CARPATHIAN J. MATH. 33 (2017), No. 2, 247 - 256
Online version at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Walk-set induced connectedness in digital spaces

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ABSTRACT. In an undirected simple graph, we define connectedness induced by a set of walks of the same lengths. We show that the connectedness is preserved by the strong product of graphs with walk sets. This result is used to introduce a graph on the vertex set $\mathbb{Z}^2$ with sets of walks that is obtained as the strong product of a pair of copies of a graph on the vertex set $\mathbb{Z}$ with certain walk sets. It is proved that each of the walk sets in the graph introduced induces connectedness on $\mathbb{Z}^2$ that satisfies a digital analogue of the Jordan curve theorem. It follows that the graph with any of the walk sets provides a convenient structure on the digital plane $\mathbb{Z}^2$ for the study of digital images.

1. INTRODUCTION

Digital images may be considered to be approximations of real ones and, therefore, to be able to study them, we need the digital spaces $\mathbb{Z}^m$, $m > 0$ an integer, to be equipped with structures that provide a connectedness behaving analogously to the connectedness in the Euclidean (real) spaces. It is one of the basic tasks of digital topology, a theory that was founded for the study of geometric and topological properties of digital images, to find such convenient structures on the digital spaces. In the classical approach to digital topology (see [6-7]), adjacency graphs with the vertex set $\mathbb{Z}^m$ are used to provide such structures. For example, the well-known 4- and 8-adjacencies are used on the digital plane $\mathbb{Z}^2$ and the 6-, 18-, and 26-adjacencies are used on the digital space $\mathbb{Z}^3$. A disadvantage of the classical approach is that the connectedness in the digital space $\mathbb{Z}^m$ given by an adjacency graph does not behave analogously to the connectedness in the Euclidean space $\mathbb{R}^m$ and so the adjacencies do not provide a satisfactory model of the Euclidean topology. Particularly, on the digital plane $\mathbb{Z}^2$, neither 2-adjacency nor 4-adjacency itself allows for an analogue of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a simple closed curve separates the real plane into precisely two components). To obtain such an analogue, we have to use a combination of the two adjacencies - see [3-4]. The present graphical software is mostly based on employing such a combination.

It was only in 1990 that E. D. Khalimsky, R. Kopperman and P. R. Meyer [2] proposed a new, topological approach to digital topology. They showed that there is a topology on $\mathbb{Z}^2$, the so-called Khalimsky topology, which allows for an analogue of the Jordan curve theorem, thus providing a convenient digital model of the Euclidean plane, and can, therefore, be used for studying and processing digital images. The topological approach has then been developed by many authors - see e.g. [9-11].

In [12], graphs with path partitions are introduced and studied where the path partitions considered are nothing but certain sets of walks in these graphs. It was shown in [12] that path partitions provide graphs with a special connectedness that allows for an analogue of the Jordan curve theorem so that these graphs may be used as convenient background structures on the digital plane for the study of digital images. In the present paper, we continue the graph-theoretic approach to digital topology discussed in [12].

Received: 23.11.2016. In revised form: 04.05.2017. Accepted: 11.05.2017
2010 *Mathematics Subject Classification.* 05C40, 68R10.
Key words and phrases. *Simple graph, strong product, walk, connectedness, digital space, Jordan curve theorem.*

But, in difference to [12], we employ sets of walks, which are more general than path partitions, and study the induced connectedness. It will be shown that the connectedness is preserved by the strong product of graphs. Thus, having the connectedness induced by a walk set in a graph on the digital line $\mathbb{Z}$, the connectedness is preserved also in the graph on $\mathbb{Z}^m$, $m > 0$ an integer, obtained as the strong product of m copies of the graph on $\mathbb{Z}$. We will discuss the connectednesses induced by certain special walk sets in the 2-adjacency graph on the digital line $\mathbb{Z}$ which generalize the connectedness given by the Khalimsky topology. The strong product of a pair of copies of this graphs with such a walk set will be shown to allow for a digital analogue of the Jordan curve theorem, i.e., to provide a convenient structure on the digital plane $\mathbb{Z}^2$ for the study of digital images.

