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On a theorem of Brian Fisher in the framework of w-distance

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ABSTRACT. In 1980. Fisher in [Fisher, B., *Results on common fixed points on complete metric spaces*, Glasgow Math. J., **21** (1980), 165–167] proved very interesting fixed point result for the pair of maps. In 1996. Kada, Suzuki and Takahashi introduced and studied the concept of w–distance in fixed point theory. In this paper, we generalize Fisher's result for pair of mappings on metric space to complete metric space with w–distance. The obtained results do not require the continuity of maps, but more relaxing condition $(C; k)$. As a corollary we obtain a result of Chatterjea.

1. INTRODUCTION

In 1980, Fisher [10] proved the following very interesting result.

Theorem 1.1. [10] *Let $f, g : X \mapsto X$ be two continuous mappings on a complete metric space (X, d) satisfying the inequality*

$$d(f^p x, g^q y) \leq \lambda \max\{d(f^r x, g^s y) \mid 0 \leq r \leq p, 0 \leq s \leq q\}, \ x, y \in X, \tag{1.1}$$

for some fixed $p, q \in n$ and $\lambda \in (0, 1)$. Then f and g have a unique common fixed point in X.

Let us point out that the maps in the above theorem are not commutative; (for very recent results related to Fisher theorem see eg., [4]).

In 1996. Kada, Suzuki and Takahashi [13] introduced and studied the concept of w–distance in fixed point theory, they gave examples of the w-distance and, among other things, generalized Caristi's fixed point theorem [2], Ekeland's variational principle [7] and the nonconvex minimization theorem by Takahashi [17]. See also ([15], [16]). For more recent, related results on w–distance see eg., ([6], [11], [12]).

Definition 1.1. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,
(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Let us recall that a real-valued function f defined on a metric space X is said to be lower semicontinuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$.

The following, very useful lemma has been proved in [13].

Lemma 1.1. *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let x, y, $z \in X$. Then the following hold:*

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(i) *if* $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p(x, y) = 0$ *and* $p(x, z) = 0$, *then* $y = z$;

(ii) *if* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* y_n *converges to* z;

(iii) *if* $p(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*

(iv) *if* $p(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

In this paper, we generalize Fisher's result for pair of mappings on metric space to w–distance on complete metric space. In contrast to Fisher's result, our results do not require the continuity of maps, but more relaxing new condition $(C; k)$. Further more, our methods of proofs are new, and even simpler than the corresponding methods in metric spaces.

We say that a map $T : X \to X$ on a metric space (X, d) obeys the condition $(C; k)$ if there is a constant $k \geq 0$ such that for every sequence $x_n \in X$,

$$x_n \to x_0 \in X \quad \Rightarrow \quad D(x_0) \leq k \cdot \limsup D(x_n),$$

where $D(x) = d(x, Tx), x \in X$. We point out that the the condition $(C; 1)$ was introduced and studied by Ćirić [5].

For the convenient of a reader we recall the following result of Chatterjea [3].

Theorem 1.2. *Let* T *be a mapping of a complete metric space* (X, d) *into itself. If for some* $\alpha \in [0, 1/2)$,

$$d(Tx, Ty) \leq \alpha \cdot d(x, Ty) + \alpha \cdot d(Tx, y) \text{ for every } x, y \in X \tag{1.2}$$

then T *has a unique fixed point* $u \in X$.

Mapping T in Theorem 1.2 is called Chattereja operator; (for more details see eg., [1], [10], [14]).

2. Main results

Now we state and prove our main results.

Theorem 2.3. *Let* S *and* T *be mappings of a complete metric space* (X, d) *into itself and let* p *be a w-distance. If* S *and* T *obey the condition* $(C; k)$, *and if for some fixed positive integers* l *and* q *and some* $\lambda \in [0, 1)$

$$\max\{p(S^l x, T^q y), p(T^q y, S^l x)) \leq \lambda \cdot \max\Big\{p(S^r x, T^s y), p(T^s y, S^r x) : 0 \leq r \leq l, 0 \leq s \leq q\Big\} \tag{2.3}$$

then S *and* T *have a unique common fixed point* $u \in X$. *Moreover,* $p(u, u) = 0$.

