4)$ hold. Then problem $(S)-(BC_1)$ has no positive solution ($u_n > 0$ for all $n = \overline{0,N-1}$, and $v_n > 0$ for all $n = \overline{1,N}$) for a_0 and b_0 sufficiently large.*

Theorem 3.6. *Assume that assumptions $(H1)-(H3)$ hold. Then problem $(S)-(BC_2)$ has at least one positive solution ($u_n > 0$ for all $n = \overline{1,N}$, and $v_n > 0$ for all $n = \overline{0,N-1}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Theorem 3.7. *Assume that assumptions $(H1)$, $(H2)$ and $(H4)$ hold. Then problem $(S)-(BC_2)$ has no positive solution ($u_n > 0$ for all $n = \overline{1,N}$, and $v_n > 0$ for all $n = \overline{0,N-1}$) for a_0 and b_0 sufficiently large.*

Theorem 3.8. *Assume that assumptions $(H1)-(H3)$ hold. Then problem $(S)-(BC_3)$ has at least one positive solution ($u_n > 0$ and $v_n > 0$ for all $n = \overline{0,N-1}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

4. An example

We consider $N = 30$, $p = 3$, $q = 2$, $a_1 = 3$, $a_2 = 1$, $a_3 = 1/2$, $\xi_1 = 5$, $\xi_2 = 15$, $\xi_3 = 25$, $b_1 = 1$, $b_2 = 1/2$, $\eta_1 = 10$, $\eta_2 = 20$, $s_n = t_n = 1$ for all $n = \overline{1,29}$. We also consider the functions $f, g : [0,\infty) \to [0,\infty)$, $f(x) = \frac{\widetilde{a}x^{\alpha_1}}{2x+3}$, $g(x) = \frac{\widetilde{b}x^{\alpha_2}}{3x+1}$, for all $x \in [0,\infty)$, with $\widetilde{a}, \widetilde{b} > 0$ and $\alpha_1, \alpha_2 > 2$. We have $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} g(x)/x = \infty$.

Therefore, we consider the system of second-order difference equations

$$(S_0) \qquad \begin{cases} \Delta^2 u_{n-1} + \dfrac{\widetilde{a}v_n^{\alpha_1}}{(2v_n+3)} = 0, & n = \overline{1,29}, \\ \Delta^2 v_{n-1} + \dfrac{\widetilde{b}u_n^{\alpha_2}}{(3u_n+1)} = 0, & n = \overline{1,29}, \end{cases}$$

with the multi-point boundary conditions

$$(BC_0) \qquad u_0 = 0, \ \ u_{30} = 3v_5 + v_{15} + v_{25}/2 + a_0, \ \ v_0 = 0, \ \ v_{30} = u_{10} + u_{20}/2 + b_0,$$

where a_0 and b_0 are positive constants.

We have $\Delta_0 = N^2 - (\sum_{i=1}^{p} a_i\xi_i)(\sum_{i=1}^{q} b_i\eta_i) = 50 > 0$. The functions $I_i,\ i=\overline{1,4}$ are given by

$$I_1(j)=\begin{cases} \frac{1}{60}(1335j-2j^2),\ 1\le j\le 10,\\ \frac{1}{60}(15300-195j-2j^2),\ 11\le j\le 20,\\ \frac{1}{30}(15300-480j-j^2),\ 21\le j\le 29,\end{cases}\quad I_2(j)=\begin{cases} \frac{111}{2}j,\ 1\le j\le 5,\\ \frac{3}{2}(180+j),\ 6\le j\le 15,\\ \frac{3}{2}(360-11j),\ 16\le j\le 25,\\ \frac{3}{2}(510-17j),\ 26\le j\le 29,\end{cases}$$

$$I_3(j)=\begin{cases} \frac{1}{30}(1140j-j^2),\ 1\le j\le 5,\\ \frac{1}{30}(5400+60j-j^2),\ 6\le j\le 15,\\ \frac{1}{30}(10800-300j-j^2),\ 16\le j\le 25,\\ \frac{1}{30}(15300-480j-j^2),\ 26\le j\le 29,\end{cases}\quad I_4(j)=\begin{cases} 15j,\ 1\le j\le 10,\\ 180-3j,\ 11\le j\le 20,\\ 360-12j,\ 21\le j\le 29.\end{cases}$$

