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Abstract linear second order differential equations with two small parameters and depending on time operators

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ABSTRACT. In a real Hilbert space H consider the following singularly perturbed Cauchy problem

$$\begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + A(t)u_{\varepsilon\delta}(t) = f(t), \quad t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases}$$

where $A(t) : V \subset H \to H$, $t \in [0,\infty)$, is a family of linear self-adjoint operators, $u_0, u_1 \in H$, $f : [0,T] \mapsto H$ and ε, δ are two small parameters.

We study the behavior of solutions $u_{\varepsilon\delta}$ to this problem in two different cases: $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$; $\varepsilon \to 0$ and $\delta \to 0$, relative to solution to the corresponding unperturbed problem.

We obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and a relationship between solutions to both problems. We establish that the solution to the perturbed problem has a singular behavior, relative to the parameters, in the neighbourhood of $t = 0$. We show the boundary layer and boundary layer function in both cases.

1. INTRODUCTION

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$, and V be a real Hilbert space endowed with the norm $||\cdot||$. Let $A(t) : V \subset H \to H$, $t \in [0,T]$, be a family of linear self-adjoint operators. Consider the following Cauchy problem:

$$(P_{\varepsilon\delta}) \qquad \begin{cases} \varepsilon u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + A(t)u_{\varepsilon\delta}(t) = f(t), \quad t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases}$$

where $u_0, u_1, f : [0,T] \to H$ and ε, δ are two small parameters. We investigate the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:
(i) $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the solutions to the following unperturbed system:

$$(P_\delta) \qquad \begin{cases} \delta l'_\delta(t) + A(t)l_\delta(t) = f(t), \quad t \in (0,T), \\ l_\delta(0) = u_0; \end{cases}$$

(ii) $\varepsilon \to 0$ and $\delta \to 0$, relative to the solutions to the following unperturbed system:

$$(P_0) \qquad A(t)v(t) = f(t), \quad t \in [0,T).$$

The problem $(P_{\varepsilon\delta})$ is the abstract model of singularly perturbed problems of hyperbolic-parabolic type. Many physical processes are described by systems of type $(P_{\varepsilon\delta})$. For example in [3], is considered the equation

$$\rho v_{tt} + \gamma v_t = \sigma \Delta v$$

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(where ρ, γ, σ are the mass density per unit area of the membrane, the coefficient of viscosity of the medium, and the tension of the membrane, respectively). This equation characterizes the vibration of a membrane in a viscous medium, and it can be rewritten as

$$\varepsilon^2 u_{tt} + u_t = \Delta u,$$

with $\varepsilon = (\rho\sigma)^{1/2}/\gamma$.

In the case when the medium is highly viscous ($\gamma \gg 1$), or the density ρ is very small, we have $\varepsilon \to 0$ and the formal "limit" of this equation will be the following first order equation

$$u_t = \Delta u.$$

Without pretending to make a complete analysis, let us mention some works dedicated to the study of singularly perturbed Cauchy problems for linear or nonlinear differential equations of second order of type $(P_{\varepsilon\delta})$. The case when $\delta = 1$ was widely studied by various mathematicians (see, e.g. [4], [5], [10], [12] and the bibliography therein). In [7] the asymptotic behavior of solutions to singular perturbation problems for second order equations, as $\varepsilon \to 0$ and $\delta \to 0$, is studied. In [2], [8], [15], some numerical results about singular behaviour of solutions to the problem $(P_{\varepsilon\delta})$ for some ordinary differential equations and their applicability in modeling of different physical and engineering processes are presented.

The framework of our paper will be determined by the following conditions:

(H1) *V is separable and densely and continuously embedded in H* i. e.

$$|u|^2 \leq \gamma ||u||^2, \quad \forall u \in V;$$

(H2) *The operators $A(t) : V \subset H \to H$ are linear, self-adjoint and positive definite for $t \in [0, T]$, i.e. there exists $\omega > 0$ such that*

$$(A(t)u, u) \geq \omega \, ||u||^2, \quad \forall u \in V, \quad \forall t \in [0, T];$$

(H3) *For each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is twice continuously differentiable on $[0, T]$ and*

$$|(A'(t)u, v)| + |(A''(t)u, v)| \leq a_0 ||u|| \, ||v||, \quad \forall u, \, v \in V, \quad \forall t \in [0, T].$$

In [6] the following results concerning the solvability of problems $(P_{\varepsilon\delta})$ and (P_δ) are proved.

Theorem 1.1. *Let $T > 0$. Let us assume that the conditions* **(H1)**, **(H2)** *are fulfilled and for each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is continuously differentiable on $[0, T]$. If $u_0 \in V$, $u_1 \in H$ and $f \in L^2(0, T; H)$, then there exists the unique function $u_{\varepsilon\delta} \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$, $A(\cdot)u_{\varepsilon\delta} \in L^2(0, T; H)$ (strong solution) which satisfies the equation a.e. on $(0, T)$ and the initial conditions from $(P_{\varepsilon\delta})$. If, in addition, $u_1 \in V$, $f \in W^{1,2}(0, T; H)$, then $A(\cdot)u_{\varepsilon\delta} \in W^{1,2}(0, T; H)$ and $u_{\varepsilon\delta} \in W^{3,2}(0, T; H) \bigcap W^{1,2}(0, T; V)$.*

Theorem 1.2. *Let $T > 0$. Let us assume that the conditions* **(H1)**, **(H2)** *are fulfilled and for each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is continuously differentiable on $[0, T]$. If $u_0 \in H$ and $f \in L^2(0, T; H)$, then there exists the unique function $l_\delta \in W^{1,2}(0, T; H) \bigcap L^2(0, T; V)$ which satisfies a. e. on $(0, T)$ the equation and the initial conditions from (P_δ).*

The problems $(P_{\varepsilon\delta})$ and (P_δ) can be rewritten as follows:

$$(\mathcal{P}_\mu) \qquad \begin{cases} \mu \, U_\mu''(s) + U_\mu'(s) + \mathbb{A}(s) U_\mu(s) = F(s), \quad s \in (0, T/\delta), \\ U_\mu(0) = u_0, \quad U_\mu'(0) = \delta u_1, \end{cases}$$

and

$$(\mathcal{P}_0) \qquad \begin{cases} L'(s) + \mathbb{A}(s)L(s) = F(s), & s \in (0, T/\delta), \\ L(0) = u_0, \end{cases}$$

where $U_\mu(s) = u_{\varepsilon\delta}(\delta\, s)$, $L(s) = l_\delta(s\delta)$, $\mathbb{A}(s) = A(s\delta)$, $F(s) = f(s\delta)$ and $\mu = \varepsilon/\delta^2$. Using the results obtained in the paper [14] for solutions of problems $(\mathcal{P}_\mu)$ and $(\mathcal{P}_0)$ we get the following two theorems.