We will work with some basic graph-theoretic concepts only - we refer to [1] for them. By a *graph* $G = (V, E)$, we understand an (undirected simple) graph (without loops) with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. Recall that a *walk* in G is a (finite) sequence $(x_i|\ i \leq n) = (x_0, x_1, ..., x_n)$ of vertices (i.e., elements of V) such that every pair of consecutive vertices is adjacent. The natural number (i.e., finite ordinal) n is called the *length* of the walk $(x_i|\ i \leq n)$. A walk $(x_i|\ i \leq n)$ in G is called a *path* if $x_i \neq x_j$ whenever $i, j \leq n, i \neq j$, and it is called a *circle* if $x_i \neq x_j$ whenever $i, j < n, i \neq j$, and $x_0 = x_n$. A subset $A \subseteq V$ is *connected* if any two different vertices $x, y \in A$ can be joined by a walk in G contained in A, i.e., there is a walk $(x_i|\ i \leq n)$ in G such that $x_0 = x$, $x_n = y$, and $x_i \in A$ for every $i \leq n$. Note that an equivalent definition of a connected subset $A \subseteq V$ is obtained by replacing "walk" with "path" in the previous one.

Recall that, given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2. A graph (V_1, E_1) is said to be an *induced subgraph* of a graph (V_2, E_2) if it is a subgraph of (V_2, E_2) such that $E_1 = E_2 \cap \{\{x, y\};\ x, y \in V_1\}$. We then briefly speak about the induced subgraph V_1 of (V_2, E_2).

In [8], the concept of a strong product of two graphs was introduced. We will extend this concept on an arbitrary family of graphs as follows:

Definition 1.1. Given graphs $G_j = (V_j, E_j,)$, $j = 1, 2, ..., m$ ($m > 0$ a natural number), we define their *strong product* to be the graph $\prod_{j=1}^m G_j = (\prod_{j=1}^m V_j, E)$ with the set of edges $E = \{\{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\}$; there exists a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $\{x_j, y_j\} \in E_j$ for every $j \in J$ and $x_j = y_j$ for every $j \in \{1, 2, ..., m\} - J\}$.

Clearly, the usual (direct) product of a family of graphs $G_j = (V_j, E_j,)$, $j = 1, 2, ..., m$, i.e., the graph $(\prod_{j=1}^m V_j, E)$ where $E = \{\{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\};\ \{x_j, y_j\} \in E_j$ for every $j = 1, 2, ..., m\}$, is a factor of the strong product of the family.

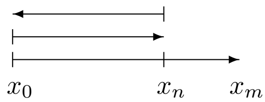
2. Graphs with walk sets

In the sequel, n will denote a natural number with $n > 1$.

Let $G = (V, E)$ be a graph. Then we denote by $\mathcal{P}_n(G)$ the set of all walks of length n in G. If $\mathcal{B} \subseteq \mathcal{P}_n(G)$ and $G_1 = (V_1, E_1)$ is an induced subgraph of G, then the set $\mathcal{B} \cap V_1^{n+1} \subseteq \mathcal{P}_n(G_1)$ will also be denoted by $\mathcal{B}$.

For every subset $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we put
$\mathcal{B}^* = \{(x_i|\ i \leq m) \in V^{m+1};\ 0 < m \leq n$ and there exists $(y_i|\ i \leq n) \in \mathcal{B}$ such that $x_i = y_i$ for every $i \leq m$ or $x_i = y_{m-i}$ for every $i \leq m\}$.
The elements of $\mathcal{B}^*$ will be called $\mathcal{B}$-*initial segments* in G. Thus, a $\mathcal{B}$-initial segment $(x_i|\ i \leq m)$ in G is a sequence consisting of the first $m + 1$ members of a walk belonging to $\mathcal{B}$

ordered according to the walk or conversely - see the following figure (with sequences represented by arrows oriented from the first to the last members):

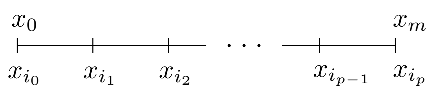


Definition 2.2. Let G_j be a graph and $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$ for every $j = 1, 2, ..., m$ ($m > 0$ a natural number). Then we define the *strong product* of the paths $\mathcal{B}_j$, $j = 1, 2, ..., m$, to be the set $\prod_{j=1}^m \mathcal{B}_j = \{((x_i^1, x_i^2, ..., x_i^m)|\ i \leq n)$; there is a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $(x_i^j|\ i \leq n) \in \mathcal{B}_j$ for every $j \in J$ and $(x_i^j|\ i \leq n)$ is a constant sequence for every $j \in \{1, 2, ..., m\} - J\}$.

