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The existence theorem of a new multi-valued mapping in metric space endowed with graph

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ABSTRACT. In this paper, we introduce a new type of multi-valued G-contraction mapping on a metric space endowed with a directed graph G and prove an existence theorem for fixed point problems in metric space endowed with a graph. Moreover, we prove fixed point theorems in partially ordered metric spaces by our main result. Some examples illustrating our main results are also present.

1. INTRODUCTION

Fixed point problem can be reduced to the problem of economic, engineering, radiothempy etc. So, we can use the process for fixed point solution to resolve such problem.

Many authors tried to use new methods to solve fixed point problem. The solution for fixed point problem in Hilbert and Banach spaces has been popular for a long time see, [5], [7], [4].

This research is to studied the fixed point problem on complete metric space endowed with graph.

Let (X, d) be a metric space and we let $CB(X)$ and $Comp(X)$ be the set of all closed bounded subsets of X and the set of all nonempty compact subsets of X, respectively. A mapping $T : X \to X$ is called contraction if there exists $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$. A point $x \in X$ is a fixed point of a multi-valued mapping $T : X \to 2^x$ if $x \in Tx$. A point $x \in X$ is a coincidence point of $g : X \to X$ and $T : X \to 2^x$ if $g(x) \in T(x)$. If g is the identity map on X, then x is a fixed point of T.

The Banach contraction principle is a very important tool for solving existence problem in many branches of mathematics and physics. It guarantees the existence and uniqueness of fixed points.

The Banach contraction principle was firstly given by as follows;

Theorem 1.1. *Let (X, d) be complete metric space and $T : X \to X$ be α-contraction. Then T has a unique fixed point.*

In 1969, Nadler [5] extended the Banach contraction principle to multi-valued contraction mapping in complete metric space as follows;

Theorem 1.2. [5] *Let (X, d) be complete metric space and let $T : X \to CB(X)$. Assume that there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y),\ \forall x, y \in X.$$

Then there exists $z \in Tz$.

Reich [7] extended Nadler's fixed point theorem by replacing k with $d(x, y)$ and $CB(X)$ with $Comp(X)$ as follows.

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Theorem 1.3. [7] *Let (X, d) be complete metric space and let $T : X \to Comp(X)$. Assume that there exists a function $\alpha : [0, \infty) \to [0, 1)$ such that $lim_{r \to t^+} sup\ \alpha(r) < 1$ for each $t \in (0, \infty)$ and*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),\ \forall x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

In 1989, Mizoguchi and Takahachi [4] relaxed the compactness assumption on the mapping to closed and bounded subset of X as follows;

Theorem 1.4. [4] *Let (X, d) be complete metric space and let $T : X \to CB(X)$. Assume that there exists a function $\alpha : [0, \infty) \to [0, 1)$ such that $lim_{r \to t^+} sup\ \alpha(r) < 1$ for each $t \in (0, \infty)$ and*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),\ \forall x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

In 2007, Berinde and Berinde [2] extended theorem 1.2 to the class of multi-valued weak contractions as follows;

Definition 1.1. [2] Let (X, d) be a metric space and $T : X \to CB(X)$ is called a multi-valued weak contraction or a multi-valued (θ,L)-weak contraction if there exists two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx),\ \forall x, y \in X.$$

For the last ten years, many research related to the existence theorem of fixed point muti-valued mappping have been extensively studied and used different techniques to prove the such theorem, see for example [3],[1].

Let a graph G be an ordered pair $(V(G), E(G))$ consisting $V(G)$ is a set of vertices of the graph and $E(G)$ be a set of its edges.

In 2008, Jachymski [3] introduced a concept of G-contraction and generalized the Banach contraction principle in a metric space endowed with a directed graph in a definition below;

Definition 1.2. [3] Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contrains all loop i.e, $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$. The mapping $f : X \to X$ is a G-contraction if f preserves edges of G, i.e.,

$$x, y \in X,\ (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G), \tag{1.1}$$

and there exists $\alpha \in (0, 1)$ such that

$$x, y \in X,\ (x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

The mapping $f : X \to X$ satisfying condition (1.1) is also called a graph-preserving mapping. He proved some certain properties on X, a G-contraction $f : X \to X$ has a fixed point if and only if there exists $x \in X$ such that $(x, f(x)) \in E(G)$.