Theorem 1.3. *Let $T > 0$, $\delta \geq \delta_0 > 0$. Let us assume that the conditions* **(H1)**, **(H2)** *and* **(H3)** *are fulfilled. If $u_0, u_1 \in V$ and $f \in W^{1,2}(0,T;H)$, then there exist constants $\varepsilon_0 = \varepsilon_0(\gamma, a_0, \omega, \delta_0)$, $\varepsilon_0 \in (0,1)$, $C = C(T, \gamma, a_0, \omega, \delta_0) > 0$, such that*

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} \leq C\, \varepsilon^{1/4}\Big(|A(0)u_0| + |A^{1/2}(0)u_1| + ||f||_{W^{1,2}(0,T;H)}\Big),\ \forall \varepsilon \in (0, \varepsilon_0],$$

where $u_{\varepsilon\delta}$ and l_δ are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ), respectively.

Theorem 1.4. *Let $T > 0$, $\delta \geq \delta_0 > 0$. Let us assume that the conditions* **(H1)–(H3)** *are fulfilled. If $u_0, A(0)u_0, u_1, f(0) \in V$ and $f \in W^{2,2}(0,T;H)$, then there exist constants $\varepsilon_0 = \varepsilon_0(\gamma, a_0, \omega, \delta_0)$, $\varepsilon_0 \in (0,1)$, $C = C(T, \gamma, a_0, \omega, \delta_0) > 0$, such that*

$$\left|\left| u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta} exp\Big\{-\frac{\delta^2\, t}{\varepsilon}\Big\}\right|\right|_{C([0,T];H)} \leq C\varepsilon^{1/4} \quad \forall \varepsilon \in (0, \varepsilon_0],$$

where $H_{\varepsilon\delta} = \delta^{-1}\, f(0) - u_1 - \delta^{-1}\, A(0)u_0$,

$$C = |A^{3/2}(0)u_0| + |A^{1/2}(0)u_1| + |A^{1/2}(0)f(0)| + ||A(\cdot)H_{\varepsilon\delta}||_{L^2(0,\infty;H)} + ||f||_{W^{2,2}(0,T;H)},$$

$u_{\varepsilon\delta}$ and l_δ are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ), respectively.

2. Main results

The main result of this paper is presented in the following theorem.

Theorem 2.5. *Let $T > 0$. Let us assume that the conditions* **(H1)–(H3)** *are fullfiled. If $u_0, u_1 \in V$ and $f \in W^{1,2}(0,T;H)$, then there exists constant $C = C(T, \gamma, a_0, \omega) > 0$, such that*

$$(2.1) \qquad ||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];H)} \leq C\, \mathcal{M}\, \Big(\frac{\varepsilon^{1/4}}{\delta^{9/4}} + \sqrt{\delta}\Big), \quad \forall \varepsilon \in (0, \mu_0\, \delta^2], \quad \forall \delta \in (0,1],$$

where $u_{\varepsilon\delta}$ and v are strong solutions to the problems $(P_{\varepsilon\delta})$ and (P_0), respectively, and

$$\mathcal{M} = |A(0)u_0| + |A^{1/2}(0)u_1| + ||f||_{W^{1,2}(0,T;H)}, \quad \mu_0 = \min\Big\{1, \frac{\omega}{6\, a_0}\Big\}.$$

The function h_δ is the solution to the problem

$$(2.2) \qquad \begin{cases} \delta h'_\delta(t) + A(t)h_\delta(t) = 0, & t \in (0,T), \\ h_\delta(0) = u_0 - A^{-1}(0)f(0), \end{cases}$$

and $|h_\delta(t)| \leq |u_0 - A^{-1}(0)f(0)|e^{-\delta t/\omega}, \quad t \in [0,T]$.

The proof of this theorem is based on two key points:
(i) *a priori* estimates of solutions to the perturbed problem $(\mathcal{P}_\mu)$, which are uniform with respect to small parameter μ;
(ii) the relationship between solutions to the problems $(\mathcal{P}_\mu)$ and $(\mathcal{P}_0)$.

In what follows we will present some results, obtained in our previous researches, which will be used to prove the last theorem.

Lemma 2.1. *Let us assume that the condition* **(H1)** *is fulfilled and the operators* $A(t)$ *satisfy conditions* **(H2)** *and* **(H3)** *with* $t \in [0, \infty)$. *If* $u_0, u_1 \in V$ *and* $F \in W^{1,2}(0, \infty; H)$, *then there exist constants* $C(\gamma, a_0, \omega) > 0$ *and* $\mu_0 = \min\left\{1; \frac{\omega}{6a_0}\right\}$, *such that for every strong solution* U_μ *to the problem* (P_μ) *the following estimate holds:*

$$\big|\big|\mathbb{A}(\cdot)U_\mu\big|\big|_{L^\infty(0,\infty;H)} + ||U_\mu||_{C^1([0,\infty);H)} + \big|\big|\mathbb{A}^{1/2}(\cdot)U_\mu\big|\big|_{W^{1,2}(0,\infty;H)} \leq C\mathcal{M}, \tag{2.3}$$

$\forall \mu \in (0, \mu_0]$, *where* $\mathcal{M} = |A(0)u_0| + |A^{1/2}(0)u_1| + ||F||_{W^{1,2}(0,\infty;H)}$.

Proof. From Theorem 1.1 it follows that the problem (P_μ) has a unique strong solution U_μ which possesses the following properties: $U_\mu \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$, $\mathbb{A}(\cdot)U_\mu \in L^2(0, T; H)$ for any $T > 0$.

