CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
33 (2017), No. 2, 219 - 232 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

On solutions of Saint-Venant's problem for elastic dipolar bodies with voids

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ABSTRACT. This study is dedicated to the Saint-Venant's problem in the context of the theory of porous dipolar bodies. We consider a right cylinder consisting of an inhomogeneous and anisotropic material. In the equilibrium equations of this problem, the axial variable is regarded as a parameter. The main result describes a class of semi-inverse solutions to the Saint-Venant's problem in terms of some generalized plane strain problems.

1. INTRODUCTION

Our study can be useful in fields of applications which deal with porous materials, such as geological materials, solid packed granular materials, the behavior of the human or animal bones and many others. The first investigations on materials with voids were published by Goodman and Cowin who in the paper [10] have created the well known granular theory. Then the studies of models of porosity structures retained the attention of many authors because many real structures can be modeled by this theory.

Also, similar studies appear in the paper [5] where the authors Cowin and Nunziato introduce, as in fact Goodman and Cowin did, an extra degree of freedom in order to investigate the mechanical behavior of porous solids in which the matrix material is elastic and the interstices are the voids of the material. This theory has found immediate applications to geological materials like rocks and soil and to manufactured porous materials, like ceramics and pressed powders. The basic feature of this theory is the introduction of a concept of material for which the bulk density is depicted as the product of two fields, the matrix material density field and the volume fraction field, respectively. See also [6]. The initial theory of Cowin and Nunziato was dedicated to materials that do not conduct heat, then in the papers by Nunziato and Cowin [21] and Ieşan [12], the thermal effect for materials with voids was proposed. The problem of generalized thermoelasticity in a thick-walled functionally graded materials cylinder with one relaxation time is presented in the paper [1], and in [2] a general solution to the field equations of the two-temperature generalized thermoelastic theory has been obtained in the context of the Green and Naghdi model. The generalized magneto-thermoelasticity theory, based on the Green-Naghdi model, is used to study the thermal shock problem of a fiber-reinforced anisotropic half-space in [23].

The consideration of the thermal effect is motivated by the fact that materials which operate at elevated temperatures will be subjected to heat flow at some time during the normal use. Thus, such heat flow will imply a non-linear temperature distribution which will give rise to the thermal stresses. Thus, the development, design and selection of materials for high temperature applications implies a great deal of care. The role of the

Received: 06.11.2016. In revised form: 31.03.2017. Accepted: 07.04.2017
2010 *Mathematics Subject Classification.* 74A15, 35A25, 74G50, 74A60.
Key words and phrases. *Thermoelasticity, dipolar body, voids, Saint-Venant's problem, semi-inverse solutions.*
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pertinent material properties and other variables which can affect the magnitude of the thermal stress must be considered.

The importance of the dipolar structure of materials was highlighted by many valuable researchers. So, the study of dipolar bodies began with the published results of R. D. Mindlin [20] as well as A. E. Green and R. S. Rivlin [11], who approached also in other papers the multipolar structures and in particular, the dipolar structures. Also M. E. Gurtin has devoted a few articles on multipolar structures. For instance, in [9], Gurtin together with E. Fried discover integral statements of the force balance, energy balance, and entropy imbalance for an interface between a body and its environment.

We want to outline that in the theory of dipolar continua the degrees of freedom for each particle are three translations and nine micro-deformations and each material point is constrained to deform homogeneously. More recently, other models of dipolar bodies ([15]-[18] and [22]) were introduced, where the volume fraction field was further used.

It should be noted that the interest and need to combine the theory of thermoelasticity of dipolar bodies with the granular theory have been emphasized since the studies on porous bodies. Moreover, in [20] the author highlights that the porosity structure of a continuous medium is influenced by the displacement field. Based on these findings, in our present paper we consider the theory of thermoelastic dipolar bodies with pores.

The Saint-Venant's problem has always been an attractive subject for many researchers who have approached it in many contexts. Over time, many studies have been published on this subject, from which we mention only a few. Among the first works on this subject, in a prominent place we find the book of Ieşan [13]. Also, we can mention the book [14] which enumerates some studies on Saint-Venant's problem. The paper [7] is a study of Saint-Venant's problem for a cylinder consisting of a homogeneous and isotropic elastic material. Also, the Saint-Venant's principle in the theory of elastic materials with voids is approached in [3]. In the context of viscoelasticity the Saint-Venant's problem was studied by Chiriţă [4].

We hope that the present study can be a first step to a better understanding of the dipolar structure and thermal stress in the study of materials with voids.

2. Basic equations

For convenience the notations and terminology chosen are almost identical to those of our study [19]. We will consider an anisotropic and inhomogeneous material which is an elastic porous and dipolar continuum and which occupies, at time $t = 0$, a regular domain B of the three-dimensional Euclidean space R^3, namely a right cylinder of length L. We denote by ∂B the boundary of B and by $\bar{B}$ the closure of B, $\bar{B} = B \cup \partial B$, where ∂B is assumed to be a piecewise smooth surface. The motion of the body will report to a fix system of rectangular Cartesian axes $Ox_i, i = 1, 2, 3$ which is chosen so that the generator of the cylinder is parallel to the Ox_3 axis and the plane x_1Ox_2 contains one of the cylinder's ends. By $D(x_3)$ we denote the interior of the bounded cross section which is made at distance x_3 from the base x_1Ox_2. We will adopt the Cartesian tensor notation. Points in B are denoted by x_j and $t \in [0, \infty)$ is the temporal variable. We will use the Einstein summation convention over repeated indices. A superposed dot stands for the derivative with respect to the t time variable, and a subscript j after a comma indicates partial differentiation with respect to the spatial argument x_j. All Greek indices are understood to range over the integers $(1, 2)$, while the Latin subscripts have the range $(1, 2, 3)$. When there is no likelihood of confusion, the spatial argument and the time argument of a function will be omitted. The regularity hypotheses on the considered functions will be implied without being stated, for clarity and simplification of the presentation. For instance, it is implied that the surface ∂B must be sufficiently regular to allow the application of the divergence theorem.

The behavior of our body will be characterized with the help of the displacement vector of components u_i, the dipolar displacement tensor of components φ_{ij} and the volume distribution function φ, which in the reference state is φ_0. In the following we will use the volume distribution function σ given by the difference $\sigma = \varphi - \varphi_0$.

Using the known procedure of Green and Rivlin we consider a new motion which differs from the given motion only by a superposed rigid motion defined by a rotation of uniform rigid body angular velocity and suppose that for the given motion, all characteristics of the body are unaltered by such superposed rigid motion. So we deduce the following kinetic relations, which give expressions of the strain measures ε_{ij}, γ_{ij}, χ_{ijk} and ϕ_i with regard to the variables of motion

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),\ \gamma_{ij} = u_{j,i} - \varphi_{ij},\ \chi_{ijk} = \varphi_{jk,i},\ \phi_i = \sigma_{,i}. \tag{2.1}$$

We restrict our considerations to the case where the materials have a center of symmetry. In the case that the body, in its reference configuration, is free from stress and has zero intrinsic equilibrated body forces and body couples, we assume that the internal energy density is a quadratic form with regards to its independent constitutive variables. Then from the principle of conservation of energy, we deduce that the internal energy density can be written in the following form

$$\begin{aligned}\Psi = {} & \frac{1}{2}C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + G_{ijmn}\varepsilon_{ij}\gamma_{mn} + F_{ijmnr}\varepsilon_{ij}\chi_{mnr} + \frac{1}{2}B_{ijmn}\gamma_{ij}\gamma_{mn} \\ & + D_{ijmnr}\gamma_{ij}\chi_{mnr} + \frac{1}{2}A_{ijkmnr}\chi_{ijk}\chi_{mnr} + \mathsf{a}_{ij}\varepsilon_{ij}\sigma + c_{ij}\gamma_{ij}\sigma + e_{ijk}\chi_{ijk}\sigma + \\ & + b_{ijk}\varepsilon_{ij}\phi_k + d_{ijk}\gamma_{ij}\phi_k + f_{ijkm}\chi_{ijk}\phi_m + \frac{1}{2}p_{ij}\phi_i\phi_j + d_i\sigma\phi_i + \frac{1}{2}\xi\sigma^2.\end{aligned} \tag{2.2}$$

As a consequence, we will use the following constitutive equations that give the expressions for the stress measures in terms of the strain measures

$$\begin{aligned}\tau_{ij} &= C_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr} + \mathsf{a}_{ij}\sigma + b_{ijk}\phi_k,\\ \eta_{ij} &= G_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} + c_{ij}\sigma + d_{ijk}\phi_k,\\ \mu_{ijk} &= F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + A_{ijkmnr}\chi_{mnr} + e_{ijk}\sigma + f_{ijkm}\phi_m,\\ \lambda_i &= b_{mni}\varepsilon_{mn} + d_{mni}\gamma_{mn} + f_{mnri}\chi_{mnr} + d_i\sigma + p_{ij}\phi_j,\\ s &= -\mathsf{a}_{ij}\varepsilon_{ij} - c_{ij}\gamma_{ij} - e_{ijk}\chi_{ijk} - \xi\sigma - d_i\phi_i.\end{aligned} \tag{2.3}$$