Proof. Fix $x \in X$. Let

$$\omega \equiv \omega(x) = \sum_{0 \leq i \leq l, 0 \leq j \leq q} p(S^i x, T^j x) + \sum_{0 \leq i \leq l, 0 \leq j \leq q} p(T^j x, S^i x) + \sum_{0 \leq i, j \leq l} p(S^i x, S^j x) + \sum_{0 \leq i, j \leq q} p(T^i x, T^j x).$$

We will prove that

$$\max\Big\{p(S^i x, T^j x), p(T^j x, S^i x)\Big\} \leq \frac{1}{1 - \lambda} \cdot \omega,$$

for every $i, j \in \mathbb{N}$. Suppose that n_0 is a natural number such that $n_0 > \max\{l, q\}$, that our hypothesis holds for every $i, j < n_o$ and let us prove that it holds for $i = n_0$ or $j = n_0$. Put

$$p(S^k x, T^{n_0} x) = \max\Big\{p(S^i x, T^{n_0} x), p(T^{n_0} x, S^i x), p(S^{n_0} x, T^j x), p(T^j x, S^{n_0} x) :$$

$$0 \le i,j \le n_0\Big\}$$

(On a similar way we can discuss the other cases)

We have to consider two cases:

(i) $k < l$. It follows from (2.3) that

$$p(S^k x, T^{n_0}x) \le p(S^k x, S^l x) + p(S^l x, T^{n_0}x) \le \omega(x) + \lambda \cdot v,$$

where

$$v = \max\Big\{p(S^r x, T^s x), p(T^s x, S^r x) : 0 \le r \le l, n_0 - q \le s \le n_0\Big\}.$$

$(i.1)$ Suppose that $v = p(S^i x, T^j x)$ or $v = p(T^j x, S^i x)$, $j < n_0$. By assumption,

$$v \le \frac{1}{1-\lambda} \cdot \omega,$$

so

$$p(S^k x, T^{n_0}x) \le \omega + \frac{\lambda}{1-\lambda} \cdot \omega = \frac{1}{1-\lambda} \cdot \omega.$$

$(i.2)$ Suppose that $v = p(S^i x, T^{n_0}x)$ or $v = p(T^{n_0}x, S^i x)$, $0 \le i \le l$. Now we have that

$$p(S^k x, T^{n_0}x) \le \omega + \lambda \cdot p(S^k x, T^{n_0}x).$$

That implies

$$(1-\lambda) \cdot p(S^k x, T^{n_0}x) \le \omega,$$

so

$$p(S^k x, T^{n_0}x) \le \frac{1}{1-\lambda} \cdot \omega.$$

(ii) $k \ge l$. It follows from inequality (2.3) that

$$p(S^k x, T^{n_o}x) \le \lambda \cdot m,$$

where

$$m = \max\Big\{p(S^r x, T^s x), p(T^s x, S^r x) : k - l \le r \le k, n_0 - q \le s \le n_0\Big\},$$

so $p(T^k x, T^{n_0}x) = 0$.

Suppose that $\epsilon > 0$. Choose $\delta > 0$ as in (3) of Definition 1.1. Let N be a natural number such that

$$\lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega \le \delta.$$

Assume that m, n are natural numbers such that $m, n > N \cdot \max\{l, q\}$. Thus, for every $i \ge \max\{m, n\}$, $i \in \mathbb{N}$, we have

$$\begin{aligned} p(S^i x, T^m x) &\le \max\{p(S^i x, T^m x), p(T^m x, S^i x)\} \le \lambda \cdot \max\Big\{p(S^r x, T^s x), p(T^s x, S^r x) : \\ &\qquad i - l \le r \le i, m - q \le s \le m\Big\} \\ &\le \lambda^2 \cdot \max\Big\{p(S^r x, T^s x), p(T^s x, S^r x) : \\ &\qquad i - 2l \le r \le i, m - 2q \le s \le m\Big\} \le \dots \\ &\le \lambda^N \cdot \max\Big\{p(S^r x, T^s x), p(T^s x, S^r x) : i - Nl \le r \le i, m - Nq \le s \le m\} \\ &\le \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega \le \delta \end{aligned}$$

and analogously holds

$$p(S^i x, T^n x) \le \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega \le \delta.$$