Denote by

$$E(U_\mu, t) = \mu^2|U'_\mu(t)|^2 + \frac{1}{2}|U_\mu(t)|^2 + \mu\big(\mathbb{A}(t)U_\mu(t), U_\mu(t)\big)+$$

$$+\int_0^t \big(\mathbb{A}(\tau)U_\mu(\tau), U_\mu(\tau)\big)\, d\tau + \mu\big(U'_\mu(t), U_\mu(t)\big) + \mu \int_0^t |U'_\mu(\tau)|^2 d\tau, \quad t \geq 0.$$

For every strong solution U_μ of problem (P_μ) we have

$$\frac{d}{dt}E(U_\mu, t) = \big(F(t), U_\mu(t) + 2\mu U'_\mu(t)\big) + \mu\big(\mathbb{A}'(t)U_\mu(t), U_\mu(t)\big), \quad \forall t \geq 0.$$

Integrating on $(0, t)$ we get

$$E(U_\mu, t) = E(U_\mu, 0) + \int_0^t \big(F(\tau), U_\mu(\tau) + 2\mu U'_\mu(\tau)\big)d\tau+$$

$$+\mu \int_0^t \big(\mathbb{A}'(\tau)U_\mu(\tau), U_\mu(\tau)\big)\, d\tau, \quad \forall t \geq 0. \tag{2.4}$$

If the conditions **(H2)** and **(H3)** are fulfilled for $t \in [0, \infty)$, then

$$\big(\mathbb{A}(t)u, u\big) = (A(\delta t)u, u) \geq \omega\, ||u||^2, \quad \forall u \in V, \quad \forall t \in [0, \infty) \tag{2.5}$$

and

$$\big|\big(\mathbb{A}'(t)u, v\big)\big| = \delta\, \big|\big(A'(\delta t)u, v\big)\big| \leq a_0\, \delta\, ||u||\, ||v||, \quad \forall u, v \in V, \quad \forall t \in [0, \infty], \tag{2.6}$$

$$\big|\big(\mathbb{A}''(t)u, v\big)\big| = \delta^2\, \big|\big(A''(\delta t)u, v\big)\big| \leq a_0\, \delta^2\, ||u||\, ||v||, \quad \forall u, v \in V, \quad \forall t \in [0, \infty], \tag{2.7}$$

Therefore

$$\int_0^t \big(F(\tau), U_\mu(\tau)\big)d\tau \leq \frac{1}{2}\int_0^t \big(\mathbb{A}(\tau)U_\mu(\tau), U_\mu(\tau)\big)d\tau + \frac{\gamma}{2\,\omega}\int_0^t |F(\tau)|^2 d\tau,$$

$$2\mu \int_0^t \big|\big(F(\tau), U'_\mu(\tau)\big)\big|d\tau \leq \mu^2 \int_0^t |U'_\mu(\tau)|^2 d\tau + \int_0^t |F(\tau)|^2 d\tau,$$

$$\mu \int_0^t \big|\big(\mathbb{A}'(\tau)U_\mu(\tau), U_\mu(\tau)\big)\big|d\tau \leq \frac{a_0\, \delta\mu}{\omega}\int_0^t \big(\mathbb{A}(\tau)U_\mu(\tau), U_\mu(\tau)\big)\, d\tau.$$

Thus

$$E(U_\mu,t)\le E(U_\mu,0)+\Big(\frac{1}{2}+\frac{a_0\,\delta\,\mu}{\omega}\Big)\int\limits_0^t\big(\mathbb{A}(\tau)U_\mu(\tau),U_\mu(\tau)\big)\,d\tau+$$

$$+\mu^2\int\limits_0^t\big|U_\mu'(\tau)\big|^2d\tau+\Big(1+\frac{\gamma}{2\omega}\Big)\int\limits_0^t\big|F(\tau)\big|^2d\tau,\quad\forall t\ge 0. \tag{2.8}$$

As

$$\mu^2|U_\mu'(t)|^2+\frac{1}{2}\,|U_\mu(t)|^2+\mu\big(U_\mu(t),U_\mu'(t)\big)\ge\frac{1}{3}\,\mu^2\,|U_\mu'(t)|^2+\frac{1}{8}\,|U_\mu(t)|^2,\quad\forall\mu\ge 0, \tag{2.9}$$

then from (2.8) it follows that

$$\mu^2|U_\mu'(t)|^2+|U_\mu(t)|^2+\int\limits_0^t\big(\mathbb{A}(\tau)U_\mu(\tau),U_\mu(\tau)\big)\,d\tau\le$$

$$\le C\,\Big(E(U_\mu,0)+\int\limits_0^t\big|F(\tau)|^2\,d\tau\Big),\quad\forall t\ge 0,\quad\delta\in(0,1],\quad\mu\in(0,\mu_0],\quad\mu_0=\min\Big\{1;\frac{\omega}{6a_0}\Big\}.$$

The last estimate implies

$$\mu\,||U_\mu'||_{L^\infty(0,\infty;H)}+||U_\mu\,||_{C([0,\infty);H)}+\big|\big|\mathbb{A}^{1/2}(\cdot)U_\mu\big|\big|_{L^2(0,\infty;H)}\le$$

$$\le C\,\mathcal{M},\quad\forall\mu\in(0,\mu_0],\quad\forall\delta\in(0,1]. \tag{2.10}$$

In what follows, let as observe that condition **(H3)** implies that $\mathbb{A}'(\cdot)\,U_\mu\in L^2(0,T;H)$ for any $T>0$. Then, from Theorem 1.1 it follows that function $V_\mu=U_\mu'$ is the strong solution to the problem

$$\begin{cases}\mu\,V_\mu''(t)+V_\mu'(s)+\mathbb{A}(t)V_\mu(t)=F'(t)-\mathbb{A}'(t)U_\mu(t),\quad t>0,\\ V_\mu(0)=u_1,\quad V_\mu'(0)=\dfrac{1}{\mu}\,\Big(f(0)-\delta u_1-A(0)u_0\Big)\end{cases}$$

and $V_\mu\in W^{2,2}(0,T;H)\bigcap L^2(0,T;V)$, $\mathbb{A}(\cdot)V_\mu\in L^2(0,T;H)$ for any $T>0$. Similarly as was obtained the equality (2.4), we get

$$E(V_\mu,t)=E(V_\mu,0)+\int\limits_0^t\big(F'(\tau)-\mathbb{A}'(\tau)U_\mu(\tau),V_\mu(\tau)+2\,\mu\,V_\mu'(\tau)\big)d\tau+$$

$$+\mu\int\limits_0^t\big(\mathbb{A}'(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau,\quad\forall t\ge 0.$$

Integrating by parts, we have

$$\int\limits_0^t\big(\mathbb{A}'(\tau)U_\mu(\tau),V_\mu'(\tau)\big)\,d\tau=\big(\mathbb{A}'(t)U_\mu(t),V_\mu(t)\big)-\big(\mathbb{A}'(0)U_\mu(0),V_\mu(0)\big)-$$

$$-\int\limits_0^t\big(\mathbb{A}''(\tau)U_\mu(\tau),V_\mu(\tau)\big)\,d\tau-\int\limits_0^t\big(\mathbb{A}'(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau,\quad\forall t\ge 0.$$