Recently, Beg and Butt [1] introduced the concept of G-contraction for multi-valued mapping as follows;

Definition 1.3. [1] Let $T : X \to CB(X)$ be a multi-valued mapping. The mapping T is called G-contraction if there exists $k \in (0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y)\ for\ all\ (x, y) \in E(G),$$

and if $u \in Tx$ and $v \in Ty$ such that

$$d(u, v) \leq kd(x, y) + \alpha\ for\ each\ \alpha > 0,$$

then $(u, v) \in E(G)$

Property A [3] For any sequence x_n in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Beg and Butt [1] proved that if (X, d) is complete metric space and (X, d) has property A, then G-contraction mapping $T : X \to CB(X)$ has a fixed point if and only if there exists $x \in X$ and $y \in Tx$ such that $(x, y) \in E(G)$.

Tiammee and Suatai [8] introduced concept of graph-preserving multi-valued mapping and a type of multi-valued weak G-contraction on a metric space endowed with graph as follows;

Definition 1.4. [8] Let X be a nonempty set, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. The multi-valued mapping $T : X \to CB(X)$ is said to be graph preserving if

$$(x, y) \in E(G) \Longrightarrow (u, v) \in E(G)$$

for all $u \in Tx$ and $v \in Ty$.

Definition 1.5. [8] Let X be a nonempty set, $G = (V(G), E(G))$ a directed graph such that $V(G) = X, g : X \to X$. The multi-valued mapping $T : X \to CB(X)$ is said to be g-graph preserving if for any $x, y \in X$, such that

$$(g(x), g(y)) \in E(G) \Longrightarrow (u, v) \in E(G)$$

for all $u \in Tx$ and $v \in Ty$.

Definition 1.6. [8] Let (X, d) be a metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, $g : X \to X$. The multi-valued mapping $T : X \to CB(X)$is said to be a multi-valued weak G-contraction with respect to g or (g, α, L)-G-contraction if there exists a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$ and $L \geq 0$ with

$$H(Tx, Ty) \leq \alpha(d(g(x), g(y)))d(g(x), g(y)) + LD(g(y), Tx),$$

for all $x, y \in X$ such that $(g(x), g(y)) \in E(G)$.

They prove the next results.

Theorem 1.5. [8] *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and let $g : X \to X$ be a surjective mapping. If $T : X \to CB(X)$ is a multi-valued mapping satisfying the following properties:*
(1) T is a g-graph-preserving mapping;
(2) there exists $x_0 \in X$ such that $(g(x_0), y) \in E(G)$ for some $y \in Tx_0$;
(3) X has Property A;
(4) T is a (g, α, L)-G-contraction;
then there exists $u \in X$ such that $g(u) \in Tu$.

Many authors tried to introduce a new techniques to prove existence of fixed point theorem such as Phon-on et al [6] introduced a new concept of weak graph preserving and proved their fixed point theorem in a complete metric space endowed with a graph, see more example [1], [8], [6].

Many authors proved their results by using the following lemma 1.1., see for more detial [1], [8], [6].

Lemma 1.1. [4] *Let (X, d) be metric space in $CB(X)$,$\{x_k\}$ be in X such that $x_k \in A_{k-1}$. Let $\alpha : [0, \infty) \to [0, 1)$ be a function satisfying $\limsup_{r \to t^+} < 1$ for every $t \in [0, \infty)$. Suppose that $d(x_{k-1}, x_k)$ is a non-increasing sequence such that*

$$H(A_{k-1}, A_k) \leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)$$
$$d(x_{k-1}, x_k) \leq H(A_{k-1}, A_k) + \alpha^{n_k}(d(x_{k-1}, x_k)),$$

where $n_1 < n_2 < ...$ and $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cuachy sequence in X.

In this paper, we introduced a new type of multi-valued G-contraction on a metric space endowed with a directed graph G and new techniques to simplify the proof of existence theorem for coincidence point of such mapping, that is, we do not use lemma 1.1 to prove our main theorem.

2. Preliminaries

In this section, we introduce basic concept and definition for proof main result.

Let (X, d) be a metric space and let CB(X) be the set of all closed bounded subsets of X. For $x \in X$ and $A, B \in CB(X)$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$
$$\delta(A, B) = \sup\{d(x, B) : x \in A\},$$
$$D(A, B) = \inf\{d(x, B) : x \in A\}.$$

The symbol $H(A, B)$ is the Pompeiu-Hausdorff metric[4], that is,

$$H(A, B) = max\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\}.$$

The following two lemmas are useful for our main results.

Lemma 2.2. [5] *Let (X, d) be metric space. If $A, B \in CB(X)$ and $a \in A$, then, for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Definition 2.7. A partial order is a binary relation $\leq$ over the set X which the followings conditions:

1. $x \leq x$ (reflexivity);
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity);

for all $x, y \in X$. A set with a partial order $\leq$ is called a partially ordered set. We write $x < y$ if $x \leq y$ and $x \neq y$.