It is evident that $\prod_{j=1}^m \mathcal{B}_j \subseteq \mathcal{P}_n(\prod_{j=1}^m G_j)$.

Given a graph G and $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we will employ the walks in G that are formed by subsequent $\mathcal{B}$-initial segments in G. More precisely, we define:

Definition 2.3. Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. A sequence $C = (x_i|\ i \leq r)$, $r > 0$ a natural number, of vertices of V is called a *$\mathcal{B}$-walk* in G if there is an increasing sequence $(i_k|\ k \leq p)$ of natural numbers with $i_0 = 0$ and $i_p = r$ such that $i_k - i_{k-1} \leq n$ and $(x_i|\ i_{k-1} \leq i \leq i_k) \in \mathcal{B}^*$ for every k with $0 < k \leq p$ (see the figure below). The sequence $(i_k|\ k \leq p)$ is said to be a *binding sequence* of C.
If the members of C are pairwise different, then C is called a *$\mathcal{B}$-path* in G.
A $\mathcal{B}$-walk C is said to be a *$\mathcal{B}$-circle* if, for every pair i_0, i_1 of different natural numbers with $i_0, i_1 \leq r$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, r\}$.



Of course, every $\mathcal{B}$-walk ($\mathcal{B}$-path, *$\mathcal{B}$-circle*) in a graph $G = (V, E)$ is a walk (path, circle) in G and both concepts coincide if $\mathcal{B} = \mathcal{P}_1(G)$.

Observe that, if $(x_0, x_1, ..., x_r)$ is a $\mathcal{B}$-walk in G, then $(x_r, x_{r-1}, ..., x_0)$ is a $\mathcal{B}$-walk in G, too ($\mathcal{B}$-walks are closed under reversion). Further, if $C_1 = (x_i|\ i \leq r)$ and $C_2 = (y_i|\ i \leq s)$ are $\mathcal{B}$-walks in G such that $x_{r-1} = y_0$, then, putting $z_i = x_i$ for all $i \leq r$ and $z_i = y_{i-r}$ for all i with $r < i \leq r + s$, we get a $\mathcal{B}$-walk $(z_i|\ i \leq r + s)$ in G ($\mathcal{B}$-walks are closed under composition). We denote the $\mathcal{B}$-walk $(z_i|\ i \leq r + s)$ by $C_1 \oplus C_2$.

Definition 2.4. Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. A set $A \subseteq V$ is said to be *$\mathcal{B}$-connected* in G if any two different vertices of G belonging to A can be joined by a *$\mathcal{B}$-walk* in G contained in A. A maximal $\mathcal{B}$-connected set in G is called a *$\mathcal{B}$-component* of G.

In particular, every $\mathcal{B}$-walk (and thus every $\mathcal{B}$-circle) in a graph G (where $\mathcal{B} \subseteq \mathcal{P}_n(G)$) is $\mathcal{B}$-connected in G. Clearly, a subset $A \subseteq V$ is connected in a graph $G = (V, E)$ if and only if it is $\mathcal{P}_1(G)$-connected in G.

Note that, given a graph $G = (V, E)$ and $\mathcal{B} \subseteq \mathcal{P}_n(G)$, the union of a finite sequence of nonempty $\mathcal{B}$-connected subsets of V is $\mathcal{B}$-connected if the intersection of every consecutive pair of the subsets is nonempty (because $\mathcal{B}$-walks are closed under composition).

Proposition 2.1. *Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$, and $Y_j \subseteq V_j$ be a subset for every $j = 1, 2, ..., m$. If Y_j is a $\mathcal{B}_j$-connected set in G_j for every $i = 1, 2, ..., n$, then $\prod_{j=1}^m Y_j$ is a $\prod_{j=1}^m \mathcal{B}_j$-connected set in $\prod_{j=1}^m G_j$.*

Proof. If $m = 1$, then the statement is trivial. Therefore, we will suppose that $m > 1$.