Then, using (2.5), (2.6), (2.7) and (2.10), we get

$$2\,\mu\Big|\int\limits_0^t \big(\mathbb{A}'(\tau)U_\mu(\tau),V_\mu'(\tau)\big)\,d\tau\Big| \le \frac{a_0\,\delta\,\mu}{\omega}\,\Big(\big|\mathbb{A}^{1/2}(0)u_0\big|^2+\big|\mathbb{A}^{1/2}(0)u_1\big|^2\Big)+$$

$$+\frac{a_0^2\,\delta^2\,\mu}{\omega}\,\big(U_\mu(t),U_\mu(t)\big)+\mu\,\big(\mathbb{A}(t)V_\mu(t),V_\mu(t)\big)+$$

$$+\frac{8\,a_0\,\delta^2\,\mu}{\omega}\int\limits_0^t \big(\mathbb{A}(\tau)U_\mu(\tau),U_\mu(\tau)\big)\,d\tau+\frac{a_0\,\delta\,\mu}{\omega}\,\Big(\frac{\delta}{8}+2\Big)\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau\le$$

$$\le C\,\mathcal{M}^2+\mu\,\big(\mathbb{A}(t)V_\mu(t),V_\mu(t)\big)+\frac{a_0\,\delta\,\mu}{\omega}\,\Big(\frac{\delta}{8}+2\Big)\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau,\quad \forall t\ge 0,$$

$$2\int\limits_0^t \big|\big(\mathbb{A}'(\tau)U_\mu(\tau),V_\mu(\tau)\big)\big|d\tau\le\frac{8\,a_0^2\,\delta}{\omega^2}\int\limits_0^t \big(\mathbb{A}(\tau)U_\mu(\tau),U_\mu(\tau)\big)\,d\tau+$$

$$+\frac{\delta}{8}\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau\le C\,\mathcal{M}^2+\frac{\delta}{8}\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau,\quad \forall t\ge 0,$$

$$\int\limits_0^t \big|\big(F'(\tau),V_\mu(\tau)+2\,\mu V_\mu'(\tau)\big)\big|d\tau\le\Big(1+\frac{2\,\gamma^2}{\omega}\Big)\int\limits_0^t \big|F'(\tau)\big|^2\,d\tau+$$

$$+\frac{1}{8}\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau+\mu^2\int\limits_0^t \big|V_\mu'(\tau)\big|^2\,d\tau,\quad \forall t\ge 0.$$

Thus, for $\delta\in(0,1]$ and $\mu\in(0,\mu_0]$ we have

$$E(V_\mu,t)\le E(V_\mu,0)+\mu\,\big(\mathbb{A}(t)V_\mu(t),V_\mu(t)\big)+$$

$$+\frac{2}{3}\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau+\mu^2\int\limits_0^t \big|V_\mu'(\tau)\big|^2d\tau+C\,\mathcal{M}^2,\quad \forall t\ge 0. \tag{2.11}$$

As the inequality (2.9) is also true for V_μ and

$$E\big(V_\mu,0\big)\le C\,\mathcal{M}^2,\quad \forall\delta\in(0,1),\quad \forall\mu\in(0,\mu_0],$$

then from (2.11) it follows that

$$\mu^2|V_\mu'(t)|^2+|V_\mu(t)|^2+\int\limits_0^t \big(\mathbb{A}(\tau)V_\mu(\tau),V_\mu(\tau)\big)\,d\tau\le$$

$$\le C\,\mathcal{M}^2,\quad \forall t\ge 0,\quad \delta\in(0,1],\quad \mu\in(0,\mu_0].$$

The last estimate implies

$$\mu\,||U_\mu''||_{L^\infty(0,\infty;H)}+||U_\mu'\,||_{C([0,\infty);H)}+\big|\big|\mathbb{A}^{1/2}(\cdot)U_\mu'\big|\big|_{L^2(0,\infty;H)}\le$$

$$\le\ C\,\mathcal{M},\quad \forall\mu\in(0,\mu_0],\quad \forall\delta\in(0,1]. \tag{2.12}$$

From (2.12), using the equation from $(\mathcal{P}_\mu)$ we get

$$\big|\big|\mathbb{A}(\cdot)U_\mu\big|\big|_{L^\infty(0,\infty;H)}\le\big|\big|F\big|\big|_{L^\infty(0,\infty;H)}+\big|\big|U_\mu'\big|\big|_{L^\infty(0,\infty;H)}+\mu\,\big|\big|U_\mu''\big|\big|_{L^\infty(0,\infty;H)}\le$$

$$\leq C\,\mathcal{M}, \quad \forall \mu \in (0, \mu_0], \quad \forall \delta \in (0, 1]. \tag{2.13}$$

Finally, using (2.10), (2.12) and (2.13), we obtain (2.3).

Lemma 2.1 is proved. □

In what follows for $\varepsilon > 0$ denote by

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon}}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big), \quad \forall \varepsilon > 0,$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 2.2. [11] *The function $K(t,\tau,\varepsilon)$ possesses the following properties:*

- **(i)** $K \in C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$;
- **(ii)** $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t > 0, \quad \forall \tau > 0;$
- **(iii)** $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0;$
- **(iv)** $K(0,\tau,\varepsilon) = \frac{1}{2\varepsilon}\exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}, \quad \forall \tau \geq 0;$
- **(v)** *For every $t > 0$ fixed and every $q, s \in \mathbb{N}$ there exist constants $C_1(q,s,t,\varepsilon) > 0$ and $C_2(q,s,t) > 0$ such that*

$$\big|\partial_t^s\partial_\tau^q K(t,\tau,\varepsilon)\big| \leq C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \forall \tau > 0;$$

- **(vi)** $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0;$
- **(vii)** *For every continuous function $\varphi : [0,\infty) \to H$ with $|\varphi(t)| \leq M\exp\{\gamma t\}$ the following equality is true:*

$$\lim_{t\to 0}\Big|\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big| = 0, \; for\ every\ \varepsilon \in \big(0,(2\gamma)^{-1}\big);$$

- **(viii)**

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0,$$

- **(ix)** *Let $q \in [0,1]$. Then*

$$\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^q\,d\tau \leq C\,\varepsilon^{q/2}\,\big(1+\sqrt{t}\big)^q, \quad \forall \varepsilon > 0, \quad \forall t \geq 0;$$

- **(x)** *Let $p \in (1,\infty]$ and $f : [0,\infty) \to H$, $f(t) \in W^{1,p}(0,\infty;H)$. Then*

$$\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big| \leq$$

$$\leq C(p)\,\|f'\|_{L^p(0,\infty;H)}\,\big(1+\sqrt{t}\big)^{\frac{p-1}{p}}\,\varepsilon^{(p-1)/2p}, \quad \forall \varepsilon > 0, \quad \forall t \geq 0.$$

