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Existence of solutions for Caputo fractional boundary value problems with integral conditions

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ABSTRACT. In this paper, we investigate the existence results for Caputo fractional boundary value problems with integral conditions. Our analysis relies on Banach's contraction principle, Leray-Schauder nonlinear alternative, Boyed and Wong fixed point theorem, and Krasnoselskii's fixed point theorem. As applications, some examples are provided to illustrate our main results.

1. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, biology, economics and control theory. They also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. There are a large number of papers dealing with the existence or multiplicity of solutions or positive solutions of boundary value problem for some fractional differential equations. For details, see [1] - [8], [12, 13, 15, 19, 20, 22, 24, 25, 26] and the references therein.

For example, in [7], Bai and Lü discussed the following fractional boundary value problem

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, D^α is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions are obtained.

In [5], Ahmad and Sivasundaram considered the following nonlinear fractional integro-differential equation with four-point nonlocal boundary conditions

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u'(0) + au(\eta_1) = 0, \quad bu'(1) + u'(\eta_2) = 0, \quad 0 < \eta_1 \leq \eta_2 < 1, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo's fractional derivative, $\mathbb{X}$ is a Banach space and $f : [0,1] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous. Applying some standard fixed point theorems, they proved the existence and uniqueness of solutions.

However, we note that among the existing literatures, few people have studied the boundary value problems of fractional differential equations with only integral conditions. So, in this paper, we discuss the existence results for the following Caputo fractional

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boundary value problems:

$$\text{(1.1)} \qquad \begin{cases} {}^cD^\alpha u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \displaystyle\int_0^1 u(t)dt, \quad u(1) = \int_0^1 tu(t)dt, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Our results are based on Banach's contraction principle, Leray-Schauder nonlinear alternative, Boyed and Wong fixed point theorem, and Krasnoselskii's fixed point theorem. To be detail, we first consider the related problem (2.3) below and find out the equivalent integral equation (2.4), and then we define an operator F by (2.8). We observe that problem (1.1) have solutions if and only if the operator F have fixed points.

It is noteworthy that, if $\alpha = 2$ then problem (1.1) recaptures the following boundary value problem:

$$\text{(1.2)} \qquad \begin{cases} u''(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \displaystyle\int_0^1 u(t)dt, \quad u(1) = \int_0^1 tu(t)dt, \end{cases}$$

of which Guezane-Lakoud et al. [17] established the existence of nontrivial solution by using Banach's contraction principle and Leray-Schauder nonlinear alternative.

The rest of this paper is organized as follows. In Section 2, we present some necessary definitions from fractional calculus theory and two useful lemmas. In Section 3, we give the main results. In the end, Section 4, some examples illustrating the results established in this paper are also presented.

2. Preliminaries

In this section, we present some necessary definitions from fractional calculus theory and two useful lemmas.

Definition 2.1. [21] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $y : (0,\infty) \to \mathbb{R}$ is defined by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is Gamma function, and $y(t)$ is pointwise defined on $(0,\infty)$.

Definition 2.2. [21] The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : (0,\infty) \to \mathbb{R}$ is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integral part of real number α, and $y(t)$ is pointwise defined on $(0,\infty)$.

Definition 2.3. [21] The Caputo's fractional derivative of order $\alpha > 0$ for a function y is defined by

$${}^cD^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integral part of real number α, and $y(t)$ is pointwise defined on $(0,\infty)$.

Lemma 2.1. [24] *For $\alpha > 0$, then*

$$I^{\alpha}({}^{c}D^{\alpha})y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}, i = 1, \ldots, n-1, (n = [\alpha] + 1)$.

Lemma 2.2. *Let $y \in \mathbb{C}([0,1], \mathbb{R})$, then the boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = \int_0^1 u(t)dt, \quad u(1) = \int_0^1 tu(t)dt \end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) =& -\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}y(s)ds \\ &+ \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^{\alpha}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \end{aligned} \tag{2.4}$$

where $1 < \alpha \leq 2$ is a real number.