The material of the cylinder is assumed to be inhomogeneous in a cross-section of the cylinder, that is, all the constitutive coefficients C_{ijmn}, G_{ijmn}, ..., ξ from (2) are functions which depend on (x_1, x_2), that is

$$C_{ijmn} = C_{ijmn}\left(x_1, x_2\right),\ G_{ijmn} = G_{ijmn}\left(x_1, x_2\right),\ \ldots.,\xi = \xi\left(x_1, x_2\right)$$

Also, these constitutive coefficients obey the following symmetry relations

$$\begin{aligned}& C_{ijmn} = C_{jimn} = C_{mnij},\ G_{ijmn} = G_{jimn},\ F_{ijmnr} = F_{jimnr},\\ & \mathsf{a}_{ij} = \mathsf{a}_{ji},\ b_{ijk} = b_{jik},\ p_{ij} = p_{ji}.\end{aligned} \tag{2.4}$$

If we denote by n_i the components of the outward unit normal to the surface ∂B, then at each regular point of ∂B we can define the components of the surface traction t_i, the components of the surface couple μ_{jk} and the equilibrated surface traction h by

$$t_i = \left(\tau_{ij} + \eta_{ij}\right)n_j,\ \mu_{jk} = \mu_{ijk}n_i,\ h = \lambda_i n_i. \tag{2.5}$$

The boundary of each cross section will be denoted by ∂D so that the lateral boundary of the cylinder is $\partial D \times [0, L]$. In the absence of body loads, the equations of equilibrium in the context of elasticity of dipolar porous bodies are (see [13])

$$(\tau_{ij} + \eta_{ij})_{,j} = 0,\ \mu_{ijk,i} + \eta_{jk} = 0. \tag{2.6}$$

In the same context, the balance of the equilibrated forces has the form

$$\lambda_{i,i} + s = 0. \tag{2.7}$$

Together with equations (6) and (7) we will consider the following lateral boundary conditions

$$t_i = 0,\ \mu_{jk} = 0,\ h = 0,\ \text{on } \partial D \times [0, L] \tag{2.8}$$

and the end boundary conditions

$$\begin{aligned} t_{3i} &= t_i^{(1)},\ \mu_{3jk} = \mu_{jk}^{(1)},\ \lambda_3 = h^{(1)},\ \text{on } D(0),\\ t_{3i} &= t_i^{(2)},\ \mu_{3jk} = \mu_{jk}^{(2)},\ \lambda_3 = h^{(2)},\ \text{on } D(L), \end{aligned} \tag{2.9}$$

where $t_i^{(1)},\ t_i^{(2)},\ \mu_{jk}^{(1)},\ \mu_{jk}^{(2)},\ h^{(1)}$ and $h^{(2)}$ are given functions on their domain of definition. It is called the Saint-Venant's problem for the domain B, the problem of determining the displacement field u_i, the dipolar displacement field φ_{ij} and the volume distribution function σ that satisfy the equations (6) and (7), the lateral boundary conditions (8) and the end boundary conditions (9).
Using the procedure proposed by Ieşan and Ciarletta in [14] we deduce the necessary and sufficient conditions for the existence of a solution to the problem of Saint-Venant, namely

$$\begin{aligned} &\int_{D(0)} t_i^{(1)} dA + \int_{D(L)} t_i^{(2)} dA = 0,\ \int_{D(0)} \mu_{ij}^{(1)} dA + \int_{D(L)} \mu_{ij}^{(2)} dA = 0,\ \int_{D(0)} \varepsilon_{ijk} x_j t_k^{(1)} dA +\\ &+ \int_{D(L)} \varepsilon_{ijk} x_j t_k^{(2)} dA = 0,\ \int_{D(0)} \varepsilon_{ijk} x_i \mu_{jk}^{(1)} dA + \int_{D(L)} \varepsilon_{ijk} x_i \mu_{jk}^{(2)} dA = 0. \end{aligned} \tag{2.10}$$

In other words, the relations (10) state that for the equilibrium of the cylinder, the tractions on the cylinder bases should have a null resultant and null torque.

Lemma 2.1. *The equations of equilibrium (6) and the balance of the equilibrated forces (7) can be written in the form*

$$\begin{aligned} &[(C_{ijmn} + G_{ijmn}) u_{n,m} + (G_{mnij} + B_{ijmn})(u_{n,m} - \varphi_{mn}) +\\ &\quad + (F_{mnrij} + D_{ijmnr}) \varphi_{nr,m} + (\mathsf{a}_{ij} + c_{ij}) \sigma + (b_{ijk} + d_{ijk}) \sigma_{,k}]_{,j} = 0,\\ &[F_{ijkmn} u_{n,m} + D_{mnijk}(u_{n,m} - \varphi_{mn}) + A_{ijkmnr} \varphi_{nr,m} + e_{ijk}\sigma +\\ &+ f_{ijkm}\sigma_{,m}]_{,i} + G_{jkmn} u_{m,n} + B_{jkmn}(u_{n,m} - \varphi_{mn}) + D_{jkmnr}\varphi_{nr,m} + c_{jk}\sigma + d_{jkm}\sigma_{,m} = 0,\\ &[b_{mni} u_{m,n} + d_{mni}(u_{n,m} - \varphi_{mn}) + f_{mnri}\varphi_{nr,m} + d_i\sigma + p_{ij}\sigma_{,j}]_{,i} -\\ &\quad - \mathsf{a}_{ij} u_{i,j} - c_{ij}(u_{j,i} - \varphi_{ij}) - e_{ijk}\varphi_{jk,i} - \xi\sigma - d_i\sigma_{,i} = 0, \end{aligned} \tag{2.11}$$

all these equations taking place in the domain $B = D \times (0, L)$.

Proof. By direct calculations, we take into account the geometric equations (1) and the constitutive equations (3), then the equations of equilibrium (6) and the balance of the equilibrated forces can be restated in the form (11). $\square$

Lemma 2.2. *The lateral boundary conditions (8) can be written in the form*

$$\begin{aligned}
&[(C_{i\alpha mn}+G_{i\alpha mn})\,u_{n,m}+(G_{mni\alpha}+B_{i\alpha mn})\,(u_{n,m}-\varphi_{mn})+\\
&\qquad+(F_{mnri\alpha}+D_{i\alpha mnr})\,\varphi_{nr,m}+(\mathsf{a}_{i\alpha}+c_{i\alpha})\,\sigma+(b_{i\alpha k}+d_{i\alpha k})\,\sigma_{,k}]\,n_\alpha=0,\\
&[F_{\alpha jkmn}u_{n,m}+D_{mn\alpha jk}\,(u_{n,m}-\varphi_{mn})+A_{\alpha jkmnr}\varphi_{nr,m}+e_{\alpha jk}\sigma+\\
&\qquad+f_{\alpha jkm}\sigma_{,m}]\,n_\alpha=0,\\
&[b_{mn\alpha}u_{m,n}+d_{mn\alpha}\,(u_{n,m}-\varphi_{mn})+f_{mnr\alpha}\varphi_{nr,m}+d_\alpha\sigma+p_{\alpha j}\sigma_{,j}]\,n_\alpha=0,
\end{aligned}\tag{2.12}$$

on $\partial D\times(0,L)$.

Proof. By direct calculations, we substitute the geometric equations (1) and the constitutive equations (3) into the lateral boundary conditions (8) and obtain (12). □

Lemma 2.3. *The end boundary conditions (9) can be written in the form*

$$\begin{aligned}
&(C_{i3mn}+G_{i3mn})\,u_{n,m}+(G_{mni3}+B_{i3mn})\,(u_{n,m}-\varphi_{mn})+\\
&\qquad+(F_{mnri3}+D_{i3mnr})\,\varphi_{nr,m}+(\mathsf{a}_{ij}+c_{ij})\,\sigma+(b_{i3k}+d_{i3k})\,\sigma_{,k}=t_i^{(1)},\ \text{on } D(0)\\
&F_{i3kmn}u_{n,m}+D_{mni3k}\,(u_{n,m}-\varphi_{mn})+A_{i3kmnr}\varphi_{nr,m}+e_{i3k}\sigma+f_{i3km}\sigma_{,m}=\mu_{jk}^{(1)},\ \text{on } D(0)\\
&b_{mn3}u_{m,n}+d_{mn3}\,(u_{n,m}-\varphi_{mn})+f_{mnr3}\varphi_{nr,m}+d_3\sigma+p_{3j}\sigma_{,j}=h^{(1)},\ \text{on } D(0)\\
&(C_{i3mn}+G_{i3mn})\,u_{n,m}+(G_{mni3}+B_{i3mn})\,(u_{n,m}-\varphi_{mn})+\\
&\qquad+(F_{mnri3}+D_{i3mnr})\,\varphi_{nr,m}+(\mathsf{a}_{i3}+c_{i3})\,\sigma+(b_{i3k}+d_{i3k})\,\sigma_{,k}=t_i^{(1)},\ \text{on } D(L)\\
&F_{i3kmn}u_{n,m}+D_{mni3k}\,(u_{n,m}-\varphi_{mn})+A_{i3kmnr}\varphi_{nr,m}+e_{i3k}\sigma+f_{i3km}\sigma_{,m}=\mu_{jk}^{(1)},\ \text{on } D(L)\\
&b_{mn3}u_{m,n}+d_{mn3}\,(u_{n,m}-\varphi_{mn})+f_{mnr3}\varphi_{nr,m}+d_3\sigma+p_{3j}\sigma_{,j}=h^{(1)},\ \text{on } D(L)
\end{aligned}\tag{2.13}$$

Proof. Using equations (1) and (3) into the end boundary conditions (9), we obtain, directly, the equations (13). □

In the following we will denote by (S-V) the problem consisting of the equations (11), the lateral boundary conditions (12) and the end boundary conditions (13). Also, we assume that the internal energy density Ψ, defined in (2), and associated to the solution of the boundary value problem (S-V), is positive definite. According to [14], if the internal energy density Ψ is positive definite, then the boundary value problem (S-V) has a unique solution, except for a rigid displacement.