Lemma 2.3. [11] *Let us assume that the condition* **(H1)** *is fulfilled and the operators $A(t)$ satisfy conditions* **(H2)** *and* **(H3)** *with $t \in [0,\infty)$. If $F \in L^\infty(0,\infty;H)$, U_μ is the strong solution to the problem $(\mathcal{P}_\mu)$ with $U_\mu \in W^{2,\infty}(0,\infty;H) \cap L^\infty(0,\infty;V)$, $\mathbb{A}(\cdot)U_\mu \in L^\infty(0,\infty;H)$, then the function w_μ, defined by*

$$w_\mu(s) = \int_0^\infty K(s,\tau,\mu)\,U_\mu(\tau)\,d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w'_\mu(s) + \mathbb{A}(s)w_\mu(s) = F_0(s,\mu) + \displaystyle\int_0^\infty K(s,\tau,\mu)\big[\mathbb{A}(s) - \mathbb{A}(\tau)\big]U_\mu(\tau)d\tau, \text{a.e. } s > 0, \\ W_\mu(0) = \varphi_\mu, \end{cases}$$

$$F_0(s,\mu) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3s}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{s}{\mu}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{s}{\mu}}\Big)\Big]\delta u_1 + \int_0^\infty K(s,\tau,\mu)\,F(\tau)\,d\tau,$$

$$\varphi_\mu = \int_0^\infty e^{-\tau}\,U_\mu(2\mu\tau)\,d\tau.$$

Proof of Theorem 2.5. During the proof, we will agree to denote by C all constants $C(T,\gamma,a_0,\omega)$. Consider the function $f \in W^{1,2}(0,T;H)$. Define on $[0,\infty)$ the function $\widetilde{f}$ as follows:

$$\widetilde{f}(t) = \begin{cases} f(t), \quad 0 \le t \le T, \\ \dfrac{2T-t}{T}f(T), \quad T < t < 2T, \\ 0, \quad t \ge 2T. \end{cases}$$

As

$$|f(t)|^2 = |f(\tau)|^2 + 2\int_\tau^t \big(f(s), f'(s)\big)ds \le$$

$$\le |f(\tau)|^2 + \int_\tau^t \big(|f(s)|^2 + |f'(s)|^2\big)ds \le |f(\tau)|^2 + ||f||^2_{W^{1,2}(0,T;H)}, \quad 0 \le \tau \le t \le T,$$

then integrating we get

$$T|f(t)|^2 \le \int_0^T |f(\tau)|^2 d\tau + T||f||^2_{W^{1,2}(0,T;H)}, \quad \forall t \in [0,T],$$

equivalent to

$$|f(t)| \le \sqrt{1 + \frac{1}{T}}\;||f||_{W^{1,2}(0,T;H)}, \quad \forall t \in [0,T].$$

Using the last estimate we obtain

$$||\widetilde{f}||_{W^{1,2}(0,\infty;H)} \le 2\sqrt{T + \frac{1}{T^2}}\;||f||_{W^{1,2}(0,T;H)}. \tag{2.14}$$

Also denote by

$$\widetilde{A}(t) = \begin{cases} A(t), \quad 0 \le t \le T, \\ A_0(t), \quad T < t \le a + T, \\ A_0(T+a), \quad t \ge a + T, \end{cases}$$

where

$$A_0(t) = A(T) + A'(T)(t-T) + \frac{1}{2}A''(T)(t-T)^2 -$$

$$-\Big[\frac{2}{3a}\,A''(T) + \frac{1}{a^2}\,A'(T)\Big]\,(t-T)^3 + \Big[\frac{1}{4a^2}\,A''(T) + \frac{1}{2a^3}\,A'(T)\Big]\,(t-T)^4,$$

and $a = \min\Big\{1, \dfrac{\omega}{8a_0}\Big\}$. If $\widetilde{\mathbb{A}}(t) = \widetilde{A}(\delta t)$, then, for each $u, v \in V$ the function $t \mapsto (\widetilde{\mathbb{A}}u, v)$ is twice continuously differentiable on $[0,\infty)$,

$$\big(\widetilde{\mathbb{A}}(t)u, u\big) \ge \frac{\omega}{2}\,||u||, \quad \forall u \in V, \quad \forall t \in [0,\infty),$$

$$\Big|(\widetilde{\mathbb{A}}'(t)u, v)\Big| + \Big|(\widetilde{\mathbb{A}}''(t)u, v)\Big| \le C\,\delta||u||\,||v||, \quad \forall u,\, v \in V, \quad \forall t \in [0,\infty), \quad \forall \delta \in (0,1]. \tag{2.15}$$

If we denote by $\widetilde{U}_\mu$ the unique strong solution to the problem $(\mathcal{P}_\mu)$, defined on $(0,\infty)$ instead of $(0,S)$ with $S=T/\delta$, $\widetilde{\mathbb{A}}$ instead of $\mathbb{A}$, $\widetilde{f}$ instead of f, and $\widetilde{F}(s)=\widetilde{f}(s\delta)$ then, from Lemma 2.1, it follows that $\widetilde{U}_\mu \in W^{2,\infty}(0,\infty;H)\cap W^{1,2}(0,\infty;V)$, $\widetilde{\mathbb{A}}(\cdot)\widetilde{U}_\mu \in L^\infty(0,\infty;H)$ and $\widetilde{U}_\mu = U_\mu$ on $(0,S)$. Moreover,

$$||\widetilde{F}||^2_{W^{1,2}(0,\infty;H)} = \int_0^\infty \big[|\widetilde{F}(s)|^2 + |\widetilde{F}'(s)|^2\big]\,ds = \int_0^\infty \Big[|\widetilde{f}(s\delta)|^2 + \Big|\frac{d\widetilde{f}}{ds}(s\delta)\Big|^2\Big]\,ds =$$

$$= \int_0^\infty \Big[\frac{1}{\delta}|\widetilde{f}(s)|^2 + \delta\,|\widetilde{f}'(s)|^2\Big]\,ds \le \Big(\delta + \frac{1}{\delta}\Big)\,||\widetilde{f}||^2_{W^{1,2}(0,\infty;H)}, \quad \forall \delta > 0.$$

Then the estimate (2.14) imply

$$||\widetilde{F}||_{W^{1,2}(0,\infty;H)} \le 2\big(\delta^{1/2} + \delta^{-1/2}\big)\sqrt{T + \frac{1}{T^2}}\,||f||_{W^{1,2}(0,T;H)} \le$$

$$\le C\mathcal{M}\delta^{-1/2}, \quad \forall \delta \in (0,1]. \tag{2.16}$$

Due to these estimates and Lemma 2.1, the following estimates

$$||\widetilde{\mathbb{A}}(\cdot)\widetilde{U}_\mu||_{L^\infty(0,\infty;H)} + ||\widetilde{U}_\mu\,||_{C^1([0,\infty;H)} + ||\widetilde{\mathbb{A}}^{1/2}(\cdot)\widetilde{U}_\mu||_{W^{1,2}(0,\infty;H)} \le$$

$$\le C\mathcal{M}\,\delta^{-1/2}, \quad \forall \mu \in (0,\mu_0], \quad \forall \delta \in (0,1], \tag{2.17}$$

are valid.