Proof. From Lemma 2.1, we obtain $u(t) + e_0 + e_1 t + I^{\alpha}y(t) = 0$. Then $u(t) = -e_0 - e_1 t - I^{\alpha}y(t)$. By the definition of I^{α}, we have

$$u(t) = -e_0 - e_1 t - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds. \tag{2.5}$$

Using the first integral condition and by the fact that $u(0) = -e_0$, we get

$$-e_0 = -e_0 - \frac{e_1}{2} - \frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t (t-s)^{\alpha-1}y(s)dsdt, \tag{2.6}$$

where

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t (t-s)^{\alpha-1}y(s)dsdt = \frac{1}{\Gamma(\alpha)}\int_0^1\int_s^1 (t-s)^{\alpha-1}y(s)dtds \\ =&\frac{1}{\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha}y(s)ds = \frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds. \end{aligned}$$

Substituting above calculation in (2.6), we have $e_1 = -\frac{2}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds$. Substituting e_1 in (2.5), we get

$$u(t) = -e_0 + \frac{2t}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds. \tag{2.7}$$

Using the second integral condition, we have

$$\begin{aligned} &-e_0 + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ =&\int_0^1 \left\{-e_0 t + \frac{2t^2}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t t(t-s)^{\alpha-1}y(s)ds\right\}dt \\ =&-\frac{e_0}{2} + \frac{2}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t t(t-s)^{\alpha-1}y(s)dsdt, \end{aligned}$$

where

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t t(t-s)^{\alpha-1}y(s)dsdt\\=&\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t (t-s)(t-s)^{\alpha-1}y(s)dsdt+\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t s(t-s)^{\alpha-1}y(s)dsdt\\=&\frac{1}{(\alpha+1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}y(s)ds+\frac{1}{\Gamma(\alpha+1)}\int_0^1 s(1-s)^{\alpha}y(s)ds.\end{aligned}$$

Then we have

$$\begin{aligned}&-e_0+\frac{2}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\=&-\frac{e_0}{2}+\frac{2}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)ds-\frac{1}{(\alpha+1)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}y(s)ds\\&-\frac{1}{\Gamma(\alpha+1)}\int_0^1 s(1-s)^{\alpha}y(s)ds,\end{aligned}$$

which means that

$$\begin{aligned}e_0=&\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)ds-\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}y(s)ds\\&+\frac{2}{\Gamma(\alpha+1)}\int_0^1 s(1-s)^{\alpha}y(s)ds.\end{aligned}$$

Substituting e_0 in (2.7), we have

$$\begin{aligned}u(t)=&-\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)ds+\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}y(s)ds\\&+\frac{2}{\Gamma(\alpha+1)}\int_0^1(t-s)(1-s)^{\alpha}y(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds.\end{aligned}$$

The proof is completed. □

Let $\mathbb{C}=\mathbb{C}([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|,t\in[0,1]\}$. Define an operator $F:\mathbb{C}\to\mathbb{C}$ by

$$\begin{aligned}(Fu)(t)=&-\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}f(s,u(s))ds+\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}f(s,u(s))ds\\&+\frac{2}{\Gamma(\alpha+1)}\int_0^1(t-s)(1-s)^{\alpha}f(s,u(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds.\end{aligned}\tag{2.8}$$

Observe that problem (1.1) have solutions if and only if the operator F have fixed points. For the sake of convenience, we set a constant Λ as

$$\Lambda=\frac{8}{3}\frac{1}{\Gamma(\alpha+2)}+\frac{2\alpha}{(\alpha+1)(\alpha+2)\Gamma(\alpha)}+\frac{\alpha+4}{(\alpha+2)\Gamma(\alpha+1)}.\tag{2.9}$$

3. Existence of solutions

In this section, we will introduce our main results. Our first result is based on Banach's fixed point theorem.

Theorem 3.1. *Assume that* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function satisfying the conditions*
(H_1) $|f(t,u)-f(t,v)|\le L\,|u-v|\,,\quad \forall\, t\in[0,1]$ *and* $u,v\in\mathbb{R}$
(H_2) $L\Lambda<1$
where L *is a Lipschitz constant, and* Λ *is defined by* (2.9). *Then problem* (1.1) *has a unique solution.*