First, we will show that the statement is true if $Y_j = (y_i^j|\, i \le p_j)$ is a $\mathcal{B}_j$-initial segment in G_j for every $j = 1, 2, ..., m$. For each $j = 1, 2, ..., m$, there is a walk $(x_i^j|\, i \le n) \in \mathcal{B}$ such that $y_i^j = x_i^j$ for all $i \le p_j$ or $y_i^j = x_{p_j - i}^j$ for all $i \le p_j$ (because $(y_i^j|\, i \le p_j)$ is a $\mathcal{B}_j$-initial segment in G_j). Let $y \in \prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ be an arbitrary element. Then, for each $j = 1, 2, ..., m$, there is a natural number q_j, $q_j < p_j$, such that $y = (y_{q_1}^1, y_{q_2}^2, ..., y_{q_m}^m)$. It follows that either $(y_{q_1 - i}^1|\ i \le q_1)$ or $(y_i^1|\ q_1 \le i \le p_1)$ is a $\mathcal{B}_1$-initial segment in G_1 with the first member $y_{q_1}^1$ and the last one x_0^1. Denote this $\mathcal{B}_1$-initial segment by $(z_i^1|\ i \le r_1)$ and put $C_1 = ((z_i^1, y_{q_2}^2, y_{q_3}^3, ..., y_{q_m}^m)|\ i \le r_1)$. Clearly, C_1 is a $\prod_{j=1}^m \mathcal{B}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$, with the first member y, and with $z_{r_1}^1 = x_0^1$. It follows that either $(y_{q_2 - i}^2|\ i \le q_2)$ or $(y_i^2|\ q_2 \le i \le p_2)$ is a $\mathcal{B}_2$-initial segment in G_2 with the first member $y_{q_2}^2$ and the last one x_0^2. Denote this $\mathcal{B}_2$-initial segment by $(z_i^2|\ i \le r_2)$ and put $C_2 = ((x_0^1, z_i^2, y_{q_3}^3, y_{q_4}^4, ..., y_{q_m}^m)|\ i \le r_2)$. Clearly, C_2 is a $\prod_{j=1}^m \mathcal{B}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ such that $z_0^2 = y_{q_2}^2$ and $z_{r_2}^2 = x_0^2$. Thus, $C_1 \oplus C_2$ is a $\prod_{j=1}^m \mathcal{B}_j$-walk in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$, with the first member y, and with the last one $(x_0^1, x_0^2, y_{q_3}^3, y_{q_4}^4, ..., y_{q_m}^m)$. Repeating this construction m-times, we get $\prod_{j=1}^m \mathcal{B}_j$-initial segments C_1, C_2,...,C_m in $\prod_{j=1}^m G_j$ with the members of each of them belonging to $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ such that $C_1 \oplus C_2 \oplus ... \oplus C_m$ is a $\prod_{j=1}^m \mathcal{B}_j$-walk in $\prod_{j=1}^m G_j$ with the first member y and the last one $(x_0^1, x_0^2, ..., x_0^m)$. We have shown that any point of $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ can be connected with the point $(x_0^1, x_0^2, ..., x_0^m)$ by a $\prod_{j=1}^m \mathcal{B}_j$-walk in the graph $\prod_{j=1}^m G_j$ contained in $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$.

Second, we will show that the statement is true if $Y_j = (x_i^j|\ i \le p_j)$ is a $\mathcal{B}_j$-walk in G_j for every $j = 1, 2, ..., m$. If $m = 1$, then the statement is trivial. Let $m > 1$. For each $j = 1, 2, ..., m$, let $(i_k^j|\ k \le q_j)$ be the binding sequence of $(x_i^j|\ i \le p_j)$, i.e., a sequence of natural numbers with $i_0^j = 0$ and $i_{q_j - 1}^j = p_j - 1$ such that $(x_i^j|\ i_k^j \le i \le i_{k+1}^j)$ is a $\mathcal{B}_j$-initial segment in G_j whenever $k \le q_j$. For every $j = 1, 2, ..., m$, putting $C_k^j = \{x_i^j;\ i_k^j \le i \le i_{k+1}^j\}$, we get $\{x_i^j;\ i \le p_j\} = \bigcup_{k<q_j} C_k^j$. Therefore, $\prod_{j=1}^m \{x_i^j;\ i \le p_j\} = \bigcup_{k_1<q_1} \bigcup_{k_2<q_2} ... \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ where $\prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m G_j$ whenever $k_j < q_j$, $j = 1, 2, ..., m$, by the previous part of the proof. Thus, for any $k_j < q_j$, $j = 1, 2, ..., m-1$, $(\prod_{j=1}^m C_{k_j}^j|\ k_m < q_m)$ is a finite sequence of connected sets with nonempty intersection of every consecutive pair of them. Hence, the set $\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m G_j$. Consequently, for every k_j with $k_j < q_j$, $j = 1, 2, ..., m-2$, $(\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j|\ k_{m-1} < q_{m-1})$ is a finite sequence of connected sets with nonempty intersection of any consecutive pair of them. Thus, the set $\bigcup_{k_{m-1}<q_{m-1}} \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m G_j$. After repeating this considerations m-times, we get the conclusion that the set $\bigcup_{k_1<q_1} \bigcup_{k_2<q_2} ... \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j = \prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ is $\prod_{j=1}^m \mathcal{B}_j$-connected in $\prod_{j=1}^m G_j$.