By Lemma 2.3, the function W_μ, defined by

$$W_\mu(s) = \int_0^\infty K(s,\tau,\mu)\,\widetilde{U}_\mu(\tau)\,d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} W'_\mu(s) + \widetilde{\mathbb{A}}(s)W_\mu(s) = \widetilde{F}_0(s,\mu) + \displaystyle\int_0^\infty K(s,\tau,\mu)\,\big[\widetilde{\mathbb{A}}(s) - \widetilde{\mathbb{A}}(\tau)\big]\widetilde{U}_\mu(\tau)\,d\tau, \text{ a.e. } s>0, \\ W_\mu(0) = \varphi_\mu, \end{cases} \tag{2.18}$$

where

$$\widetilde{F}_0(s,\mu) = \delta f_0(s,\mu)u_1 + \int_0^\infty K(s,\tau,\mu)\,\widetilde{F}(\tau)\,d\tau,$$

$$f_0(s,\mu) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3s}{4\mu}\Big\}\lambda\Big(\sqrt{\frac{s}{\mu}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{s}{\mu}}\Big)\Big],$$

$$\varphi_\mu = \int_0^\infty e^{-\tau}\,\widetilde{U}_\mu(2\mu\tau)\,d\tau.$$

Using the property $(\mathbf{x})$ from Lemma 2.2 and (2.17), we obtain that

$$||\widetilde{U}_\mu - W_\mu||_{C([0,\,s];\,H)} \le C\,\mathcal{M}\,\mu^{1/4}\,\delta^{-1/2}\sqrt{1+\sqrt{s}} \le$$

$$\le C\,\mathcal{M}\,\frac{\varepsilon^{1/4}}{\delta^{5/4}}, \quad \forall \varepsilon > 0, \quad \forall \delta \in (0,1], \quad \forall s \in [0,S]. \tag{2.19}$$

Denote by $R(s,\mu)=\tilde{L}(s)-W_\mu(s)$, where $\tilde{L}$ is the strong solution to the problem $(\mathcal{P}_0)$ with $\tilde{f}$ instead of f, $T=\infty$ and W_μ is the strong solution of (2.18). Then, due to Theorem 1.2, $R(\cdot,\mu)\in W^{1,2}(0,\infty;\,H)$ and R is the strong solution in H to the problem

$$\begin{cases} R'(s,\mu)+\widetilde{\mathbb{A}}(s)R(s,\mu)=\mathcal{F}(s,\mu)-\int\limits_0^\infty K(s,\tau,\mu)\,\big[\widetilde{\mathbb{A}}(s)-\widetilde{\mathbb{A}}(\tau)\big]\widetilde{U}_\mu(\tau)\,d\tau, \ \text{a.e. } t>0, \\ R(0,\mu)=u_0-\varphi_\mu, \end{cases}$$

where

$$\mathcal{F}(s,\mu)=\tilde{F}(s)-\int\limits_0^\infty K(s,\tau,\mu)\tilde{F}(\tau)\,d\tau-\delta\,f_0(s,\mu)\,u_1. \tag{2.20}$$

Taking the inner product in H by R and then integrating, we obtain

$$|R(s,\mu)|^2+2\int\limits_0^s\Big|\widetilde{\mathbb{A}}^{1/2}(\xi)R(\xi,\mu)\Big|^2\,d\xi\le|R(0,\mu)|^2+2\int\limits_0^s|\mathcal{F}(\xi,\mu)|\,|R(\xi,\mu)|\,d\xi-$$

$$-2\int\limits_0^s\int\limits_0^\infty K(\xi,\tau,\mu)\Big(\big[\widetilde{\mathbb{A}}(\xi)-\widetilde{\mathbb{A}}(\tau)\big]\widetilde{U}_\mu(\tau),R(\xi,\mu)\Big)\,d\tau d\xi,\quad\forall s\ge 0. \tag{2.21}$$

Using condition (2.15), property (**ix**) from Lemma 2 and (2.17), we get

$$\int\limits_0^s\int\limits_0^\infty K(\xi,\tau,\mu)\Big|\Big(\big[\widetilde{\mathbb{A}}(\xi)-\widetilde{\mathbb{A}}(\tau)\big]\widetilde{U}_\mu(\tau),R(\xi,\mu)\Big)\Big|\,d\tau d\xi\le$$

$$\le C\,\delta^{1/2}\,\mathcal{M}\int\limits_0^s||R(\xi,\mu)||\int\limits_0^\infty K(\xi,\tau,\mu)\,|\xi-\tau|\,d\tau\,d\xi\le$$

$$\le C\,\delta^{1/2}\,\mu^{1/2}\,\mathcal{M}\int\limits_0^s\Big|\widetilde{\mathbb{A}}^{1/2}(\xi)R(\xi,\mu)\Big|\,(1+\sqrt{\xi})\,d\xi\le$$

$$\le C\,\delta\,\mu\,\mathcal{M}^2\int\limits_0^s(1+\sqrt{\xi}\,)^2\,d\xi+\int\limits_0^s\Big|\widetilde{\mathbb{A}}^{1/2}(\xi)R(\xi,\mu)\Big|^2\,d\xi\le$$

$$\le C\,\mathcal{M}^2\,\frac{\varepsilon}{\delta^3}+\int\limits_0^s\Big|\widetilde{\mathbb{A}}^{1/2}(\xi)R(\xi,\mu)\Big|^2\,d\xi,\quad\forall s\in[0,S],\quad\forall\varepsilon\in(0,\mu_0\delta^2],\quad\forall\delta\in(0,1].$$

Then from (2.21) it follows that

$$|R(s,\mu)|^2+\int\limits_0^s\Big|\widetilde{\mathbb{A}}^{1/2}(\xi)R(\xi,\mu)\Big|^2\,d\xi\le|R(0,\mu)|^2+C\,\mathcal{M}^2\,\frac{\varepsilon}{\delta^3}+$$

$$+2\int\limits_0^s|\mathcal{F}(\xi,\mu)|\,|R(\xi,\mu)|\,d\xi,\quad\forall s\in[0,S],\quad\forall\varepsilon\in(0,\mu_0\delta^2],\quad\forall\delta\in(0,1].$$