Proof. We transform problem (1.1) into a fixed point problem $u=Fu$, where $F:\mathbb{C}\to\mathbb{C}$ is defined by (2.8). Assume that $\sup\limits_{t\in[0,1]}|f(t,0)|=M$, and choose a constant R satisfying

$$R\ge\frac{M\Lambda}{1-L\Lambda}. \tag{3.10}$$

First, we will show that $FB_R\subset B_R$, where $B_R=\{u\in\mathbb{C}:\|u\|\le R\}$. For any $u\in B_R$, we have

$$\begin{aligned}\|Fu\|\le&(L\|u\|+M)\sup_{t\in[0,1]}\left|\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}ds+\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}ds\right.\\&\left.+\frac{2}{\Gamma(\alpha+1)}\int_0^1(t-s)(1-s)^{\alpha}ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\right|\\\le&(LR+M)\Lambda\le R.\end{aligned}$$

Therefore $FB_R\subset B_R$.

Next, we will show that F is a contraction. For any $u,v\in\mathbb{C}$ and for each $t\in[0,1]$, we have

$$\begin{aligned}\|Fu-Fv\|\le\sup_{t\in[0,1]}&\left\{L\|u-v\|\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}ds+L\|u-v\|\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}ds\right.\\&\left.+L\|u-v\|\frac{2}{\Gamma(\alpha+1)}\int_0^1(t-s)(1-s)^{\alpha}ds\;+L\|u-v\|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\right\}\\\le&L\Lambda\|u-v\|.\end{aligned}$$

As $L\Lambda<1$, therefore F is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Remark 3.1. In Theorem 3.1, if $\alpha=2$, then $\Lambda=\dfrac{8}{3}\dfrac{1}{\Gamma(4)}+\dfrac{1}{3\Gamma(2)}+\dfrac{3}{2\Gamma(3)}$. Problem (1.1) reduces to problem (1.2) and problem (1.2) has a unique solution.

Next, we prove the existence of solutions of problem (1.1) by using the following Leray-Schauder nonlinear alternative:

Theorem 3.2. (*Nonlinear Alternative for Single Valued Maps*) [16] *Let* $\mathbb{E}$ *be a Banach space,* $\mathbb{C}$ *be a closed convex subset of* $\mathbb{E}$, $\mathbb{U}$ *be an open subset of* $\mathbb{C}$, *and* $0\in\mathbb{U}$. *Suppose that* $F:\overline{\mathbb{U}}\to\mathbb{C}$ *is a continuous, compact (that is,* $F(\overline{\mathbb{U}})$ *is a relatively compact subset of* $\mathbb{C}$) *map. Then, either*
(1) *F has a fixed point in* $\overline{\mathbb{U}}$ *or*
(2) *there is a* $u\in\partial\mathbb{U}$ $(the\ boundary\ of\ \mathbb{U}\ in\ \mathbb{C})$ *and* $\lambda\in(0,1)$ $with\ u=\lambda F(u)$.

Theorem 3.3. *Assume that:*
(H_3) *there exists a continuous nondecreasing function* $\psi:[0,\infty)\to(0,\infty)$ *and a function* $p\in L^1([0,1],\mathbb{R}^+)$ *such that* $|f(t,u)|\le p(t)\psi(\|u\|),\ for\ each\ (t,u)\in[0,1]\times\mathbb{R}$;
(H_4) *there exists a constant* $M>0$ *such that* $\dfrac{M}{\psi(\|u\|)\|p\|_{L^1}\Lambda}>1,\ where\ \|p\|_{L^1}\neq0.$
Then problem (1.1) *has at least one solution.*

Proof. We define $F : \mathbb{C} \to \mathbb{C}$ as in (2.8). The proof consists of several steps.

(1) F maps bounded sets into bounded sets in $\mathbb{C}([0,1],\mathbb{R})$.

Let $B_K = \{u \in \mathbb{C}([0,1],\mathbb{R}) : \|u\| \le K\}$ be a bounded set in $\mathbb{C}([0,1],\mathbb{R})$ and $u \in B_K$. Then we have

$$\begin{aligned}
|Fu(t)| \le & \frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^\alpha |f(s,u(s))|\,ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}|f(s,u(s))|\,ds \\
& + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^\alpha |f(s,u(s))|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))|\,ds \\
\le & \psi(\|u\|)\|p\|_{L^1}\left\{\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^\alpha ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}ds\right. \\
& \left. + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^\alpha ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\right\} \\
\le & \psi(\|u\|)\|p\|_{L^1}\Lambda.
\end{aligned}$$

Thus

$$\|Fu\| \le \psi(K)\|p\|_{L^1}\Lambda.$$

(2) F maps bounded sets into equicontinuous sets of $\mathbb{C}([0,1],\mathbb{R})$.