Finally, let Y_j be a connected set in G_j for every $j \in \{1, 2, ..., m\}$ and let $(x_1, x_2, ..., x_m)$, $(y_1, y_2, ..., y_m) \in \prod_{j=1}^m G_j$ be arbitrary points. Then, for every $j \in \{1, 2, ..., m\}$, there is a $\mathcal{B}_j$-walk $(z_i^j|\ i \le p_j)$ in G_j joining the points x_j and y_j which is contained in Y_j. Hence, the set $\prod_{j=1}^m \{z_i^j|\ i \le p_j\}$ contains the points $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_m)$ and is a connected set in $\prod_{j=1}^m G_j$ by the previous part of the proof. Thus, there is a $\mathcal{B}_j$-walk C in $\prod_{j=1}^m G_j$ joining the points $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_m)$ which is contained in

$\prod_{j=1}^{m}\{z_i^j|\, i \le p_j\}$. Since $\prod_{j=1}^{m}\{z_i^j|\, i \le p_j\} \subseteq \prod_{j=1}^{m} Y_j$, C is contained in $\prod_{j=1}^{m} Y_j$. Therefore, $\prod_{j=1}^{m} Y_j$ is a $\prod_{j=1}^{m} \mathcal{B}_j$-connected set in $\prod_{j=1}^{m} G_j$. The proof is complete. □

3. Connectedness in $\mathbb{Z}^2$ induced by the strong product of certain walk sets in the 2-adjacency graph on $\mathbb{Z}$

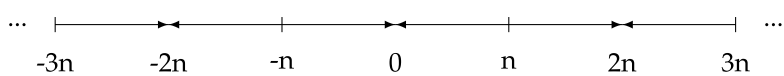
To obtain possible applications of the introduced concept of $\mathcal{B}$-connectedness (in a graph G where $\mathcal{B} \subseteq \mathcal{P}_n(G)$) in digital topology, we focus on a particular graph G on the digital plane $\mathbb{Z}^2$ and certain sets of walks of the same lengths in the graph. We will show that this graph allows for a digital analogue of the Jordan curve theorem. To this end, we define:

Definition 3.5. Let G be a graph on $\mathbb{Z}^2$ and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. A finite $\mathcal{B}$-connected subset $J \subseteq \mathbb{Z}^2$ is called a *$\mathcal{B}$-Jordan curve* in G if the following two conditions are satisfied:

(1) For every $z \in J$, there are precisely two elements of J adjacent to z.
(2) The induced subgraph $\mathcal{Z}^2 - J$ of G has precisely two $\mathcal{B}$-components

By the *2-adjacency graph* on $\mathbb{Z}$ we understand the graph $\mathbb{Z}_2 = (\mathbb{Z}, A_2)$ where $A_2 = \{\{p,q\};\ p,q \in \mathbb{Z},\ |p-q| = 1\}$.

From now on, $\mathcal{B}$ will denote the walk set $\mathcal{B} \subseteq \mathcal{P}_n(\mathbb{Z}_2)$ given as follows:
$\mathcal{B} = \{(x_i|\ i \le n) \in \mathcal{P}_n(\mathbb{Z}_2);$ there is an odd number $l \in \mathbb{Z}$ such that $x_i = ln + i$ for all $i \le n$ or $x_i = ln - i$ for all $i \le n\}$.
Thus, the walks belonging to $\mathcal{B}$ are the arithmetic sequences $(x_i|\ i \le n)$ of integers with the difference equal to 1 or -1 and with $x_0 = ln$ where $l \in \mathbb{Z}$ is an odd number (so that $\mathcal{B}$ is a set of paths) - see the following figure where the walks belonging to $\mathcal{B}$ are represented as arrows (oriented from the first to the last members of the sequences):



It may easily be seen that $\mathbb{Z}$ is a $\mathcal{B}$-connected set in $\mathbb{Z}_2$. For $\mathcal{B} \subseteq \mathcal{P}_1(\mathbb{Z}_2)$, the $\mathcal{B}$-connectedness coincide with the connectedness given by the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k-1, 2k, 2k+1\};\ k \in \mathbb{Z}\}$ - cf. [5].