3. The main results

In several articles on Saint-Venant's problem (see for instance [4]) the state of the generalized plane strain for the interior of the cross section is defined.

According to this, the displacement field $\mathbf{u}$, the dipolar displacement field $\boldsymbol{\varphi}$ and the volume distribution σ depend on D only on x_1 and x_2

$$u_i=u_i(x_1,x_2)\,,\ \varphi_{ij}=\varphi_{ij}(x_1,x_2)\,,\ \sigma=\sigma(x_1,x_2)\,,\ (x_1,x_2)\in D\tag{3.14}$$

where the domain $D\subset R^2$ is a cross section of the considered cylinder.
As a consequence, also, other functions involved in the (S-V) problem depend only on x_1 and x_2. So, as far as the stress measures are concerned, we have

$$\begin{aligned}
&\tau_{ij}=\tau_{ij}(x_1,x_2)\,,\ \eta_{ij}=\eta_{ij}(x_1,x_2)\,,\ \mu_{ijk}=\mu_{ijk}(x_1,x_2)\,,\\
&\qquad\lambda_i=\lambda_i(x_1,x_2)\,,\ s=s(x_1,x_2)\,,\ (x_1,x_2)\in D
\end{aligned}$$

and if we denote by $\mathbf{U}$ the components of the displacement field $\mathbf{u}$, the dipolar displacement field $\boldsymbol{\varphi}$ and the volume distribution σ in the domain D, that is, $\mathbf{U}=(u_i,\varphi_{ij},\sigma)$, then the constitutive equations (3) can be rewritten in the form

$$
\begin{aligned}
\tau_{ij}(\mathbf{U}) &= C_{ijm\alpha}u_{m,\alpha} + G_{\alpha nij}\left(u_{n,\alpha} - \varphi_{\alpha n}\right) + F_{\alpha nrij}\varphi_{nr,\alpha} + \mathsf{a}_{ij}\sigma + b_{ij\alpha}\sigma_{,\alpha},\\
\eta_{ij}(\mathbf{U}) &= G_{ijm\alpha}u_{m,\alpha} + B_{ij\alpha n}\left(u_{n,\alpha} - \varphi_{\alpha n}\right) + D_{ij\alpha nr}\varphi_{nr,\alpha} + c_{ij}\sigma + d_{ij\alpha}\sigma_{,\alpha},\\
\mu_{ijk}(\mathbf{U}) &= F_{ijkm\alpha}u_{m,\alpha} + D_{\alpha nijk}\left(u_{n,\alpha} - \varphi_{\alpha n}\right) + A_{ijk\alpha nr}\varphi_{nr,\alpha} + e_{ijk}\sigma + f_{ijk\alpha}\sigma_{,\alpha},\\
\lambda_i(\mathbf{U}) &= b_{m\alpha i}u_{m,\alpha} + d_{\alpha ni}\left(u_{n,\alpha} - \varphi_{\alpha n}\right) + f_{\alpha nri}\varphi_{nr,\alpha} + d_i\sigma + p_{i\alpha}\sigma_{,\alpha},\\
s(\mathbf{U}) &= -\mathsf{a}_{\alpha j}u_{j,\alpha} - c_{\alpha j}\left(u_{j,\alpha} - \varphi_{\alpha j}\right) - e_{\alpha jk}\varphi_{jk,\alpha} - \xi\sigma - d_\alpha\sigma_{,\alpha}.
\end{aligned}
\tag{3.15}
$$

Now we are able to adapt the (S-V) problem above, for the domain D and its boundary ∂D, as a plane problem. We denote by (P-S-V) this plane problem and it consists of

- the equilibrium equations

$$
\begin{aligned}
&\left(\tau_{i\alpha}(\mathbf{U}) + \eta_{i\alpha}(\mathbf{U})\right)_{,\alpha} + f_i = 0,\ \mu_{\alpha jk,\alpha}(\mathbf{U}) + \eta_{jk}(\mathbf{U}) + g_{jk} = 0,\\
&\lambda_{\alpha,\alpha}(\mathbf{U}) + s(\mathbf{U}) + l = 0,\ in\ D
\end{aligned}
\tag{3.16}
$$

where $f_i = f_i\left(x_1, x_2\right)$ are the components of the body force, $g_{jk} = g_{jk}\left(x_1, x_2\right)$ are the components of the dipolar body force and $l = l\left(x_1, x_2\right)$ is the extrinsic equilibrated force;

- the boundary conditions

$$
\left(\tau_{i\alpha}(\mathbf{U}) + \eta_{i\alpha}(\mathbf{U})\right)n_\alpha = \tilde{t}_i,\ \mu_{\alpha jk}(\mathbf{U})n_\alpha = \tilde{m}_{jk},\ \lambda_\alpha(\mathbf{U})n_\alpha = \tilde{h},\ on\ \partial D
\tag{3.17}
$$

where $\tilde{t}_i$ are the components of the boundary traction, $\tilde{m}_{jk}$ are the components of the boundary couple traction and $\tilde{h}$ is the equilibrated boundary traction.
We can find another form of the problem (P-S-V) consisting of (16) and (17) by taking into account the constitutive equations (15). So, the equilibrium equations take the form

$$
\begin{aligned}
\mathcal{F}_i(\mathbf{U}) \equiv\ & \left[\left(C_{i\alpha m\beta} + G_{i\alpha m\beta}\right)u_{m,\beta} + \left(G_{\beta ni\alpha} + B_{i\alpha\beta n}\right)\left(u_{n,\beta} - \varphi_{\beta n}\right) + \right.\\
& \left. + \left(F_{\beta nri\alpha} + D_{i\alpha\beta nr}\right)\varphi_{nr,\beta} + \left(\mathsf{a}_{i\alpha} + c_{i\alpha}\right)\sigma + \left(b_{i\alpha\beta} + d_{i\alpha\beta}\right)\sigma_{,\beta}\right]_{,\alpha} = -f_i,\\
\mathcal{G}_{jk}(\mathbf{U}) \equiv\ & \left[F_{\alpha jkm\beta}u_{m,\beta} + D_{\beta n\alpha jk}\left(u_{n,\beta} - \varphi_{\beta n}\right) + A_{\alpha jk\beta nr}\varphi_{nr,\beta} + e_{\alpha jk}\sigma + f_{\alpha jk\beta}\sigma_{,\beta}\right]_{,\alpha} +\\
& + G_{jkm\alpha}u_{m,\alpha} + B_{jk\alpha n}\left(u_{n,\alpha} - \varphi_{\alpha n}\right) + D_{jk\alpha nr}\varphi_{nr,\alpha} + c_{jk}\sigma + d_{jk\alpha}\sigma_{,\alpha} = -g_{jk}\\
\mathcal{L}(\mathbf{U}) \equiv\ & \left[b_{m\beta\alpha}u_{m,\beta} + d_{\beta n\alpha}\left(u_{n,\beta} - \varphi_{\beta n}\right) + f_{\beta nr\alpha}\varphi_{nr,\beta} + d_\alpha\sigma + p_{\alpha\beta}\sigma_{,\beta}\right]_{,\alpha} -\\
& - \mathsf{a}_{\alpha j}u_{j,\alpha} - c_{\alpha j}\left(u_{j,\alpha} - \varphi_{\alpha j}\right) - e_{\alpha jk}\varphi_{jk,\alpha} - \xi\sigma - d_\alpha\sigma_{,\alpha} = -l
\end{aligned}
\tag{3.18}
$$

and the boundary conditions become

$$
\begin{aligned}
\mathcal{T}_i(\mathbf{U}) \equiv\ & \left[\left(C_{i\alpha m\beta} + G_{i\alpha m\beta}\right)u_{m,\beta} + \left(G_{\beta ni\alpha} + B_{i\alpha\beta n}\right)\left(u_{n,\beta} - \varphi_{\beta n}\right) + \right.\\
& \left. + \left(F_{\beta nri\alpha} + D_{i\alpha\beta nr}\right)\varphi_{nr,\beta} + \left(\mathsf{a}_{i\alpha} + c_{i\alpha}\right)\sigma + \left(b_{i\alpha\beta} + d_{i\alpha\beta}\right)\sigma_{,\beta}\right]n_\alpha = \tilde{t}_i,\\
\mathcal{M}_{jk}(\mathbf{U}) \equiv\ & \left[F_{\alpha jkm\beta}u_{m,\beta} + D_{\beta n\alpha jk}\left(u_{n,\beta} - \varphi_{\beta n}\right) + A_{\alpha jk\beta nr}\varphi_{nr,\beta} + e_{\alpha jk}\sigma + f_{\alpha jk\beta}\sigma_{,\beta}\right]n_\alpha = \tilde{m}_{jk},\\
\mathcal{H}(\mathbf{U}) \equiv\ & \left[b_{m\beta\alpha}u_{m,\beta} + d_{\beta n\alpha}\left(u_{n,\beta} - \varphi_{\beta n}\right) + f_{\beta nr\alpha}\varphi_{nr,\beta} + d_\alpha\sigma + p_{\alpha\beta}\sigma_{,\beta}\right]n_\alpha = \tilde{h}.
\end{aligned}
\tag{3.19}
$$