Hence, $d(T^n x, T^m x) \leq \epsilon$, and $T^n x$ is a Cauchy sequence. Since (X, d) is a complete metric space, $T^n x$ is convergent, say $\lim_{n\to\infty} T^n x = u \in X$. Since T obeys the condition $(C; k)$, we have

$$d(u, Tu) \leq k \cdot \limsup d(T^n x, TT^n x) = 0,$$

so u is a fixed point of T.

From the fact that p is lower semi-continuous, for $n > N \cdot \max\{l, q\}$ we have

$$p(S^n x, u) \leq \liminf{}_m p(S^n x, T^m x) \leq \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega. \tag{2.4}$$

On the other hand, for $n, m, r > N \cdot \max\{l, q\}$ using (2.3), (2.4) and the definition of w-distance, we have

$$\begin{aligned} p(S^n x, S^m x) &\leq p(S^n x, T^r x) + p(T^r x, S^m x) \\ &\leq \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega + \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega = 2\lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega \end{aligned} \tag{2.5}$$

From (2.4), (2.9) and Lemma 1.1 we conclude that sequence $S^n x$ converges to u. Because S obeys the condition $(C; k)$ we obtain that u is a fixed point of S too.

Moreover,

$$\begin{aligned} p(u, u) &= p(S^l u, T^q u) \\ &\leq \lambda \cdot \max\left\{p(S^r u, T^s u), p(T^s u, S^r u) : 0 \leq r \leq l, 0 \leq s \leq q\right\} \\ &\leq \lambda \cdot p(u, u), \end{aligned}$$

and therefore, $p(u, u) = 0$.

Let us prove the uniqueness of u. If $Tv = v$, then

$$\begin{aligned} p(u, v) &= p(S^l u, T^q v) \\ &\leq \lambda \cdot \max\left\{p(S^r u, T^s v), p(T^s v, S^r u) : 0 \leq r \leq l, 0 \leq s \leq q\right\} \\ &\leq \lambda \max\{p(u, v), p(v, u)\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} p(v, u) &= p(T^q v, S^l u) \\ &\leq \lambda \cdot \max\left\{p(S^r v, T^s u), p(T^s u, S^r v) : 0 \leq r \leq l, 0 \leq s \leq q\right\} \\ &\leq \lambda \max\{p(u, v), p(v, u)\}. \end{aligned}$$

Thus, $p(u, v) = p(v, u) = 0$ and $p(u, u) = 0$, and it follows from Lemma 1.1 that $u = v$.

□

From the above theorem, in the special case $S = T$, and $l = q = 1$ we have

Corollary 2.1. *Let T be a mapping of a complete metric space (X, d) into itself and let p be a w-distance. If T obeys the condition $(C; k)$, and if for some $\lambda \in [0, 1)$*

$$p(Tx, Ty) \leq \lambda \cdot \max\left\{p(T^s y, T^r x) : 0 \leq r, s \leq 1\right\}, \tag{2.6}$$

for every $x, y \in X$, then T has a unique fixed point $u \in X$. Moreover, $p(u, u) = 0$.

Now, from the above corollary we obtain the following result connected with the Chatterjea operator.

Corollary 2.2. *Let T be a mapping of a complete metric space (X, d) into itself, let T obeys the condition $(C; k)$, and let p be a w-distance. If for some $\alpha \in [0, 1/2)$*

$$p(Tx, Ty) \leq \max\{\alpha \cdot p(x, Ty) + \alpha \cdot p(Tx, y), \alpha \cdot p(x, Tx) + \alpha \cdot p(y, Tx),$$
$$\alpha \cdot p(Tx, x) + \alpha \cdot p(Tx, y), \alpha \cdot p(Tx, x) + \alpha \cdot p(y, Tx)\},$$

for every $x, y \in X$, then T has a unique fixed point $u \in X$. Moreover, $p(u, u) = 0$.