In the sequel, m will denote (similarly to n) a natural number with $m > 0$. Using results of the previous section, we may propose new structures on the digital spaces convenient for the study of digital images. Such a structure on $\mathbb{Z}^m$ is obtained as the strong product of m copies of the 2-adjacency graph on $\mathbb{Z}$ with the walk set given by the strong product of m-copies of the walk set $\mathcal{B}$. More formally, we may consider the graph $G^m = \prod_{j=1}^{m} G_j$ on $\mathbb{Z}^m$, where G_j is the 2-adjacency graph on $\mathbb{Z}$ for every $j \in \{1,2,...,m\}$, with the walk set $\mathcal{B}^m \subseteq \mathcal{P}_n(G^m)$ given by $\mathcal{B}^m = \prod_{j=1}^{m} \mathcal{B}_j$ where $\mathcal{B}_j = \mathcal{B}$ for every $j \in \{1,2,...,m\}$ (note that every walk from $\mathcal{B}^m$ is a path). Of course, G^1 is the 2-adjacency graph on $\mathbb{Z}$ and G^2 and G^3 coincide with the well known 8-adjacency graph on $\mathbb{Z}^2$ and 26-adjacency graph on $\mathbb{Z}^3$, i.e., the graphs $(\mathbb{Z}^2, A_8)$ where $A_8 = \{\{(x_1,y_1),(x_2,y_2)\};\ (x_1,y_1),(x_2,y_2) \in \mathbb{Z}^2,\ \max\{|x_1 - x_2|, |y_1 - y_2|\} = 1\}$ and $(\mathbb{Z}^3, A_{26})$ where $A_{26} = \{\{(x_1,y_1,z_1),(x_2,y_2,z_2)\}; (x_1,y_1,z_1),(x_2,y_2,z_2) \in \mathbb{Z}^3,\ \max\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\} = 1\}$, respectively.

As an immediate consequence of Proposition 1 we get:

Theorem 3.1. *$\mathbb{Z}^m$ is a connected set in G^m.*

In the case $\mathcal{B} \subseteq \mathcal{P}_1(\mathbb{Z}_2)$, the $\mathcal{B}^m$-connectedness in the graph G^m (on $\mathcal{Z}^m$) coincides with the connectedness in the Khalimsky topology on $\mathbb{Z}^m$ and $\mathcal{B}^2$-Jordan curves in G^2 coincide with the Jordan curves in the Khalimsky topology on $\mathcal{Z}^2$ introduced in [2]. The connectedness in the Khalimsky topology (on $\mathcal{Z}^m$) coincides with the connectedness in

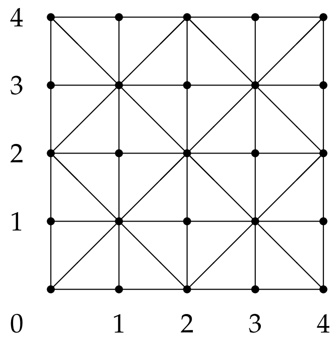


FIGURE 1. A section of the connectedness graph of the Khalimsky topology on the digital plane.

the so-called connectedness graph of the topology. In Figure 1, the connectedness graph of the Khalimsky topology on the digital plane $\mathbb{Z}^2$ is demonstrated.

Since the Khalimsky topology is well known, we will suppose that $\mathcal{B} \subseteq \mathcal{P}_n(\mathbb{Z}_2)$ where $n > 1$. Furthermore, we will restrict our considerations to $m = 2$ because this case is the most important one with respect to possible applications in digital topology. Thus, we will focus on the graph G^2 with the walk set $\mathcal{B}^2$.