We assume that the functions f_i, g_{jk}, l, $\tilde{t}_i$, $\tilde{m}_{jk}$ and $\tilde{h}$ satisfy, on their domain of definition, the necessary conditions of regularity for the existence of a solution of the above plane problem (for example, as required in the paper [8]).
Recall that, according to [14], the necessary and sufficient conditions for the existence of a solution of the plane problem are that the resultant and the torque of the supply loads are null, that is

$$
\begin{aligned}
&\int_D f_i dA + \int_{\partial D}\tilde{t}_i ds = 0,\ \int_D g_{3i} dA + \int_{\partial D}\tilde{m}_{3i} ds = 0,\\
&\int_D \varepsilon_{3\alpha\beta}x_\alpha f_\beta dA + \int_{\partial D}\varepsilon_{3\alpha\beta}x_\alpha\tilde{t}_\beta ds = 0,\ \int_D \varepsilon_{3\alpha\beta}x_\alpha g_{3\beta} dA + \int_{\partial D}\varepsilon_{3\alpha\beta}x_\alpha\tilde{m}_{3\beta} ds = 0.
\end{aligned}
\tag{3.20}
$$

In order to obtain a generalized plane problem which corresponds to the system of equations (11) and to the lateral boundary conditions (12), we will consider that the balance equations (11) are satisfied on the plane domain D, and the lateral boundary conditions (12) are satisfied on the plane curve ∂D, the variable x_3 being considered as a parameter, $x_3 \in (0, L)$. As a consequence, on the cross-section D of the cylinder acts a resultant force having the components

$$(\mathcal{R}_i(\mathbf{U}), \mathcal{R}_{jk}(\mathbf{U}))\,,\ where\ \mathcal{R}_i(\mathbf{U})=\int_D (\tau_{3i}+\eta_{3i})\,(\mathbf{U})dA,\ \mathcal{R}_{jk}(\mathbf{U})=\int_D \mu_{3jk}(\mathbf{U})dA, \tag{3.21}$$

and a resultant moment of the traction of components

$$\mathcal{M}_i(\mathbf{U})=\int_D \varepsilon_{ijk}x_j\,(\tau_{3k}+\eta_{3k})\,(\mathbf{U})dA + \int_D \varepsilon_{ijk}x_j\mu_{33k}(\mathbf{U})dA. \tag{3.22}$$

As a particular case, from (22) we deduce that

$$\begin{aligned}\mathcal{M}_\alpha(\mathbf{U}) &= \varepsilon_{3\alpha\beta}\int_D x_\beta\,(\tau_{33}+\eta_{33})\,(\mathbf{U})dA + \varepsilon_{3\alpha\beta}\int_D x_\beta\mu_{333}(\mathbf{U})dA - \\ &\quad -x_3\varepsilon_{3\alpha\beta}\int_D (\tau_{3\beta}+\eta_{3\beta})\,(\mathbf{U})dA - x_3\varepsilon_{3\alpha\beta}\int_D \mu_{33\beta}(\mathbf{U})dA, \\ \mathcal{M}_3(\mathbf{U}) &= \varepsilon_{3\alpha\beta}\int_D x_\alpha\,(\tau_{3\beta}+\eta_{3\beta})\,(\mathbf{U})dA + \varepsilon_{3\alpha\beta}\int_D x_\alpha\mu_{33\beta}(\mathbf{U})dA.\end{aligned} \tag{3.23}$$

In the following theorem we will find a sufficient condition which allows the expression of the solution of Saint-Venant's problem in terms of the generalized plane strain.

Theorem 3.1. *If* $\mathbf{U}$ *is a solution of Saint-Venant's problem and the corresponding resultant force* $(\mathcal{R}_i(\mathbf{U}), \mathcal{R}_{jk}(\mathbf{U}))$ *and resultant moment of the traction* $\mathcal{M}_3(\mathbf{U})$ *are independent of* x_3, *then* $\mathbf{U}$ *can be expressed in terms of the generalized plane strain.*

Proof. First, the plane boundary value problem consisting of equations (11) on D and the lateral boundary conditions (12) on ∂D will be written in another form, by separating the terms that refer to x_3 which is considered as a parameter. So, equations (11) become

$$\begin{aligned}&[(C_{i\alpha m\beta} + G_{i\alpha m\beta})\,u_{m,\beta} + (G_{\beta ni\alpha} + B_{i\alpha\beta n})\,(u_{n,\beta} - \varphi_{\beta n}) + \\ &\quad + (F_{\beta nri\alpha}+D_{i\alpha\beta nr})\,\varphi_{nr,\beta}+(\mathsf{a}_{i\alpha}+c_{i\alpha})\,\sigma+(b_{i\alpha\beta}+d_{i\alpha\beta})\,\sigma_{,\beta}]_{,\alpha} + \\ &\quad + [(C_{i\alpha m3} + G_{i\alpha m3})\,u_{m,3} + (G_{3ni\alpha} + B_{i\alpha 3n})\,(u_{n,3} - \varphi_{3n}) + \\ &\quad + (F_{3nri\alpha}+D_{i\alpha 3nr})\,\varphi_{nr,3}+(b_{i\alpha 3}+d_{i\alpha 3})\,\sigma_{,3}]_{,\alpha} + \tau_{3i,3}(\mathbf{U}) + \eta_{3i,3}(\mathbf{U}) = 0, \\ &[F_{\alpha jkm\beta}u_{m,\beta}+D_{\beta n\alpha jk}\,(u_{n,\beta} - \varphi_{\beta n})+A_{\alpha jk\beta nr}\varphi_{nr,\beta}+e_{\alpha jk}\sigma+f_{\alpha jk\beta}\sigma_{,\beta}]_{,\alpha} + \\ &\quad +G_{jkm\alpha}u_{m,\alpha}+B_{jk\alpha n}\,(u_{n,\alpha}-\varphi_{\alpha n})+D_{jk\alpha nr}\varphi_{nr,\alpha}+c_{jk}\sigma+d_{jk\alpha}\sigma_{,\alpha}+ \\ &\quad +[F_{\alpha jkm3}u_{m,3}+D_{3n\alpha jk}\,(u_{n,3} - \varphi_{3n})+A_{\alpha jk3nr}\varphi_{nr,3}+f_{\alpha jk3}\sigma_{,3}]_{,\alpha}+\mu_{3jk,3}(\mathbf{U})=0 \\ &[b_{m\beta\alpha}u_{m,\beta} + d_{\beta n\alpha}\,(u_{n,\beta} - \varphi_{\beta n}) + f_{\beta nr\alpha}\varphi_{nr,\beta} + d_\alpha\sigma + p_{\alpha\beta}\sigma_{,\beta}]_{,\alpha} - \\ &\quad -\mathsf{a}_{\alpha j}u_{j,\alpha} - c_{\alpha j}\,(u_{j,\alpha} - \varphi_{\alpha j}) - e_{\alpha jk}\varphi_{jk,\alpha} - \xi\sigma - d_\alpha\sigma_{,\alpha} + \\ &\quad +[b_{m3\alpha}u_{m,3}+d_{3n\alpha}\,(u_{n,3}-\varphi_{3n})+f_{3nr\alpha}\varphi_{nr,3}+p_{\alpha 3}\sigma_{,3}]_{,\alpha}+\lambda_{3,3}(\mathbf{U})=0,\ in\ D\end{aligned} \tag{3.24}$$

and the lateral boundary conditions (12) are written in the form

$$\begin{aligned}&[(C_{i\alpha m\beta}+G_{i\alpha m\beta})\,u_{m,\beta}+(G_{\beta ni\alpha}+B_{i\alpha\beta n})\,(u_{n,\beta}-\varphi_{\beta n})+\\&\quad+(F_{\beta nri\alpha}+D_{i\alpha\beta nr})\,\varphi_{nr,\beta}+(\mathsf{a}_{i\alpha}+c_{i\alpha})\,\sigma+(b_{i\alpha\beta}+d_{i\alpha\beta})\,\sigma_{,\beta}]\,n_\alpha=\\&\quad=-\,[(C_{i\alpha m3}+G_{i\alpha m3})\,u_{m,3}+(G_{3ni\alpha}+B_{i\alpha 3n})\,(u_{n,3}-\varphi_{3n})+\\&\quad+(F_{3nri\alpha}+D_{i\alpha 3nr})\,\varphi_{nr,3}+(b_{i\alpha 3}+d_{i\alpha 3})\,\sigma_{,3}]\,n_\alpha,\end{aligned}$$

$$\begin{aligned}&[F_{\alpha jkm\beta}u_{m,\beta}+D_{\beta n\alpha jk}\,(u_{n,\beta}-\varphi_{\beta n})+A_{\alpha jk\beta nr}\varphi_{nr,\beta}+e_{\alpha jk}\sigma+f_{\alpha jk\beta}\sigma_{,\beta}]\,n_\alpha=\\&\quad=-\,[F_{\alpha jkm3}u_{m,3}+D_{3n\alpha jk}\,(u_{n,3}-\varphi_{3n})+A_{\alpha jk3nr}\varphi_{nr,3}+f_{\alpha jk3}\sigma_{,3}]\,n_\alpha,\\&[b_{m\beta\alpha}u_{m,\beta}+d_{\beta n\alpha}\,(u_{n,\beta}-\varphi_{\beta n})+f_{\beta nr\alpha}\varphi_{nr,\beta}+d_\alpha\sigma+p_{\alpha\beta}\sigma_{,\beta}]\,n_\alpha=\\&\quad=-\,[b_{m3\alpha}u_{m,3}+d_{3n\alpha}\,(u_{n,3}-\varphi_{3n})+f_{3nr\alpha}\varphi_{nr,3}+p_{\alpha 3}\sigma_{,3}]\,n_\alpha,\ \textit{on}\ \partial D\end{aligned}\tag{3.25}$$