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$ and B_K be a bounded set of $\mathbb{C}([0,1],\mathbb{R})$ as before, then for $u \in B_k$ we have

$$\begin{aligned}
& |Fu(r_2) - Fu(r_1)| \\
= & \left|\frac{2}{\Gamma(\alpha+1)}\int_0^1 (r_2-r_1)(1-s)^\alpha f(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{r_1}(r_2-s)^{\alpha-1}f(s,u(s))ds\right. \\
& \left. - \frac{1}{\Gamma(\alpha)}\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}f(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^{r_1}(r_1-s)^{\alpha-1}f(s,u(s))ds\right| \\
\le & \left|\frac{2}{\Gamma(\alpha+1)}\int_0^1 (r_2-r_1)(1-s)^\alpha f(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}f(s,u(s))ds\right. \\
& \left. + \frac{1}{\Gamma(\alpha)}\int_0^{r_1} c_1(r_2-r_1)\left[(r_1-s)^{\alpha-2} - (r_2-s)^{\alpha-2}\right] f(s,u(s))ds\right| \\
\le & \frac{2}{\Gamma(\alpha+1)}\int_0^1 |r_2-r_1|(1-s)^\alpha p(s)\psi(K)ds + \frac{1}{\Gamma(\alpha)}\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}p(s)\psi(K)ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{r_1}\left\{c_1|r_2-r_1|\left[(r_1-s)^{\alpha-2} - (r_2-s)^{\alpha-2}\right]p(s)\psi(K)\right\}ds,
\end{aligned}$$

where c_1 is a positive constant. As $r_2 - r_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $u \in B_K$. Thus F is equicontinuous. As F satisfies the above assumptions, therefore, it follows by the Arzela-Ascoli theorem that $F : \mathbb{C}([0,1],\mathbb{R}) \to \mathbb{C}([0,1],\mathbb{R})$ is completely continuous.

(3) Let $\lambda \in (0,1)$ and let $u = \lambda F u$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
|u(t)| = & |\lambda Fu(t)| \\
\le & \frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^\alpha |f(s,u(s))|\,ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}|f(s,u(s))|\,ds \\
& + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^\alpha |f(s,u(s))|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))|\,ds \\
\le & \psi(\|u\|)\|p\|_{L^1}\Lambda,
\end{aligned}$$

and consequently

$$\frac{\|u\|}{\psi(\|u\|)\|p\|_{L^1}\Lambda} \leq 1.$$

In view of (H_4), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in \mathbb{C}([0,1],\mathbb{R}) : \|u\| < M\}.$$

Note that the operator $F : \overline{U} \to \mathbb{C}([0,1],\mathbb{R})$ is continuous and completely continuous (which is well known to be compact restricted to bounded sets). From the choice of U, there is no $u \in \partial U$ such that $u = \lambda F u$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $u \in \overline{\mathbb{U}}$, which is a solution of the problem (1.1). This completes the proof. □

Remark 3.2. In Theorem 3.3, if $\alpha = 2$, then $\Lambda = \frac{8}{3}\frac{1}{\Gamma(4)} + \frac{1}{3\Gamma(2)} + \frac{3}{2\Gamma(3)}$ and M satisfies $\frac{M}{\Lambda\psi(\|u\|)\|p\|_{L^1}} > 1$, where $\|p\|_{L^1} = \int_0^1 p(s)ds \neq 0$. Problem (1.1) reduces to Problem (1.2) and Problem (1.2) has at least one solution.

The third result is based on Boyed and Wong fixed point theorem below.