Definition 2.8. Let $(X, \prec)$ be a partially ordered set. For each $A, B \subset X$,

$$A \prec B \; if \; a < b \; for \; any \; a \in A, b \in B.$$

Definition 2.9. Let (X, d) be metric space endowed with a parial order $\leq$. Let $g : X \to X$ be surjective and $T : X \to CB(X)$, T is said to be g-incresing if for any $x, y \in X$,

$$g(x) < g(y) \Longrightarrow Tx \prec Ty.$$

In the case $g = I_X$, the identity map, the mapping T is called an increasing mapping.

We define a new type of multi-valued mapping on a metric space endowed with a directed graph G as follow:

Definition 2.10. Let (X, d) be a metric space and $G = (V(G), E(G))$ be directed graph such that $V(G) = X$ and let $g : X \to X$ be a mapping. The multi-valued mapping $T : X \to CB(X)$ is said to be (α, β, L, g)-G-contraction if there exists $\alpha, \beta \in \left(0, \frac{1}{3}\right)$ with $\alpha < \beta$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \alpha d(g(x), g(y)) + \beta D(g(x), Ty) + LD(g(y), Tx) \tag{2.2}$$

for all $x, y \in X$ and $(g(x), g(y)) \in E(G)$.

Example 2.1. Let $X = \{0,1,2,3,4,5\}$ and $d(x,y) = |x-y|$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = X$ and
$E(G) = \{(0,2),(0,4),(1,3),(1,5),(2,5),(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$
Let $T : X \to CB(X)$. and $g : X \to X$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \\ 2 & \text{if } x = 3 \\ 5 & \text{if } x = 4 \\ 4 & \text{if } x = 5 \end{cases}$$

and

$$T(x) = \begin{cases} \{1\} & \text{if } x = 0,2,4 \\ \{0,2,4\} & \text{if } x = 1,3 \\ \{3,5\} & \text{if } x = 5 \end{cases}$$

Then T is $(\frac{1}{6}, \frac{1}{4}, 3, g)$-G-contraction mapping.
Solution Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (0,2)$,then $x = 0, y = 3$. It follows that $\sup_{u\in T0} d(u, T3) = 1$ and $\sup_{v\in T3} d(v, T0) = 3$. From the definition of Pompeiu-Hausdorff, we have $H(Tx,Ty) = 3$, $D(g(0), T3) = 0$ and $D(g(3), T0) = 1$.

Put $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$ and $L = 3$.
Then $H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + 3(g(y), Tx)$.

If $(g(x), g(y)) = (0,4)$,then $x = 0, y = 5$. It follows that $\sup_{u\in T0} d(u, T5) = 2$ and $\sup_{v\in T5} d(v, T0) = 4$. From the definition of Pompeiu-Hausdorff, we have $H(Tx,Ty) = H(T0,T5) = 4$, $D(g(0), T5) = 3$ and $D(g(5), T0) = 3$.

Put $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$ and $L = 3$.
Then $H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + 3(g(y), Tx)$.

If $(g(x), g(y)) = (1,3)$,then $x = 1, y = 2$. It follows that $\sup_{u\in T1} d(u, T2) = 3$ and $\sup_{v\in T2} d(v, T1) = 1$. From the definition of Pompeiu-Hausdorff, we have $H(Tx,Ty) = H(T1,T2) = 3$, $D(g(1), T2) = 0$ and $D(g(2), T1) = 1$.

Put $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$ and $L = 3$.
Then $H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + 3(g(y), Tx)$.

If $(g(x), g(y)) = (1,5)$,then $x = 1, y = 4$. It follows that $\sup_{u\in T1} d(u, T4) = 3$ and $\sup_{v\in T4} d(v, T1) = 1$. From the definition of Pompeiu-Hausdorff, we have $H(Tx,Ty) = H(T1,T4) = 3$, $D(g(1), T4) = 0$ and $D(g(4), T1) = 1$.

Put $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$ and $L = 3$.
Then $H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + 3(g(y), Tx)$.

If $(g(x), g(y)) = (2,5)$,then $x = 3, y = 4$. It follows that $\sup_{u\in T3} d(u, T4) = 3$ and $\sup_{v\in T4} d(v, T3) = 1$. From the definition of Pompeiu-Hausdorff, we have $H(Tx,Ty) = H(T3,T4) = 3$, $D(g(3), T4) = 1$ and $D(g(4), T3) = 1$.