Let us prove that Chatterjea operator obeys the condition $(C; k)$.

Lemma 2.2. *Let T be a Catterjea operator as in Theorem 1.2. Then T obeys the condition $(C; k)$ for $k = (1 + \alpha)/(1 - \alpha)$.*

Proof. Assume that $x_n, x_0 \in X$ and $x_n \to x_0$ as $n \to \infty$. Now,

$$\begin{aligned} d(Tx_0, x_0) &\leq d(Tx_0, Tx_n) + d(Tx_n, x_n) + d(x_n, x_0) \\ &\leq \alpha \cdot [d(x_0, Tx_n) + d(Tx_0, x_n)] + d(Tx_n, x_n) + d(x_n, x_0) \\ &\leq \alpha \cdot [d(x_0, x_n) + d(x_n, Tx_n) + d(Tx_0, x_n)] + d(Tx_n, x_n) + d(x_n, x_0). \end{aligned}$$

Hence,

$$d(Tx_0, x_0) \leq \alpha \cdot [\limsup d(x_n, Tx_n) + d(Tx_0, x_0)] + \limsup d(Tx_n, x_n),$$

and

$$d(Tx_0, x_0) \leq \frac{1 + \alpha}{1 - \alpha} \cdot \limsup d(x_n, Tx_n).$$

□

Remark 2.1. From Corollary 2.2 and Lemma 2.2 we obtain Theorem 1.2.

The following example shows that we can not apply Theorem 2.3 if the condition $(C; k)$ is not satisfied.

Example 2.1. Let (X, d) be a metric space where $X = [0, 1]$ and $d(x, y) = |x - y|, x, y \in X$ and let $T : X \to X$ be a function defined as follows: $Tx = 1$ if $x = 0$ and $Tx = \frac{x}{2}$ for $x \neq 0$ (see [9]). Then for $p, q \geq 2$ we have $d(T^p x, T^q y) \leq \frac{1}{2} d(T^{p-1}x, T^{q-1}y)$ for every $x, y \in X$. Nevertheless, T does not obey the condition $(C; k)$ at 0, so T does not have a fixed point.

We now give an example, to show that it is possible to apply our result with the condition $(C; k)$, but Fisher result does not apply because the functions are not continuous.

Example 2.2. Let (X, d) be a metric space where $X = [0, 3] \cup [4, 5]$ and $d(x, y) = |x - y|, x, y \in X$. Let us define $T : X \to X$, by $Tx = 0, x \in [0, 3], Tx = 3, x \in [4, 5)$ and $T(5) = 4\frac{1}{4}$ and $S : X \to X$ by $Sx = 0, x \in [0, 3], Sx = 3, x \in [4, 5)$ and $S(5) = 4\frac{1}{8}$. Let us prove that there is $\lambda \in [0, 1)$ such that

$$d(Tx, S^2 y) \leq \lambda \cdot \max\left\{d(T^s x, S^r y) : 0 \leq r \leq 1, 0 \leq s \leq 2\right\} \tag{2.7}$$

holds. It is obvious that $S^2 x = 0$ for $x \in [0, 5)$ and $S^2(5) = 3$.

If $x \in [0, 3]$ and $y = 5$, then $d(Tx, S^2 y) = 3$. Furthermore, $d(Tx, Sy) = 4\frac{1}{8}$. Hence $d(Tx, S^2 y) \leq \frac{8}{11} \cdot d(Tx, Sy)$.

If $x \in [4, 5)$ and $y \in [0, 5)$, then $d(Tx, S^2 y) = 3$. Since, $d(x, S^2 y) \geq 4$, we have $d(Tx, S^2 y) \leq \frac{3}{4} \cdot d(x, S^2 y)$.

If $x = 5$ and $y \in [0, 5)$, then $d(T5, S^2 y) = 4\frac{1}{4}$. From $d(5, S^2 y) = 5$ we have $d(Tx, S^2 y) \leq \frac{17}{20} \cdot d(5, S^2 y)$.