In this way, we can consider that the boundary value problem consisting of (24) and (25) is a generalized plane strain boundary value problem, with the particular loads on D

$$\begin{aligned}&f_i=[(C_{i\alpha m3}+G_{i\alpha m3})\,u_{m,3}+(G_{3ni\alpha}+B_{i\alpha 3n})\,(u_{n,3}-\varphi_{3n})+\\&\quad+(F_{3nri\alpha}+D_{i\alpha 3nr})\,\varphi_{nr,3}+(b_{i\alpha 3}+d_{i\alpha 3})\,\sigma_{,3}]_{,\alpha}+\tau_{3i,3}(\mathbf{U})+\eta_{3i,3}(\mathbf{U}),\\&g_{jk}=[F_{\alpha jkm3}u_{m,3}+D_{3n\alpha jk}\,(u_{n,3}-\varphi_{3n})+A_{\alpha jk3nr}\varphi_{nr,3}+f_{\alpha jk3}\sigma_{,3}]_{,\alpha}+\mu_{3jk,3}(\mathbf{U}),\\&l=[b_{m3\alpha}u_{m,3}+d_{3n\alpha}\,(u_{n,3}-\varphi_{3n})+f_{3nr\alpha}\varphi_{nr,3}+p_{\alpha 3}\sigma_{,3}]_{,\alpha}+\lambda_{3,3}(\mathbf{U}),\end{aligned}\tag{3.26}$$

and the particular tractions on ∂D

$$\begin{aligned}&\tilde{t}_i=-\,[(C_{i\alpha m3}+G_{i\alpha m3})\,u_{m,3}+(G_{3ni\alpha}+B_{i\alpha 3n})\,(u_{n,3}-\varphi_{3n})+\\&\quad+(F_{3nri\alpha}+D_{i\alpha 3nr})\,\varphi_{nr,3}+(b_{i\alpha 3}+d_{i\alpha 3})\,\sigma_{,3}]\,n_\alpha,\\&\tilde{m}_{jk}=-\,[F_{\alpha jkm3}u_{m,3}+D_{3n\alpha jk}\,(u_{n,3}-\varphi_{3n})+A_{\alpha jk3nr}\varphi_{nr,3}+f_{\alpha jk3}\sigma_{,3}]\,n_\alpha,\\&\tilde{h}=-\,[b_{m3\alpha}u_{m,3}+d_{3n\alpha}\,(u_{n,3}-\varphi_{3n})+f_{3nr\alpha}\varphi_{nr,3}+p_{\alpha 3}\sigma_{,3}]\,n_\alpha\end{aligned}\tag{3.27}$$

Consistent with the necessary and sufficient conditions (20) we must have

$$\begin{aligned}&\int_D(\tau_{3i,3}(\mathbf{U})+\eta_{3i,3}(\mathbf{U}))\,dA=0,\ \int_D\mu_{3jk,3}(\mathbf{U})dA=0,\\&\int_D\varepsilon_{3\alpha\beta}x_\alpha\,(\tau_{3\beta,3}(\mathbf{U})+\eta_{3\beta,3}(\mathbf{U}))\,dA=0,\ \int_D\varepsilon_{3\alpha\beta}x_\alpha\mu_{33\beta,3}(\mathbf{U})dA=0.\end{aligned}\tag{3.28}$$

If we take into account the constitutive equations (3) and the fact that all the constitutive coefficients are functions which depend on (x_1,x_2), we deduce that

$$\tau_{3i,3}(\mathbf{U})+\eta_{3i,3}(\mathbf{U})=\tau_{3i}(\mathbf{U}_{,\mathbf{3}})+\eta_{3i}(\mathbf{U}_{,\mathbf{3}}),\ \mu_{3jk,3}(\mathbf{U})=\mu_{3jk}(\mathbf{U}_{,\mathbf{3}})\tag{3.29}$$

and, therefore, we can write (28) in the form

$$\begin{aligned}&\int_D(\tau_{3i}(\mathbf{U}_{,\mathbf{3}})+\eta_{3i}(\mathbf{U}_{,\mathbf{3}}))\,dA=0,\ \int_D\mu_{3jk}(\mathbf{U}_{,\mathbf{3}})dA=0,\\&\int_D\varepsilon_{3\alpha\beta}x_\alpha\,(\tau_{3\beta}(\mathbf{U}_{,\mathbf{3}})+\eta_{3\beta}(\mathbf{U}_{,\mathbf{3}}))\,dA=0,\ \int_D\varepsilon_{3\alpha\beta}x_\alpha\mu_{33\beta}(\mathbf{U}_{,\mathbf{3}})dA=0.\end{aligned}\tag{3.30}$$

Considering relations (30), (21)-(23) we deduce that a sufficient condition which allows the expression of the solution of Saint-Venant's problem in terms of the generalized plane strain is the fact that the resultant forces $(\mathcal{R}_i(\mathbf{U}),\mathcal{R}_{jk}(\mathbf{U}))$ and the resultant moment of the traction $\mathcal{M}_3(\mathbf{U})$ are independent of x_3. The proof of the theorem is complete. □

With the help of the results from Theorem 1, we obtain the next result.

Theorem 3.2. *If* $\mathbf{U}$ *is a solution of the Saint-Venant's problem so that the resultant forces* $(\mathcal{R}_i(\mathbf{U}),\mathcal{R}_{jk}(\mathbf{U}))$ *and the resultant moment of the traction* $\mathcal{M}_3(\mathbf{U})$ *are independent of* x_3*, then* $\mathcal{M}_1(\mathbf{U})$ *and* $\mathcal{M}_2(\mathbf{U})$ *are also independent of* x_3*.*

Proof. If we take into account the equations (6) and (7) and the lateral boundary conditions (8), we can obtain the relations

$$\left(\int_D x_\alpha \left(\tau_{33}(\mathbf{U}) + \eta_{33}(\mathbf{U})\right) dA\right)_{,3} = \int_D x_\alpha \left(\tau_{33,3}(\mathbf{U}) + \eta_{33,3}(\mathbf{U})\right) dA =$$

$$= -\int_D x_\alpha \left(\tau_{\beta 3,\beta}(\mathbf{U}) + \eta_{\beta 3,\beta}(\mathbf{U})\right) dA = -\int_D \left(x_\alpha \left(\tau_{\beta 3}(\mathbf{U}) + \eta_{\beta 3}(\mathbf{U})\right)\right)_{,\beta} dA + \tag{3.31}$$

$$+\int_D (\tau_{\alpha 3}(\mathbf{U}) + \eta_{\alpha 3}(\mathbf{U}))\, dA = -\int_{\partial D} x_\alpha \left(\tau_{\beta 3}(\mathbf{U}) + \eta_{\beta 3}(\mathbf{U})\right) n_\beta ds + \mathcal{R}_\alpha(\mathbf{U}) = \mathcal{R}_\alpha(\mathbf{U})$$

the last result being obtained with the help of the divergence theorem.
Now, we use the relation (23) in order to obtain the following relations

$$\mathcal{M}_{\alpha,3}(\mathbf{U}) = \varepsilon_{3\alpha\beta}\left(\int_D x_\beta \left(\tau_{33}(\mathbf{U}) + \eta_{33}(\mathbf{U})\right) dA\right)_{,3} - \varepsilon_{3\alpha\beta}\mathcal{R}_\beta(\mathbf{U}) - x_3\varepsilon_{3\alpha\beta}\mathcal{R}_{\beta,3}(\mathbf{U}) \tag{3.32}$$

Finally, we use the assumptions of the theorem and combine the results from relations (31) and (32) so that we are led to the desired result and the proof of Proposition 1 is concluded. □

It should be noted that taking into account that $\mathcal{R}_\alpha(\mathbf{U})$ is independent of x_3, by direct calculations, from (31) we deduce that

$$\int_D x_\alpha \left(\tau_{33}(\mathbf{U}_{,33}) + \eta_{33}(\mathbf{U}_{,33})\right) dA = 0. \tag{3.33}$$

Now we want to approach a class of semi-inverse solutions to Saint-Venant's problem, which can be expressed in terms of a state of the generalized plane strain.

With a suggestion given by (39) and (40) we will consider those solutions of Saint-Venant's problem having the property that the expressions of the displacemnt $\mathbf{u}_{,3}$ and the dipolar displacement $\boldsymbol{\varphi}_{,3}$ are the same as a rigid displacement and the volume distribution function σ is independent of x_3.