Hence, we deduce that assumptions $(H1)$, $(H2)$ and $(H4)$ are satisfied. In addition, by using the above functions $I_i,\ i=\overline{1,4}$, we obtain $A := \sum_{i=1}^{29} I_1(i) \approx 3974.8333333$, $B := \sum_{i=1}^{29} I_2(i) = 5962.5$, $C := \sum_{i=1}^{29} I_3(i) \approx 4124.83333333$, $D := \sum_{i=1}^{29} I_4(i) = 2700$, and then $L = \max\{A+B, C+D\} = A+B \approx 9937.33333333$. We choose $c_0 = 1$ and if we select $\widetilde{a}$, $\widetilde{b}$ satisfying the conditions $\widetilde{a} < \frac{5}{L}$, $\widetilde{b} < \frac{4}{L}$, then we conclude that $f(x) < 1/L$, $g(x) < 1/L$ for all $x \in [0,1]$. For example, if $\widetilde{a} \le 5\cdot 10^{-4}$ and $\widetilde{b} \le 4\cdot 10^{-4}$, then the above conditions for f and g are satisfied. So, assumption $(H3)$ is also satisfied. By Theorems 3.2 and 3.3 we deduce that problem $(S_0)-(BC_0)$ has at least one positive solution for sufficiently small $a_0 > 0$ and $b_0 > 0$, and no positive solution for sufficiently large a_0 and b_0.

By the proofs of Theorems 3.2 and 3.3 we can find some intervals for a_0 and b_0 such that problem $(S_0)-(BC_0)$ has at least one positive solution, or it has no positive solution. We consider $\widetilde{a} = 5\cdot 10^{-4}$, $\widetilde{b} = 4\cdot 10^{-4}$, $c_0 = 1$, $L = A+B$ (as above), $\alpha_1 = 3$ and $\alpha_2 = 4$. Then the sequences $(h_n)_{n=\overline{0,30}}$ and $(k_n)_{n=\overline{0,30}}$ from (3.7) are $h_n = \frac{n}{20}(12a_0 + 17b_0)$ and $k_n = \frac{n}{5}(3b_0 + 2a_0)$ for all $n = \overline{0,30}$. If we choose $a_0 \le \min\left\{\frac{1}{24}\left[f^{-1}\left(\frac{1}{L}\right)-1\right], \frac{1}{36}\left[g^{-1}\left(\frac{1}{L}\right)-1\right]\right\}$ and $b_0 \le \min\left\{\frac{1}{36}\left[f^{-1}\left(\frac{1}{L}\right)-1\right], \frac{1}{51}\left[g^{-1}\left(\frac{1}{L}\right)-1\right]\right\}$, then the inequalities (3.11) are satisfied. Because $f^{-1}(\frac{1}{L}) \approx 1.00242$ and $g^{-1}(\frac{1}{L}) \approx 1.00194$, then for $a_0 \le 5.37\cdot 10^{-5}$ and $b_0 \le 3.79\cdot 10^{-5}$ our problem $(S_0)-(BC_0)$ has at least one positive solution.

Now we choose $c = 4$ (the constant from the beginning of the proof of Theorem 3.3), and then we obtain $\sum_{i=4}^{26} I_2(i) = 5476.5$ and $R \approx 0.01027116$ (given by (3.12)). For $\widetilde{R} := 2R + 0.1$, the inequalities $\frac{f(x)}{x} \ge \widetilde{R}$ and $\frac{g(y)}{y} \ge \widetilde{R}$ are satisfied for $x \ge M_0' \approx 483.66463109$ and $y \ge M_0'' \approx 30.23301425$, respectively. We consider $M_0 = \max\{M_0', M_0''\} = M_0'$, and then for $a_0 \ge \frac{5M_0}{16}$ and $b_0 \ge \frac{5M_0}{24}$, the inequalities (3.13) are satisfied. Therefore, if $a_0 \ge 151.1452$ and $b_0 \ge 100.7635$, our problem $(S_0)-(BC_0)$ has no positive solution.

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DEPARTMENT OF MATHEMATICS
BAYLOR UNIVERSITY
ONE BEAR PLACE 97328, WACO, TEXAS, 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu

DEPARTMENT OF MATHEMATICS
GH. ASACHI TECHNICAL UNIVERSITY
BLVD. CAROL I 11, 700506 IASI, ROMANIA
E-mail address: rluca@math.tuiasi.ro

GH. ASACHI TECHNICAL UNIVERSITY
FACULTY OF COMPUTER ENGINEERING AND AUTOMATIC CONTROL
STREET PROF.DR.DOC. DIMITRIE MANGERON 27, 700050 IASI, ROMANIA
E-mail address: alexandru.tudorache93@gmail.com