If $x = 5$ and $y = 5$, then $d(T5, S^2 5) = 1\frac{1}{4}$. Since $d(5, S^2 5) = 3$ and $\frac{5}{4} \leq \frac{17}{20} \cdot 3$, $d(T5, S^2 5) \leq \frac{17}{20} \cdot d(5, S^2 5)$. So, T and S satisfy the Fisher quasi-contraction, where $p = 1$, $q = 2$, and $\lambda = \frac{17}{20}$. It is easy to prove that operators T and S obey the condition $(C; 1)$.

For example, $\frac{3}{4} = d(5, T5) \leq \limsup d(x_n, Tx_n) \leq \lim x_n - 3 = 2$, where $x_n \in X$ is any sequence such that $x_n \to 5$, as $n \to \infty$. By Theorem 2.3, T and S have a unique fixed point in X, and this fixed point is $x = 0$. On the other hand, T and S are not continuous, so they do not satisfy the conditions of Fisher theorem.

In the next result we suppose that $l = 1$, and S is an arbitrary function.

Theorem 2.4. *Let S and T be mappings of a complete metric space (X, d) into itself, assume that T obeys the condition $(C : k)$ and let p be a w-distance. If for some fixed positive integer q and some $\lambda \in [0, 1)$*

$$\max\{p(Sx, T^q y), p(T^q y, Sx)) \leq \lambda \cdot \max\Big\{p(S^r x, T^s y), p(T^s y, S^r x) : 0 \leq r \leq 1, 0 \leq s \leq q\Big\}, \tag{2.8}$$

then S and T have a unique common fixed point $u \in X$. Moreover, $p(u, u) = 0$.

Proof. Let x be an arbitrary point in X. As in the proof of Theorem 2.3, we show that the sequences $S^n x$ and $T^n x$ converge to some $u \in X$. Since T obeys the condition $(C : k)$ we know that u is a fixed point of T. It is clear that $T^n u = u$ for every $n \in \mathbb{N}$ and that sequences $T^n u$ and $S^n u$ converge to u. Using the same notation as in the proof of Theorem 2.3, when we put u instead of x, for $n > N \cdot \max\{l, q\}$ we have

$$p(u, u) = p(T^n u, u) \leq \liminf_m p(T^n u, S^m u) \leq \lambda^N \cdot \frac{1}{1-\lambda} \cdot \omega, \tag{2.9}$$

so $p(u, u) = 0$. On the other hand, using (2.8) we obtain

$$p(Su, u) = p(Su, T^q u) \leq \lambda \cdot \max\Big\{p(S^r u, T^s u), p(T^s u, S^r u) : 0 \leq r \leq 1, 0 \leq s \leq q\Big\} \leq \lambda \cdot \max\Big\{p(Su, u), p(u, Su)\Big\}$$

and

$$p(u, Su) = p(T^q u, Su) \leq \lambda \cdot \max\Big\{p(S^r u, T^s u), p(T^s u, S^r u) : 0 \leq r \leq 1, 0 \leq s \leq q\Big\} \leq \lambda \cdot \max\Big\{p(Su, u), p(u, Su)\Big\}.$$

Hence, $p(u, Su) = p(Su, u) = 0$. From $p(u, u) = p(u, Su) = 0$ and Lemma 1.1 we conclude that $Su = u$. □

Remark 2.2. We are very grateful to a referee that we can generalize our result on the way that instead of condition $(C; k)$ we can use the condition $(C; k)$ with respect to p denoted by $(C; k)_p$ as follows:

If $\limsup_n\{m > n : p(x_n, x_m)\} = 0$ and x_n converges to x_0, then

$$p(x_0, Tx_0) \leq k \cdot \limsup_n p(x_n, Tx_n). \tag{2.10}$$

Theorem 2.3. is still valid under the assumption of $(C; k)$ wrt p instead of $(C; k)$ wrt d. Because (2) of Definition 1.1 implies $\lim_{n\to\infty} p(T^n x, u) = 0$. $(C; k)$ wrt p implies $p(u, Tu) = 0$, by (1) and (3) of Definition 1.1 we obtain $Tu = u$.

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