Let us denote by $(S-V)_s$ the class of these solutions and by $\mathbf{U}^0$ an element from this class, $\mathbf{U}^0 = \left(\mathbf{u}^0, \boldsymbol{\varphi}^0, \sigma^0\right) \in (S-V)_s$. If we take into account (21)-(23), (29) and (30) then we are led to the conclusion that

$$\left(\mathcal{R}_\alpha\left(\mathbf{U^0}\right)\right)_{,3} = 0,\ \left(\mathcal{R}_3\left(\mathbf{U^0}\right)\right)_{,3} = 0,\ \left(\mathcal{M}_i\left(\mathbf{U^0}\right)\right)_{,3} = 0. \tag{3.34}$$

In the case of a rigid displacement and a dipolar rigid displacement, by direct integration, we deduce that

$$\begin{aligned} u^0_\alpha &= -\frac{1}{2}a_\alpha x_3^2 - \varepsilon_{3\alpha\beta}a_4 x_\beta x_3 + v_\alpha(x_1,x_2), \\ u^0_3 &= (a_1x_1 + a_2x_2 + a_3)\, x_3 + v_3(x_1,x_2), \\ \varphi^0_{\alpha k} &= -\frac{1}{2}b_{\alpha k} x_3^2 - \varepsilon_{3\alpha\beta}b_{4k}x_\beta x_3 + w_{\alpha k}(x_1,x_2), \\ \varphi^0_{3k} &= (b_{1k}x_1 + b_{2k}x_2 + b_{3k})\, x_3 + w_{3k}(x_1,x_2), \\ \sigma^0 &= \psi(x_1,x_2), \end{aligned} \tag{3.35}$$

for $\mathbf{U}^0 = \left(\mathbf{u}^0, \boldsymbol{\varphi}^0, \sigma^0\right) \in (S-V)_s$. As usual, $\alpha = 1,2$. The coefficients a_m and b_{mk} in (35) are arbitrary constants, for $m = 1,2,3,4$ and $k = 1,2,3$. Also, v_i and w_{ik} are arbitrary functions independent of x_3, for $i,k = 1,2,3$.
If we substitute (35) into (3) we find the components of the stress in this class of solutions,

namely

$$
\begin{aligned}
&\tau_{ij}\left(\mathbf{U}^0\right)=C_{ij33}\left(a_1x_1+a_2x_2+a_3\right)-a_4C_{ij\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\\
&\qquad+F_{ijk33}\left(b_{1k}x_1+b_{2k}x_2+b_{3k}\right)-b_{4k}F_{ij\alpha33}\varepsilon_{3\alpha\beta}x_\beta+\tau_{ij}\left(\mathbf{U}\right),\\
&\eta_{ij}\left(\mathbf{U}^0\right)=G_{ij33}\left(a_1x_1+a_2x_2+a_3\right)-a_4G_{ij\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\\
&\qquad+D_{ijk33}\left(b_{1k}x_1+b_{2k}x_2+b_{3k}\right)-b_{4k}D_{ij\alpha33}\varepsilon_{3\alpha\beta}x_\beta+\eta_{ij}\left(\mathbf{U}\right),\\
&\mu_{ijk}\left(\mathbf{U}^0\right)=F_{ijk33}\left(a_1x_1+a_2x_2+a_3\right)-a_4F_{ijk\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\\
&\qquad+A_{ijkm33}\left(b_{1m}x_1+b_{2m}x_2+b_{3m}\right)-b_{4m}A_{ijkm\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\mu_{ijk}\left(\mathbf{U}\right),\\
&\lambda_i\left(\mathbf{U}^0\right)=b_{33i}\left(a_1x_1+a_2x_2+a_3\right)-a_4b_{i\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\\
&\qquad+f_{ik33}\left(b_{1k}x_1+b_{2k}x_2+b_{3k}\right)-b_{4k}f_{ik\alpha3}\varepsilon_{3\alpha\beta}x_\beta+\lambda_i\left(\mathbf{U}\right),\\
&\mathtt{s}\left(\mathbf{U}^0\right)=-\mathtt{a}_{33}\left(a_1x_1+a_2x_2+a_3\right)-a_4\mathtt{a}_{\alpha3}\varepsilon_{3\alpha\beta}x_\beta-\\
&\qquad-e_{33k}\left(b_{1k}x_1+b_{2k}x_2+b_{3k}\right)-b_{4k}e_{\alpha3k}\varepsilon_{3\alpha\beta}x_\beta-\mathtt{s}\left(\mathbf{U}\right).
\end{aligned}
\tag{3.36}
$$

The expressions for $\tau_{ij}(\mathbf{U})$, $\eta_{ij}(\mathbf{U})$, $\mu_{ijk}(\mathbf{U})$, $\lambda_i(\mathbf{U})$ and $s(\mathbf{U})$ are those from (15).
Considering the relations (36), the equilibrium equations (18) receive the form

$$
\begin{aligned}
&\mathcal{F}_i^0\left(\mathbf{U}^0\right)=\mathcal{F}_i\left(\mathbf{U}\right)+\\
&\quad+\left[\left(C_{i\alpha33}+G_{i\alpha33}\right)a_\beta x_\beta+\left(C_{i\alpha33}+G_{i\alpha33}\right)a_3-a_4\varepsilon_{3\gamma\beta}\left(C_{i\alpha\gamma3}+G_{i\alpha\gamma3}\right)x_\beta\right]_{,\alpha}+\\
&\quad+\left[(F_{i\alpha k33}+D_{i\alpha k33})b_{\beta k}x_\beta+(F_{i\alpha k33}+D_{i\alpha k33})b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}\left(F_{i\alpha k\gamma3}+D_{i\alpha k\gamma3}\right)x_\beta\right]_{,\alpha}=0,\\
&\mathcal{G}_{ij}^0\left(\mathbf{U}^0\right)=\mathcal{G}_{ij}\left(\mathbf{U}\right)+\left[F_{ij\alpha33}a_\beta x_\beta+F_{ij\alpha33}a_3-a_4\varepsilon_{3\gamma\beta}F_{ij\alpha\gamma3}x_\beta\right]_{,\alpha}+\\
&\quad+\left[A_{ij\alpha k33}b_{\beta k}x_\beta+A_{ij\alpha k33}b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}A_{ij\alpha k\gamma3}x_\beta\right]_{,\alpha}=0,\\
&\mathcal{L}^0\left(\mathbf{U}^0\right)=\mathcal{L}\left(\mathbf{U}\right)+\\
&\quad+\left[\left(b_{33\alpha}+d_{33\alpha}\right)a_\beta x_\beta+\left(b_{33\alpha}+d_{33\alpha}\right)a_3-a_4\varepsilon_{3\gamma\beta}\left(b_{3\gamma\alpha}+d_{3\gamma\alpha}\right)x_\beta\right]_{,\alpha}+\\
&\quad+\left[f_{\alpha k33}b_{\beta k}x_\beta+f_{\alpha k33}b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}f_{\alpha k\gamma3}x_\beta\right]_{,\alpha}-\\
&\quad-\mathtt{a}_{33}a_\beta x_\beta-\mathtt{a}_{33}a_3+a_4\varepsilon_{3\gamma\beta}\mathtt{a}_{\gamma3}x_\beta-e_{33k}b_{\beta k}x_\beta-e_{33k}b_{3k}+b_{4k}e_{\gamma3k}\varepsilon_{3\gamma\beta}x_\beta=0,
\end{aligned}
\tag{3.37}
$$

which are satisfied in the domain D.
Also, with the help of (36), the boundary conditions (19) receive the form

$$
\begin{aligned}
&\mathcal{T}_i^0\left(\mathbf{U}^0\right)=\mathcal{T}_i\left(\mathbf{U}\right)+\\
&\quad+\left[\left(C_{i\alpha33}+G_{i\alpha33}\right)a_\beta x_\beta+\left(C_{i\alpha33}+G_{i\alpha33}\right)a_3-a_4\varepsilon_{3\gamma\beta}\left(C_{i\alpha\gamma3}+G_{i\alpha\gamma3}\right)x_\beta\right]n_\alpha+\\
&\quad+\left[(F_{i\alpha k33}+D_{i\alpha k33})b_{\beta k}x_\beta+(F_{i\alpha k33}+D_{i\alpha k33})b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}\left(F_{i\alpha k\gamma3}+D_{i\alpha k\gamma3}\right)x_\beta\right]n_\alpha=0\\
&\mathcal{M}_{ij}^0\left(\mathbf{U}^0\right)=\mathcal{M}_{ij}\left(\mathbf{U}\right)+\left[F_{ij\alpha33}a_\beta x_\beta+F_{ij\alpha33}a_3-a_4\varepsilon_{3\gamma\beta}F_{ij\alpha\gamma3}x_\beta\right]n_\alpha+\\
&\quad+\left[A_{ij\alpha k33}b_{\beta k}x_\beta+A_{ij\alpha k33}b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}A_{ij\alpha k\gamma3}x_\beta\right]n_\alpha=0,\\
&\mathcal{H}^0\left(\mathbf{U}^0\right)=\mathcal{H}\left(\mathbf{U}\right)+\\
&\quad+\left[\left(b_{33\alpha}+d_{33\alpha}\right)a_\beta x_\beta+\left(b_{33\alpha}+d_{33\alpha}\right)a_3-a_4\varepsilon_{3\gamma\beta}\left(b_{3\gamma\alpha}+d_{3\gamma\alpha}\right)x_\beta\right]n_\alpha+\\
&\quad+\left[f_{\alpha k33}b_{\beta k}x_\beta+f_{\alpha k33}b_{3k}-b_{4k}\varepsilon_{3\gamma\beta}f_{\alpha k\gamma3}x_\beta\right]n_\alpha=0
\end{aligned}
\tag{3.38}
$$