Put $\alpha = \frac{1}{6}, \beta = \frac{1}{4}$ and $L = 3$.
Then $H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + 3(g(y), Tx)$.

If $(g(x), g(y)) \in \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$. It obvious that
$H(Tx,Ty) \leq \frac{1}{6}d(g(x), g(y)) + \frac{1}{4}(g(x), Ty) + (3)(g(y), Tx)$. Hence T is $(\frac{1}{6}, \frac{1}{4}, 3, g)$-G-contraction mapping.

3. Main results

Theorem 3.6. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be directed graph such that $V(G) = X$, and let $g : X \to X$ be a surjective mapping. If $T : X \to CB(X)$ is a*

multi-valued mapping satisfying the following properties :
1) T is a g-graph-preserving mapping ;
2) there exists $x_0 \in X$ such that $(g(x_0), y) \in E(G)$ for some $y \in Tx_0$;
3) X has Property A;
4) T is a (α, β, L, g)-G-contraction;
Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof. Let $x_0 \in X$. Since g is surjective, there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$.
By 2), we have $(g(x_0), g(x_1)) \in E(G)$. Since $(\beta - \alpha)d(g(x_0), g(x_1)) > 0$ and Lemma 2.2, there exists $g(x_2) \in Tx_1$ such that

$$d(g(x_1), g(x_2)) \leq H(Tx_0, Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)). \tag{3.3}$$

Since $(g(x_0), g(x_1)) \in E(G), g(x_1) \in Tx_0, g(x_2) \in Tx_1$ and T is g-graph-preserving mapping, then $(g(x_1), g(x_2)) \in E(G)$. From (3.3) and T is (α, β, L, g)-G-contraction, we have

$$\begin{aligned} d(g(x_1), g(x_2)) &\leq H(Tx_0, Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)) \\ &\leq \alpha d(g(x_0), g(x_1)) + \beta D(g(x_0), Tx_1) + LD(g(x_1), Tx_0) + \\ &\quad (\beta - \alpha)d(g(x_0), g(x_1)) \\ &= \alpha d(g(x_0), g(x_1)) + \beta d(g(x_0), Tx_1) + (\beta - \alpha)d(g(x_0), g(x_1)) \\ &= \beta d(g(x_0), g(x_1)) + \beta d(g(x_0), g(x_2)) \\ &\leq \beta[d(g(x_0), g(x_1)) + d(g(x_1), g(x_2))] + \beta d(g(x_0), g(x_1)) \\ &= 2\beta d(g(x_0), g(x_1)) + \beta d(g(x_1), g(x_2)). \end{aligned}$$

It implies that $d(g(x_1), g(x_2)) \leq \frac{2\beta}{1-\beta} d(g(x_0), g(x_1))$.
Since $(\beta - \alpha)d(g(x_1), g(x_2)) > 0$ and lemma 2.2, there exists $g(x_3) \in Tx_2$ such that

$$d(g(x_2), g(x_3)) \leq H(Tx_1, Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)). \tag{3.4}$$

Since $(g(x_1), g(x_2)) \in E(G), g(x_2) \in Tx_1, g(x_3) \in Tx_2$ and T is g-graph-preserving mapping, we have $(g(x_2), g(x_3)) \in E(G)$. From (3.4)and T is (α, β, L, g)-G-contraction, we have

$$\begin{aligned} d(g(x_2), g(x_3)) &\leq H(Tx_1, Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)) \\ &\leq \alpha d(g(x_1), g(x_2)) + \beta D(g(x_1), Tx_2) + LD(g(x_2), Tx_1) + \\ &\quad (\beta - \alpha)d(g(x_1), g(x_2)) \\ &= \alpha d(g(x_1), g(x_2)) + \beta d(g(x_1), Tx_2) + (\beta - \alpha)d(g(x_1), g(x_2)) \\ &= \beta d(g(x_1), g(x_2)) + \beta d(g(x_1), g(x_3)) \\ &\leq \beta[d(g(x_1), g(x_2)) + d(g(x_2), g(x_3))] + \beta d(g(x_1), g(x_2)) \\ &= 2\beta d(g(x_1), g(x_2)) + \beta d(g(x_2), g(x_3)). \end{aligned}$$