Definition 3.4. [23] Let $\mathbb{E}$ be a Banach space and let $A : \mathbb{E} \to \mathbb{E}$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\rho) < \rho$ for all $\rho > 0$ with the following property:

$$\|Ax - Ay\| \leq \Psi(\|x - y\|), \quad \forall x, y \in \mathbb{E}.$$

Lemma 3.3. *(Boyed and Wong)* [11] *Let $\mathbb{E}$ be a Banach space and let $A : \mathbb{E} \to \mathbb{E}$ be a nonlinear contraction. Then, A has a unique fixed point in $\mathbb{E}$.*

Theorem 3.4. *Suppose that*

(H_5) *there exists a continuous function $h : [0,1] \to \mathbb{R}^+$ such that*

$$|f(t,x) - f(t,y)| \leq h(t)\frac{|x-y|}{G + |x-y|}$$

for all $t \in [0,1]$ and $x, y \geq 0$, where

$$\begin{aligned} G =& \frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^\alpha h(s)ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1}h(s)ds \\ &+ \frac{2}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha+1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds. \end{aligned}$$

Then, problem (1.1) *has a unique solution.*

Proof. Let the operator $F : \mathbb{C} \to \mathbb{C}$ be defined as (2.8). We define a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\rho) = \frac{G\rho}{G+\rho}, \quad \forall \rho \geq 0,$$

such that $\Psi(0) = 0$ and $\Psi(\rho) < \rho$, for all $\rho > 0$. Let $u, v \in \mathbb{C}$. Then, we get

$$|f(s,u(s)) - f(s,v(s))| \leq h(s)\frac{|u-v|}{G + |u-v|}.$$

Thus

$$
\begin{aligned}
&|Fu(t)-Fv(t)|\\
\leq&\left\{\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}h(s)ds+\frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha+1}h(s)ds\right.\\
&\left.+\frac{2}{\Gamma(\alpha+1)}\int_0^1(t-s)(1-s)^{\alpha}h(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds\right\}\times\frac{\|u-v\|}{G+\|u-v\|}\\
\leq&\frac{G\|u-v\|}{G+\|u-v\|},\quad \forall t\in[0,1].
\end{aligned}
$$

This implies that $\|Fu-Fv\|\leq\Psi(\|u-v\|)$. Hence, F is a nonlinear contraction. Therefore, by Lemma 3.3, the operator F has a unique fixed point in $\mathbb{C}$, which is a unique solution of problem (1.1). □

Remark 3.3. In Theorem 3.4, if $\alpha=2$, then

$$
\begin{aligned}
G=&\frac{8}{3}\frac{1}{\Gamma(3)}\int_0^1(1-s)^2h(s)ds+\frac{4}{3}\frac{1}{\Gamma(2)}\int_0^1(1-s)^3h(s)ds\\
&+\frac{2}{\Gamma(3)}\int_0^1(1-s)^3h(s)ds+\frac{1}{\Gamma(2)}\int_0^1(1-s)h(s)ds.
\end{aligned}
$$

Problem (1.1) reduces to Problem (1.2) and Problem (1.2) has a unique solution.

As the fourth result, we prove the existence of solutions of (1.1) by using Krasnoselskii's fixed point theorem below.

Theorem 3.5. [18] *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*
(1) $Ax+By\in K$ *whenever* $x,y\in K$,
(2) A *is compact and continuous,*
(3) B *is a contraction mapping.*
Then, there exists $z\in K$ such that $z=Az+Bz$.

Theorem 3.6. *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying (H_1) and the following assumption holds:*
(H_6) $|f(t,u)|\leq\mu(t),\ \forall(t,u)\in[0,1]\times\mathbb{R}$, *and* $\mu\in\mathbb{L}^1([0,1],\mathbb{R}^+)$.
If

$$
L\left\{\frac{8}{3}\frac{1}{\Gamma(\alpha+2)}+\frac{2\alpha}{(\alpha+1)(\alpha+2)\Gamma(\alpha)}+\frac{2}{(\alpha+2)\Gamma(\alpha+1)}\right\}<1, \tag{3.11}
$$

then problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Setting $\max_{t\in[0,1]}|\mu(t)|=\|\mu\|$ and choosing a constant

$$R\geq\|\mu\|\Lambda,$$

where Λ is given by (2.9), and define $B_R=\{u\in\mathbb{C}:\|\mu\|\leq R\}$.
We define the operators F_1 and F_2 on the ball B_R as

$$(F_1u)(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds$$

$$(F_2u)(t) = -\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha} f(s,u(s))ds + \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1} f(s,u(s))ds$$
$$+ \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^{\alpha} f(s,u(s))ds.$$