Applying Lemma of Brézis (see, e.g., [9]), we get

$$|R(s,\mu)| + \left(\int_0^s \left| A^{1/2} R(\xi,\mu) \right|^2 d\xi \right)^{1/2} \leq$$

$$\leq C \Big(|R(0,\mu)| + \mathcal{M}\, \frac{\varepsilon^{1/2}}{\delta^{3/2}} + \int_0^s |\mathcal{F}(\xi,\mu)|\, d\xi \Big), \; \forall s \in [0,S],\; \forall \varepsilon \in (0,\mu_0\delta^2],\; \forall \delta \in (0,1]. \tag{2.22}$$

Using (2.17), we obtain

$$\big|R(0,\mu)\big| \leq \int_0^\infty e^{-\tau} \Big| \tilde{U}_\mu(2\mu\tau) - u_0 \Big|\, d\tau \leq \int_0^\infty e^{-\tau} \int_0^{2\mu\tau} \big| \tilde{U}'_\mu(\xi) \big|\, d\xi\, d\tau \leq$$

$$\leq C\, \mu\, \mathcal{M}\, \delta^{-1/2} = C\, \mathcal{M}\, \frac{\varepsilon}{\delta^{5/2}}, \quad \forall \varepsilon \in (0,\mu_0\, \delta^2], \quad \forall \delta \in (0,1]. \tag{2.23}$$

In what follows, we will estimate $\big|\mathcal{F}(s,\mu)\big|$. Using the property $(\mathbf{x})$ from Lemma 2.2 and (2.16), we have

$$\Big| \tilde{F}(s) - \int_0^\infty K(s,\tau,\mu)\, \tilde{F}(\tau)\, d\tau \Big| \leq C \|\widetilde{F}'\|_{L^2(0,\infty\,;\,H)} \big(1+\sqrt{s}\big)^{\frac{1}{2}} \mu^{\frac{1}{4}} \leq$$

$$\leq C\, \mathcal{M}\, \frac{\varepsilon^{1/4}}{\delta^{5/4}}, \quad \forall \mu > 0, \quad \forall s > 0. \tag{2.24}$$

Since $e^\xi \lambda(\sqrt{\xi}) \leq C, \quad \forall \xi \geq 0$, the estimates

$$\int_0^s e^{3\xi/4\mu} \lambda\big(\sqrt{\xi/\mu}\big)\, d\xi = \mu \int_0^{s/\mu} e^{3\,\xi/4}\, \lambda\big(\sqrt{\xi}\big)\, d\xi \leq C\, \mu \int_0^\infty e^{-\xi/4}\, d\xi \leq C\mu, \quad \forall s \geq 0,$$

$$\int_0^s \lambda\Big(\frac{1}{2}\sqrt{\frac{\xi}{\mu}}\Big)\, d\xi \leq \mu \int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\xi}\Big)\, d\xi \leq C\, \mu, \quad \forall s \geq 0, \quad \forall \mu > 0,$$

hold, then

$$\Big| \delta \int_0^s f_0(\xi,\mu)\, u_1 d\xi \Big| \leq C\, \delta\mu |u_1| \leq C\, \mathcal{M}\, \frac{\varepsilon}{\delta}, \quad \forall \varepsilon > 0, \quad \forall \delta > 0, \quad \forall s \geq 0. \tag{2.25}$$

Using (2.24) and (2.25), from (2.20), we obtain

$$\int_0^s \big| \mathcal{F}(\xi,\mu) \big|\, d\xi \leq \int_0^s \Big| \tilde{F}(\xi) - \int_0^\infty K(\xi,\tau,\mu)\, \tilde{F}(\tau)\, d\tau \Big| d\xi + C\, \mathcal{M}\, \frac{\varepsilon}{\delta} \leq$$

$$\leq C\, \mathcal{M}\, \Big(S \frac{\varepsilon^{1/4}}{\delta^{5/4}} + \frac{\varepsilon}{\delta} \Big) \leq C\, \mathcal{M}\, \Big(\frac{\varepsilon^{1/4}}{\delta^{9/4}} + \frac{\varepsilon}{\delta} \Big), \quad \forall s \in [0,S], \quad \forall \varepsilon > 0, \quad \delta > 0. \tag{2.26}$$

From (2.22), using (2.23) and (2.26), we get the estimate

$$||R||_{C([0,\,S];\,H)} \leq C\, \mathcal{M}\, \frac{\varepsilon^{1/4}}{\delta^{9/4}}, \quad \forall \varepsilon \in (0,\mu_0\, \delta^2], \quad \forall \delta \in (0,1]. \tag{2.27}$$

Consequently, from (2.19) and (2.27), we deduce

$$||\tilde{U}_\mu - \tilde{L}||_{C([0,S];H)} \le ||\tilde{U}_\mu - W_\mu||_{C([0,S];H)} + ||R||_{C([0,S];H)} \le$$

$$\le C\,\mathcal{M}\,\frac{\varepsilon^{1/4}}{\delta^{9/4}}, \quad \forall \varepsilon \in (0, \mu_0\,\delta^2], \quad \forall \delta \in (0,1]. \tag{2.28}$$

Since $U_\mu(s) = \tilde{U}_\mu(s)$, $L(s) = \tilde{L}(s)$, for all $s \in [0,S]$, $U_\mu(s) = u_{\varepsilon\delta}(\delta\, s)$ and $L(s) = l_\delta(\delta\, s)$, from (2.28) we get

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} \le C\,\mathcal{M}\,\frac{\varepsilon^{1/4}}{\delta^{9/4}}, \quad \forall \varepsilon \in (0, \mu_0\,\delta^2], \quad \forall \delta \in (0,1]. \tag{2.29}$$

In what follows, let us denote by $R_1(t,\delta) = l_\delta(t) - v(t) - h_\delta(t)$, where l_δ is the solution to the problem (P_δ), v is the solution to the problem (P_0) and h_δ is the solution to the problem

$$\begin{cases} \delta h_\delta'(t) + A(t) h_\delta(t) = 0, \quad t \in (0,T), \\ h_\delta(0) = u_0 - A^{-1}(0) f(0). \end{cases}$$

Due to Theorem 1.2 and condition (**H3**), from the last statements, we deduce that R_1 is the strong solution to the problem

$$\begin{cases} \delta R_1'(t,\delta) + A(t) R_1(t,\delta) = -\delta A^{-1}(t)\big(f'(t) - A'(t)A^{-1}(t)\ f(t)\big), \quad t \in (0,T), \\ R_1(0) = 0. \end{cases}$$