which are satisfied on the surface ∂D.
In this way, the boundary value problem defined by (18) and (19) is replaced by the boundary value problem defined by (37) and (38).
If we take into account the conditions (20), we deduce that the necessary and sufficient conditions for the boundary value problem (37) and (38) to have a solution $\mathbf{U}=(u_i,\varphi_{jk},\sigma)$ are satisfied for any $a_s,\ s=1,2,3,4$ and $b_{sk},\ s=1,2,3,4,\ k=1,2,3$.
Now, we will consider three particular forms of the boundary value problem (37) and (38), namely in the case when $a_i=\delta_{ij}, a_4=0$ and $b_{ik}=\delta_{ij}, b_{4k}=0$, where j is

the number of the particular problem. The corresponding solutions will be denoted by $\mathbf{U}^{(s)} = \left(u_i^{(s)}, \varphi_{ij}^{(s)}, \sigma^{(s)}\right)$, $s = 1, 2, 3$. Also, we will denote by $\mathbf{U}^{(4)} = \left(u_i^{(4)}, \varphi_{ij}^{(4)}, \sigma^{(4)}\right)$ a solution of the boundary value problem (37), (38) in the particular case when $a_i = 0,\ i = 1, 2, 3,\ a_4 = 1$ and $b_{ik} = 0,\ i = 1, 2, 3,\ b_{4k} = 1$.
In other words, the functions $\mathbf{U}^{(s)} = \left(u_i^{(s)}, \varphi_{ij}^{(s)}, \sigma^{(s)}\right),\ s = 1, 2, 3, 4$ satisfy the equations

$$(3.39)\quad \mathcal{F}_i\left(\mathbf{U}^{(s)}\right) + f_i^{(s)} = 0,\ \mathcal{G}_{ij}\left(\mathbf{U}^{(s)}\right) + g_{ij}^{(s)} = 0,\ \mathcal{L}\left(\mathbf{U}^{(s)}\right) + l^{(s)} = 0,\ in\ D,$$

and the boundary conditions

$$(3.40)\qquad \mathcal{T}_i\left(\mathbf{U}^{(s)}\right) = \tilde{T}_i^{(s)},\ \mathcal{G}_{ij}\left(\mathbf{U}^{(s)}\right) = \tilde{M}_{ij}^{(s)},\ \mathcal{L}\left(\mathbf{U}^{(s)}\right) = \tilde{L}^{(s)},\ on\ \partial D.$$

In (39) and (40) we used the notations

$$\begin{aligned}
f_i^{(\beta)} &= \left[\left(C_{i\alpha 33} + G_{i\alpha 33}\right) x_\beta + \left(F_{i\alpha k33} + D_{i\alpha k33}\right)\delta_{k\beta} x_\beta\right]_{,\alpha},\\
f_i^{(3)} &= \left[\left(C_{i\alpha 33} + G_{i\alpha 33}\right) + \left(F_{i\alpha k33} + F_{i\alpha k33}\right)\delta_{k3}\right]_{,\alpha},\\
f_i^{(4)} &= -\left[\varepsilon_{3\gamma\beta}\left(C_{i\alpha\gamma 3} + G_{i\alpha\gamma 3}\right) x_\beta + \varepsilon_{3\gamma\beta}\left(F_{i\alpha k\gamma 3} + D_{i\alpha k\gamma 3}\right)\delta_{k3} x_\beta\right]_{,\alpha},\\
g_{ij}^{(\beta)} &= \left[\left(F_{ij\alpha 33} a_\beta + A_{ij\alpha k33} b_{\beta k}\right) x_\beta\right]_{,\alpha},\ g_{ij}^{(3)} = \left(F_{ij\alpha 33} a_3 + A_{ij\alpha k33} b_{3k}\right)_{,\alpha},\\
g_{ij}^{(4)} &= -\left[\varepsilon_{3\gamma\beta}\left(F_{ij\alpha\gamma 3} + \delta_{3k} A_{ij\alpha k\gamma 3}\right) x_\beta\right]_{,\alpha},\\
l^{(\beta)} &= \left[\left(b_{\alpha 33} + d_{\alpha 33}\right) x_\beta + f_{\alpha k33}\delta_{k\beta} x_\beta\right]_{,\alpha} - \left(\mathsf{a}_{33} + e_{k33}\delta_{k3}\right) x_\beta,\\
l^{(3)} &= \left[\left(b_{\alpha 33} + d_{\alpha 33}\right) + f_{\alpha k33}\delta_{k3}\right]_{,\alpha} - \left(\mathsf{a}_{33} + e_{k33}\delta_{k3}\right),\\
(3.41)\quad l^{(4)} &= -\left[\varepsilon_{3\gamma\beta}\left(b_{3\gamma\alpha} + d_{3\gamma\alpha} + f_{\alpha k\gamma 3}\right) x_\beta\right]_{,\alpha} + \varepsilon_{3\gamma\beta}\left(\mathsf{a}_{\gamma 3} + e_{\gamma 3k}\delta_{k3}\right),\\
\tilde{T}_i^{(\beta)} &= -\left[\left(C_{i\alpha 33} + G_{i\alpha 33}\right) x_\beta + \left(F_{i\alpha k33} + D_{i\alpha k33}\right)\delta_{k\beta} x_\beta\right] n_\alpha,\\
\tilde{T}_i^{(3)} &= -\left[\left(C_{i\alpha 33} + G_{i\alpha 33}\right) + \left(F_{i\alpha k33} + F_{i\alpha k33}\right)\delta_{k3}\right] n_\alpha,\\
\tilde{T}_i^{(4)} &= \left[\varepsilon_{3\gamma\beta}\left(C_{i\alpha\gamma 3} + G_{i\alpha\gamma 3}\right) x_\beta + \varepsilon_{3\gamma\beta}\left(F_{i\alpha k\gamma 3} + D_{i\alpha k\gamma 3}\right)\delta_{k3} x_\beta\right] n_\alpha,\\
\tilde{M}_{ij}^{(\beta)} &= \left[\left(F_{ij\alpha 33} a_\beta + A_{ij\alpha k33} b_{\beta k}\right) x_\beta\right] n_\alpha,\ \tilde{M}_{ij}^{(3)} = \left(F_{ij\alpha 33} a_3 + A_{ij\alpha k33} b_{3k}\right) n_\alpha,\\
\tilde{M}_{ij}^{(4)} &= \left[\varepsilon_{3\gamma\beta}\left(F_{ij\alpha\gamma 3} + \delta_{3k} A_{ij\alpha k\gamma 3}\right) x_\beta\right] n_\alpha,\ \tilde{L}^{(\beta)} = -\left[\left(b_{\alpha 33} + d_{\alpha 33}\right) x_\beta + f_{\alpha k33}\delta_{k\beta} x_\beta\right] n_\alpha,\\
\tilde{L}^{(3)} &= -\left[\left(b_{\alpha 33} + d_{\alpha 33}\right) + f_{\alpha k33}\delta_{k3}\right] n_\alpha,\ \tilde{L}^{(4)} = \left[\varepsilon_{3\gamma\beta}\left(b_{3\gamma\alpha} + d_{3\gamma\alpha} + f_{\alpha k\gamma 3}\right) x_\beta\right] n_\alpha.
\end{aligned}$$

Of course, if $\mathbf{U}^{(s)}$ are the solutions of the problems above, then we can write

$$(3.42)\qquad \mathbf{U} = \sum_{s=1}^{4} a_s \mathbf{U}^{(s)},$$

because all the problems formulated above are linear. Therefore, if $\mathbf{W}^0 = \left(\mathbf{u}^0, \boldsymbol{\varphi}^0, \sigma^0\right)$ is a solution from our class, $\mathbf{W}^0 \in (S - V)_s$, then we have

$$(3.43)\qquad \mathbf{W}^{(0)} = \sum_{s=1}^{4} a_s \mathbf{W}^{(s)},$$

where the components of the solutions are defined by

$$
\begin{aligned}
&u_\alpha^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + v_\alpha^{(\beta)},\ u_3^{(\beta)} = x_\beta x_3 + v_3^{(\beta)},\ u_\alpha^{(3)} = v_\alpha^{(3)},\\
&u_3^{(3)} = x_3 + v_3^{(3)},\ u_\alpha^{(4)} = \varepsilon_{3\beta\alpha}x_\beta x_3 + v_\alpha^{(4)},\ u_3^{(4)} = v_3^{(4)},\\
&\varphi_{\alpha 3}^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + w_{\alpha 3}^{(\beta)},\ \varphi_{33}^{(\beta)} = x_\beta x_3 + w_{33}^{(\beta)},\ \varphi_{\alpha 3}^{(3)} = w_{\alpha 3}^{(3)},\\
&\varphi_{33}^{(3)} = x_3 + w_{33}^{(3)},\ \varphi_{\alpha 3}^{(4)} = \varepsilon_{3\beta\alpha}x_\beta x_3 + w_{\alpha 3}^{(4)},\ \varphi_{33}^{(4)} = w_{33}^{(4)}.
\end{aligned}
\tag{3.44}
$$