For $u, v \in B_R$, we have

$$\begin{aligned}\|F_1u + F_2v\| \leq &\|\mu\| \sup_{t\in[0,1]} \left\{\frac{8}{3}\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha} ds \right.\\ &+ \frac{2\alpha}{\alpha+1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1} ds + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (t-s)(1-s)^{\alpha} ds\\ &\left. + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds\right\}\\ \leq &\|\mu\|\Lambda \leq R.\end{aligned}$$

Therefore, $F_1u + F_2v \in B_R$. In view of condition (3.11), it follows that F_2 is a contraction mapping.

Next, we will show that F_1 is compact and continuous. Continuity of f together with the assumption (H_6) implies that the operator F_1 is continuous and uniformly bounded on B_R. We define $\sup_{(t,u)\in[0,1]\times B_R} |f(t,u)| = f_{max} < \infty$. Then, for $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ and $u \in B_R$, we have

$$\begin{aligned}&|F_1u(t_2) - F_1u(t_1)|\\ \leq &f_{max}\left\{\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} c_2\,|t_2-t_1|\left[(t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}\right] ds\right\}.\end{aligned}$$

where c_2 is a positive constant. Actually, as $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to be zero. So F_1 is relatively compact on B_R. Hence, by the Arzela-Ascoli Theorem, F_1 is compact on B_R. Thus all the assumption of Theorem 3.5 are satisfied and the conclusion of Theorem 3.5 implies that problem (1.1) has at least one solution on $[0,1]$. This completes the proof. □

Remark 3.4. In Theorem 3.6, if $\alpha = 2$, then $L\left\{\frac{8}{3}\frac{1}{\Gamma(4)} + \frac{1}{3\Gamma(2)} + \frac{1}{2\Gamma(3)}\right\} < 1$. Problem (1.1) reduces to Problem (1.2) and Problem (1.2) has at least one solution.

4. Examples

Example 4.1. Consider the following boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) + \dfrac{|u(t)|\sin 3t}{5(1+t)^2(1+|u^6(t)|)e^t} = 0, \quad 0 < t < 1,\\ u(0) = \displaystyle\int_0^1 u(t)dt, \quad u(1) = \int_0^1 tu(t)dt. \end{cases} \tag{4.12}$$

Here, $f(t,u(t)) = \dfrac{|u(t)|\sin 3t}{5(1+t)^2(1+|u^6(t)|)e^t}$, $\alpha = \dfrac{3}{2}$. We find that

$$\Lambda = \frac{8}{3}\frac{1}{\Gamma(\frac{7}{2})} + \frac{12}{35}\frac{1}{\Gamma(\frac{3}{2})} + \frac{11}{7}\frac{1}{\Gamma(\frac{5}{2})} \approx 2.3714$$

Since, $|f(t,u) - f(t,v)| \leq \dfrac{1}{5}|u-v|$, then (H_1) is satisfied with $L = \frac{1}{5}$. We find that $L\Lambda \approx 0.47428 < 1$. Hence, by Theorem (3.1), problem (4.12) has a unique solution on $[0,1]$.

Example 4.2. Consider the following boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) + \dfrac{(t+1)\,|u|}{1+|u|} = 0, \quad 0 < t < 1 \\ u(0) = \displaystyle\int_0^1 u(t)dt, \quad u(1) = \int_0^1 tu(t)dt. \end{cases} \tag{4.13}$$

Here, $f(t,u(t)) = \dfrac{(t+1)\,|u|}{1+|u|}, \alpha = \dfrac{3}{2}$. Choosing $h(t) = t+1$, we find that

$$\begin{aligned} G =& \frac{8}{3}\frac{1}{\Gamma(\frac{5}{2})}\int_0^1 (1-s)^{\frac{3}{2}}(s+1)ds + \left(\frac{6}{5}\frac{1}{\Gamma(\frac{3}{2})} + \frac{2}{\Gamma(\frac{5}{2})}\right)\int_0^1 (1-s)^{\frac{5}{2}}(s+1)ds \\ &+ \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{\frac{1}{2}}(1+s)ds \\ \approx& 3.0830 \end{aligned}$$

Here, $|f(t,u) - f(t,v)| \le \dfrac{(t+1)\,|u-v|}{3.0830 + |u-v|}$. Therefore, by Theorem 3.4, the problem (4.13) has a unique solution on $[0,1]$.