Taking the inner product in H by R_1 and then integrating, we obtain

$$\delta\,|R_1(t,\delta)|^2 + 2\int_0^t \left|A^{1/2}(\tau)\,R_1(\tau,\delta)\right|^2\,d\tau =$$

$$-2\delta\int_0^t \Big(A^{-1}(\tau)\big(f'(\tau) - A'(\tau)A^{-1}(\tau)\ f(\tau)\big), R_1(\tau,\delta)\Big)\,d\tau, \quad t \in (0,T). \tag{2.30}$$

Using conditions (**H1**), (**H2**) and (**H3**), we get

$$2\delta\int_0^t \left|\big(A^{-1}(\tau)\,f'(\tau), R_1(\tau,\delta)\big)\right|\,d\tau \le 2\,\delta\,\frac{\gamma}{\omega}\int_0^t |f'(\tau)|\,|R_1(\tau,\delta)|\,d\tau \le$$

$$\le \frac{1}{2}\int_0^t |A^{1/2} R_1(\tau,\delta)|^2\,d\tau + \frac{2\delta^2\gamma^3}{\omega^3}\int_0^t |f'(\tau)|^2\,d\tau$$

and

$$2\delta\int_0^t \left|\big(A'(\tau)A^{-1}(\tau)f(\tau), A^{-1}(\tau)R_1(\tau,\delta)\right|\,d\tau \le 2\delta a_0\int_0^t ||A^{-1}(\tau)f(\tau)||\,||A^{-1}(\tau)R_1(\tau,\delta)||d\tau \le$$

$$\le \frac{1}{2}\int_0^t \left|A^{1/2}\,R_1(\tau,\delta)\right|^2\,d\tau + \frac{2\delta^2\,a_0^2\,\gamma^3}{\omega^5}\int_0^t |f(\tau)|^2\,d\tau.$$

Then from (2.30) we obtain

$$\delta\,|R_1(t,\delta)|^2 \le C\,\mathcal{M}^2\,\delta^2, \quad \forall t \in [0,T], \quad \forall \delta \in (0,1].$$

Consequently, we get

$$|R_1(t,\delta)| \leq C\,\mathcal{M}\,\sqrt{\delta}, \quad \forall t \in [0,T], \quad \forall \delta \in (0,1]. \tag{2.31}$$

Thus, the estimate (2.1) is a simple consequence of (2.29) and (2.31). Theorem 2.5 is proved. □

Remark 2.1. From estimate (2.1) it follows that $||u_{\varepsilon\delta} - v||_{C([0,T];H)} \to 0$ only if $u_0 - A^{-1}(0)f(0) = 0$. In the opposite case the solution $u_{\varepsilon\delta}$ has a singular behavior relative to parameters ε and δ. In the neighbourhood of $t = 0$ this behavior is defined by the boundary layer function h_δ, which is solution to the problem (2.2).

Remark 2.2. If, in conditions of Theorem 2.5, $f = 0$, then $v = 0$, $R_1 = 0$, and $l_\delta = h_\delta$. Consequently, the estimate (2.1) in this case takes the form

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} \leq C\left(|A(0)u_0| + |A^{1/2}(0)u_1|\right)\frac{\varepsilon^{1/4}}{\delta^{5/4}},\ \forall \varepsilon \in (0,\mu_0\,\delta^2],\ \forall \delta \in (0,1].$$

Finally, let us consider the following example. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$ we consider the following boundary-value problem:

$$\begin{cases} \varepsilon\,\partial_t^2 u_{\varepsilon\delta} + \delta\,\partial_t u_{\varepsilon\delta} + A(x,t,\partial_x)\,u_{\varepsilon\delta} = f(x,t), \quad x \in \Omega,\ t \in (0,T), \\ u_{\varepsilon\delta}(x,0) = u_0(x), \quad \partial_t\,u_{\varepsilon\delta}(x,0) = u_1(x), \quad x \in \overline{\Omega}, \\ u_{\varepsilon\delta}\big|_{\partial\Omega} = 0, \quad t \in [0,T), \end{cases} \tag{2.32}$$

where $\varepsilon > 0$ and δ are small positive parameters, $u_{\varepsilon\delta}, f : [0,T) \to L^2(\Omega)$ and $A(x,t,\partial_x)$, is defined as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$A(x,t,\partial_x)u(x) = -\sum_{i,j=1}^{n} \partial_{x_i}\Big(a_{ij}(x,t)\partial_{x_j}u(x,t)\Big) + a(x,t)\,u(x,t),\ u \in D(A).$$

In this case the corresponding problem (P_0) takes the form:

$$\begin{cases} A(x,t,\partial_x)v(x,t) = f(x,t), \quad x \in \Omega,\ t \in (0,T), \\ v\big|_{\partial\Omega} = 0, \quad t \in [0,T). \end{cases} \tag{2.33}$$

Let us assume that the coefficients a_{ij} and a satisfy the following conditions:

$$\begin{cases} a_{ij},\, a \in C^2\big(\overline{\Omega} \times [0,T]\big), \quad a(x,t) \geq 0, \quad \forall (x,t) \in \overline{\Omega} \times [0,T], \\ a_{ij}(x,t) = a_{ji}(x,t), \quad \forall i,j = \overline{1,n}, \quad \forall (x,t) \in \overline{\Omega} \times [0,T], \\ \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j \geq \omega\,||\xi||^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall (x,t) \in \overline{\Omega} \times [0,T], \quad \omega > 0. \end{cases} \tag{2.34}$$

It is not difficult to see that conditions (2.34) provide the achievement of conditions (**H1**), (**H2**) and (**H3**). Consequently, from Theorem 2.5 we obtain the following theorem.

Theorem 2.6. *Let $T > 0$. Let us assume that the conditions (2.34) are fulfilled. If $u_0, u_1 \in H^2(\Omega) \bigcap H_0^1(\Omega)$ and $f \in W^{1,2}(0,T;L^2(\Omega))$, then there exists constants $C(T,\gamma,a_0,\omega) > 0$ and $\mu_0 = \left\{1; \frac{\omega}{6a_0}\right\}$, such that*

$$||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];L^2(\Omega))} \leq C\,\mathcal{M}\left(\frac{\varepsilon^{1/4}}{\delta^{9/4}} + \sqrt{\delta}\right), \quad \forall \varepsilon \in (0,\mu_0\,\delta^2], \quad \forall \delta \in (0,1],$$

where $u_{\varepsilon\delta}$ and v are strong solutions to the problems (2.32) and (2.33), respectively,

$$||h_\delta(t)||_{L^2(\Omega)} \leq C\,\mathcal{M}\,e^{-\delta t/\omega}, \quad \forall t \in [0,T],$$

$$\mathcal{M} = ||u_0||_{H^2(\Omega)} + ||u_1||_{H_0^1(\Omega)} + ||f||_{W^{1,2}(0,T;L^2(\Omega))}.$$

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