It implies that $d(g(x_2), g(x_3)) \leq \frac{2\beta}{1-\beta} d(g(x_1), g(x_2)) \leq (\frac{2\beta}{1-\beta})^2 d(g(x_0), g(x_1))$.
By the same way, we have

$$d(g(x_k), g(x_{k+1})) \leq (\tfrac{2\beta}{1-\beta})^k d(g(x_0), g(x_1)),$$

$g(x_{k+1}) \in Tx_k$ and $(g(x_k), g(x_{k+1})) \in E(G)$, for all $k \in \mathbb{N}$.
For any numbers $p \in \mathbb{N}$, we have

$$\begin{aligned} d(g(x_{k+p}), g(x_k)) &\le \sum_{n=k}^{k+p-1} d(g(x_{n+1}, g(x_n) \\ &\le \sum_{n=k}^{k+p-1} (\frac{2\beta}{1-\beta})^n d(g(x_1), g(x_0)) \\ &\le \frac{(\frac{2\beta}{1-\beta})^k}{1-(\frac{2\beta}{1-\beta})} d((g(x_1), g(x_0)). \qquad (3.5) \end{aligned}$$

From $(\frac{2\beta}{1-\beta})^k \to 0$ as $k \to \infty$, and (3.5), we can conclude that $\{g(x_k)\}_{k=0}^{\infty}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $\lim_{k\to\infty} g(x_k) = g(u)$.
From $(g(x_k), g(x_{k+1})) \in E(G)$ for all $k \in \mathbb{N}$ and assumption 3), there is a subsequence $g(x_{k_n})$ of $g(x_k)$ such that $(g(x_{k_n}), g(u)) \in E(G)$.
From 4) and $g(x_{k+1}) \in Tx_k$, we have

$$\begin{aligned} D(g(u), Tu) &\le d(g(u), g(x_{k_n+1})) + D(g(x_{k_n+1}), Tu) \\ &\le d(g(u), g(x_{k_n+1})) + H(Tx_{k_n}, Tu) \\ &\le d(g(u), g(x_{k_n+1})) + \alpha d(g(x_{k_n}), g(u)) + \beta D(g(x_{k_n}), Tu) + LD(g(u), Tx_{k_n}) \\ &\le d(g(u), g(x_{k_n+1})) + \alpha d(g(x_{k_n}), g(u)) + \beta(d(g(x_{k_n}), g(u)) + D(g(u), Tu)) + \\ &\quad + Ld(g(u), Tx_{k_n}). \end{aligned}$$

It follows that

$$\begin{aligned} D(g(u), Tu) &\le (\frac{1}{1-\beta})(d(g(u), g(x_{k_n+1})) + (\alpha+\beta) d(g(x_{k_n}), g(u)) + Ld(g(u), Tx_{k_n})) \\ &\le (\frac{1}{1-\beta})(d(g(u), g(x_{k_n+1})) + (\alpha+\beta) d(g(x_{k_n}), g(u)) + Ld(g(u), g(x_{k_n+1}))). \end{aligned}$$

Since $g(x_{k_n})$ converges to $g(u)$ as $n \to \infty$, thus $D(g(u), Tu) \le 0$. Then $D(g(u), Tu) = 0$. Since Tu is closed, we conclude that $g(u) \in Tu$. □

Corollary 3.1. *Let (X, d) be a metric space endowed with a partial order $\le, g : X \to X$ be surjective and $T : X \to CB(X)$ be a multivalued mapping. Suppose that*
1) T is a g-increasing ;
2) there exists $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$;
3) for each sequence $\{x_k\}$ such that $g(x_k) < g(x_{k+1})$ for all $k \in \mathbb{N}$ and $g(x_k)$ converge to $g(x)$, for some $x \in X$, then $g(x_k) < g(x)$ for all $k \in \mathbb{N}$;
4) there exists $\alpha, \beta \in (0, 1)$ with $0 < \alpha < \beta < \frac{1}{3}$ and $L \ge 0$ such that

$$H(Tx, Ty) \le \alpha d(g(x), g(y)) + \beta D(g(x), Ty) + LD(g(y), Tx)$$

for any $x, y \in X$ with $g(x) < g(y)$;
5) the metric d is complete.
Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof. Let $G = (V(G), E(G))$ by $V(G) = X$ and $E(G) = \{(x, y) : x < y\}$. Let $x, y \in X$ such that $(g(x), g(y)) \in E(G)$. Then $g(x) < g(y)$ so $Tx \prec Ty$. For any $u \in Tx$ and $v \in Ty$,we have $u < v$, i.e., $(u, v) \in E(G)$. So T is graph-preserving. By assumption 2),ther exist x_0 and $u \in Tx_0$ such that $g(x_0) < u$, thus $(g(x_0), u) \in E(G)$. Hence 2) of theorem 3.6 is satisfied. It's obviously 3) and 4) of theorem 3.6 are also satisfied. Therefore Corollary is obtained directly by theorem 3.6. □

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