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REFERENCES

[1] Abbas, S. and Benchohra, M., *Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative*, Commun. Math. Anal., **7** (2009), No. 2, 62–72

[2] Agarwal, R. P., Benchohra, M. and Hamani, S., *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109** (2010), No. 3, 973–1033

[3] Agarwal, R. P., Hristova, S. and O'Regan, D., *Stability of solutions to impulsive Caputo fractional differential equations*, Electron. J. Differential Equations, **2016**, Paper No. 58, 22 pp.

[4] Ahmad, B. and Sivasundaram, S., *Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions*, Commun. Appl. Anal., **13** (2009), No. 1, 121–127

[5] Ahmad, B. and Sivasundaram, S., *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl. Math. Comput., **217** (2010), No. 2, 480–487

[6] Anguraj, A. et al., *On new existence results for fractional integro-differential equations with impulsive and integral conditions*, Comput. Math. Appl., **66** (2014), No. 12, 2587–2594

[7] Bai, Z. and Lü, H., *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., **311** (2005), No. 2, 495–505

[8] Balachandran, K. and Trujillo, J. J., *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal., **72** (2010), No. 12, 4587–4593

[9] Benchohra, M., Graef, J. R. and Hamani, S., *Existence results for boundary value problems with non-linear fractional differential equations*, Appl. Anal., **87** (2008), No. 7, 851–863

[10] Benchohra, M., Hamani, S., and Ntouyas, S. K., *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal., **71** (2009), No. 7-8, 2391–2396

[11] Boyd, D. W. and Wong, J. S. W., *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458–464

[12] Cabada, A. and Hamdi, Z., *Nonlinear fractional differential equations with integral boundary value conditions*, Appl. Math. Comput., **228** (2014), 251–257

[13] Cabada, A. and Wang, G., *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., **389** (2012), No. 1, 403–411

[14] Chen, D. and Liu, W., *Chaotic behavior and its control in a fractional-order energy demand-supply system*, J. Comput. Nonlinear Dynam, **11** (2016), No. 6, Art. 061010, 7 pp.

[15] Chen, K., Liu, W. and Park, J., *Modified models for love and their dynamical properties*, Miskolc Math. Notes, **17** (2016), No. 1, 119–132

[16] Granas, A. and Dugundji, J., *Fixed point theory*, Springer Monographs in Mathematics, Springer, New York, 2003

[17] Guezane-Lakoud, A., Hamidane, N. and Khaldi, R. *Existence and uniqueness of solution for a second order boundary value problem*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., **62** (2013), No. 1, 121–129

[18] Krasnoselskiĭ, M. A., *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk (N.S.), **10** (1955), No. 1(63), 123–127

[19] Li, Y.-N., Sun, H.-R. and Feng, Z., *Fractional abstract Cauchy problem with order* $\alpha \in (1, 2)$, Dyn. Partial Differ. Equ., **13** (2016), No. 2, 155–177

[20] Liu, W., Wen, W. and Park, J., *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), No. 3, 766–777

[21] Podlubny, I., *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999

[22] Qarout, D., Ahmad, B. and Alsaedi, A., *Existence theorems for semi-linear Caputo fractional differential equations with nonlocal discrete and integral boundary conditions*, Fract. Calc. Appl. Anal., **19** (2016), No. 2, 463–479

[23] Sitthiwirattham, T., Tariboon, J. and Ntouyas, S. K., *Three-point boundary value problems of nonlinear second-order q-difference equations involving different numbers of q*, J. Appl. Math., **2013**, Art. ID 763786, 12 pp.

[24] Sun, Y. and Yan, K., *Existence of solutions for fractional differential equation three-point boundary value problems*, J. Appl. Anal. Comput., **6** (2016), No 4, 939–949

[25] Thongsalee, N., Ntouyas, S. K. and Tariboon, J., *Nonlinear Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober fractional integral conditions*, Fract. Calc. Appl. Anal., **19** (2016), No. 2, 480–497

[26] Zhang, S., *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl., **252** (2000), No. 2, 804–812

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