According to (43) and (36), the components of the stress become

$$
\begin{aligned}
&\tau_{ij}\left(\mathbf{W}^{(0)}\right) = \sum_{r=1}^{4} a_r\tau_{ij}\left(\mathbf{W}^{(r)}\right),\ \eta_{ij}\left(\mathbf{W}^{(0)}\right) = \sum_{r=1}^{4} a_r\eta_{ij}\left(\mathbf{W}^{(r)}\right),\\
&\mu_{ijk}\left(\mathbf{W}^{(0)}\right) = \sum_{r=1}^{4} a_r\mu_{ijk}\left(\mathbf{W}^{(r)}\right),\ \lambda_i\left(\mathbf{W}^{(0)}\right) = \sum_{r=1}^{4} a_r\lambda_i\left(\mathbf{W}^{(r)}\right),\\
&\mathsf{s}\left(\mathbf{W}^{(0)}\right) = \sum_{r=1}^{4} a_r\mathsf{s}\left(\mathbf{W}^{(r)}\right),
\end{aligned}
\tag{3.45}
$$

where

$$
\begin{aligned}
&\tau_{ij}\left(\mathbf{W}^{(\alpha)}\right) = C_{ij33}x_\alpha + F_{ij\beta 33}\delta_{\beta 3}x_\alpha + \tau_{ij}\left(\mathbf{U}^{(\alpha)}\right),\\
&\tau_{ij}\left(\mathbf{W}^{(3)}\right) = C_{ij33} + F_{ij\beta 33}\delta_{\beta 3} + \tau_{ij}\left(\mathbf{U}^{(3)}\right),\\
&\tau_{ij}\left(\mathbf{W}^{(4)}\right) = -\varepsilon_{3\alpha\beta}C_{ij\alpha 3}x_\beta - \varepsilon_{3\gamma\beta}F_{ijk\gamma 3}\delta_{k3}x_\beta + \tau_{ij}\left(\mathbf{U}^{(4)}\right),\\
&\eta_{ij}\left(\mathbf{W}^{(\alpha)}\right) = G_{ij33}x_\alpha + D_{ij\beta 33}\delta_{\beta 3}x_\alpha + \eta_{ij}\left(\mathbf{U}^{(\alpha)}\right),\\
&\eta_{ij}\left(\mathbf{W}^{(3)}\right) = G_{ij33} + D_{ij\beta 33}\delta_{\beta 3} + \eta_{ij}\left(\mathbf{U}^{(3)}\right),\\
&\eta_{ij}\left(\mathbf{W}^{(4)}\right) = -\varepsilon_{3\alpha\beta}G_{ij\alpha 3}x_\beta - \varepsilon_{3\gamma\beta}D_{ijk\gamma 3}\delta_{k3}x_\beta + \eta_{ij}\left(\mathbf{U}^{(4)}\right),\\
&\mu_{ijk}\left(\mathbf{W}^{(\alpha)}\right) = F_{ijk33}x_\alpha + A_{ijk\beta 33}\delta_{\beta 3}x_\alpha + \mu_{ijk}\left(\mathbf{U}^{(\alpha)}\right),\\
&\mu_{ijk}\left(\mathbf{W}^{(3)}\right) = F_{ijk33} + A_{ijk\beta 33}\delta_{\beta 3} + \mu_{ijk}\left(\mathbf{U}^{(3)}\right),\\
&\mu_{ijk}\left(\mathbf{W}^{(4)}\right) = -\varepsilon_{3\alpha\beta}F_{ijk\alpha 3}x_\beta - \varepsilon_{3\gamma\beta}A_{ijkm\gamma 3}\delta_{m3}x_\beta + \mu_{ijk}\left(\mathbf{U}^{(4)}\right),\\
&\lambda_i\left(\mathbf{W}^{(\alpha)}\right) = b_{33i}x_\alpha + f_{i\beta 33}\delta_{\beta 3}x_\alpha + \lambda_i\left(\mathbf{U}^{(\alpha)}\right),\\
&\lambda_i\left(\mathbf{W}^{(3)}\right) = b_{33i} + f_{i\beta 33}\delta_{\beta 3} + \lambda_i\left(\mathbf{U}^{(3)}\right),\\
&\lambda_i\left(\mathbf{W}^{(4)}\right) = -\varepsilon_{3\alpha\beta}b_{3\alpha i}x_\beta - \varepsilon_{3\gamma\beta}f_{ik\gamma 3}\delta_{k3}x_\beta + \lambda_i\left(\mathbf{U}^{(4)}\right),\\
&\mathsf{s}\left(\mathbf{W}^{(\alpha)}\right) = -\mathsf{a}_{33}x_\alpha - e_{33\beta}\delta_{\beta 3}x_\alpha + \mathsf{s}\left(\mathbf{U}^{(\alpha)}\right)\\
&\mathsf{s}\left(\mathbf{W}^{(3)}\right) = -\mathsf{a}_{33} - e_{33\beta}\delta_{\beta 3} + \mathsf{s}\left(\mathbf{U}^{(3)}\right)\\
&\mathsf{s}\left(\mathbf{W}^{(4)}\right) = \varepsilon_{3\alpha\beta}\mathsf{a}_{\alpha 3}x_\beta + \varepsilon_{3\gamma\beta}e_{3\gamma k}\delta_{k3}x_\beta + \mathsf{s}\left(\mathbf{U}^{(4)}\right)
\end{aligned}
\tag{3.46}
$$

If we use the relations (39), (45) and (46) we deduce that

$$\tau_{\alpha i,\,\alpha}\left(\mathbf{W}^{(s)}\right)+\eta_{\alpha i,\,\alpha}\left(\mathbf{W}^{(s)}\right)=0,\ \mu_{\alpha jk,\,\alpha}\left(\mathbf{W}^{(s)}\right)+\eta_{jk}\left(\mathbf{W}^{(s)}\right)=0,$$

$$\lambda_{\alpha,\,\alpha}\left(\mathbf{W}^{(s)}\right)+\mathtt{s}\left(\mathbf{W}^{(s)}\right)=0,\ in\ D \tag{3.47}$$

Similarly, using relations (40), (45) and (46) we get

$$\tau_{\alpha i}\left(\mathbf{W}^{(s)}\right)n_\alpha+\eta_{\alpha i}\left(\mathbf{W}^{(s)}\right)n_\alpha=0,\ \mu_{\alpha jk}\left(\mathbf{W}^{(s)}\right)n_\alpha=0,\ \lambda_\alpha\left(\mathbf{W}^{(s)}\right)n_\alpha=0,\ on\ \partial D. \tag{3.48}$$

Now, we take into consideration the relations (22), (45), (47) and (48) so we get the following two equations

$$\mathcal{R}_\alpha\left(\mathbf{W}^{(0)}\right)-\int_D\sum_{s=1}^{4}\left[\tau_{3\alpha}\left(\mathbf{W}^{(s)}\right)+\eta_{3\alpha}\left(\mathbf{W}^{(s)}\right)\right]dA-$$

$$-\int_D\sum_{s=1}^{4}\left\{\left[x_\alpha\tau_{3\beta}\left(\mathbf{W}^{(s)}\right)\right]_{,\beta}+\left[x_\alpha\eta_{3\beta}\left(\mathbf{W}^{(s)}\right)\right]_{,\beta}\right\}dA=0, \tag{3.49}$$

$$\mathcal{R}_{\alpha\beta}\left(\mathbf{W}^{(0)}\right)-\int_D\sum_{s=1}^{4}\mu_{3\alpha\beta}\left(\mathbf{W}^{(s)}\right)dA-\int_D\sum_{s=1}^{4}\left[x_\alpha\ \mu_{3\beta\gamma}\left(\mathbf{W}^{(s)}\right)\right]_{,\gamma}dA=0. \tag{3.50}$$

We conclude our considerations noting that if $\left(\mathbf{u}^0,\varphi^0,\sigma^0\right)$ is a solution from our class, that is $\left(\mathbf{u}^0,\varphi^0,\sigma^0\right)\in(S-V)_s$, then, by using the notations

$$C_{3s}=\int_D\left[\tau_{33}\left(\mathbf{W}^{(s)}\right)+\eta_{33}\left(\mathbf{W}^{(s)}\right)\right]dA,\ C_{\beta s}=\int_D x_\beta\left[\tau_{33}\left(\mathbf{W}^{(s)}\right)+\eta_{33}\left(\mathbf{W}^{(s)}\right)\right]dA,$$

$$C_{4s}=\int_D\varepsilon_{3\alpha\beta}x_\alpha\left[\tau_{3\beta}\left(\mathbf{W}^{(s)}\right)+\eta_{3\beta}\left(\mathbf{W}^{(s)}\right)\right]dA,\ s=1,2,3,4$$

we obtain

$$\mathcal{R}_3\left(\mathbf{W}^{(0)}\right)=\sum_{s=1}^{4}a_sC_{3s},\ \mathcal{M}_\alpha\left(\mathbf{W}^{(0)}\right)=\sum_{s=1}^{4}\varepsilon_{3\alpha\beta}a_sC_{\beta s},\ \mathcal{M}_3\left(\mathbf{W}^{(0)}\right)=\sum_{s=1}^{4}a_sC_{4s}.$$

4. CONCLUSIONS

We presented an approach to the Saint-Venant's problem for a cylinder consisting of a dipolar porous material. To this end we reformulate the equilibrium equations as an operator over the cross section of the cylinder, considering the axial variable as a parameter. Thus, we could find the conditions in which the solution of Saint-Venant's problem might be treated as a generalized plane strain problem. Finally, we propose a class of semi-inverse solutions to Saint-Venant's problem.

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