We denote by $G(\mathcal{B}^2)$ the factor of the 8-adjacency graph on $\mathbb{Z}^2$ whose edges are those $\{(x_1, y_1), (x_2, y_2)\} \in A_8$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:
$x_1 - y_1 = x_2 - y_2 = 2kn,$
$x_1 - y_1 = x_2 - y_2 = 2kn,$
$x_1 = x_2 = 2kn,$
$y_1 = y_2 = 2kn.$
A section of the graph $G(\mathcal{B}^2)$ is demonstrated in Figure 2 where only the vertices $(2kn, 2ln)$, $k, l \in \mathbb{Z}$, are marked out (by bold dots) and thus, on every edge drawn between two such vertices, there are $2n - 1$ more (non-displayed) vertices, so that the edges represent $2n$ edges in the graph $G(\mathcal{B}^2)$. Clearly, every circle C in $G(\mathcal{B}^2)$ is a $\mathcal{B}^2$-connected set in G^2 because it is a $\mathcal{B}^2$-circle in G^2. Indeed, C consists (i.e., is the union) of a finite sequence of paths in $\mathcal{B}^2$, hence $\mathcal{B}^2$-initial segments, such that every two consecutive paths have a point in common.

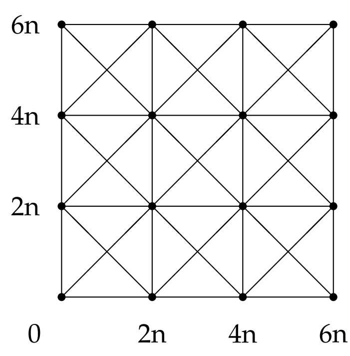


FIGURE 2. A section of the graph $G(\mathcal{B}^2)$.

Definition 3.6. A circle J in the graph $G(\mathcal{B}^2)$ is said to be *n-fundamental* if, whenever $((2k+1)n,(2l+1)n) \in J$ for some $k,l \in \mathbb{Z}$, one of the following two conditions is true:
$\{((2k+1)n-1,(2l+1)n-1),(2k+1)n+1,(2l+1)n+1))\} \subseteq J$,
$\{((2k+1)n-1,(2l+1)n+1),(2k+1)n+1,(2l+1)n-1))\} \subseteq J$.

The n-fundamental circles are just the circles in the graph demonstrated in Figure 2 that turn only at (some of) the vertices marked out by the bold dots.

Theorem 3.2. *If J is an n-fundamental circle in the graph $G(\mathcal{B}^2)$, then J is a $\mathcal{B}^2$-Jordan curve in $G(\mathcal{B}^2)$ such that one $\mathcal{B}^2$-component of the induced subgraph $\mathbb{Z}^2 - J$ of $G(\mathcal{B}^2)$ is finite, the other one is infinite, and the union of any of them with J is a $\mathcal{B}^2$-connected set in $G(\mathcal{B}^2)$.*

Proof. Let J be an n-fundamental circle in $G(\mathcal{B}^2)$. It is evident that the condition (1) in Definition 3.1 is satisfied. For every point $z = ((2k+1)n,(2l+1)n)$, $k,l \in \mathbb{Z}$, each of the following four subsets of $\mathbb{Z}^2$ is called an *n-fundamental triangle* (given by z):
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \le x+2ln-2kn\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \ge 4ln+2kn-x\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \ge x+2ln-2kn\}$,
$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \le 4ln+2kn-x\}$.
The points of any n-fundamental triangle form a segment of the shape of a (digital) rectangular triangle. Obviously, in each of the four n-fundamental triangles given by z, z is the middle point of the hypotenuse of the triangle. Every line segment constituting an edge of any of the four triangles represents precisely $2n+1$ points forming the corresponding edge of the corresponding n-fundamental triangle. Clearly, the edges of any n-fundamental triangle form a circle in the graph $G(\mathcal{B}^2)$, hence a $\mathcal{B}^2$-circle in G^2. The four types of 2-fundamental triangles are demonstrated in Figure 3.

We will show that every n-fundamental triangle is $\mathcal{B}^2$-connected in G^2 and so is also every set obtained from an n-fundamental triangle by subtracting some of its edges. Let $z = ((2k+1)n,(2l+1)n)$, $k,l \in \mathbb{Z}$, be a point and consider the n-fundamental triangle $T = \{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \le x+2ln-2kn\}$. Then T is the (digital) triangle ABC with the vertices $A = (2kn,2ln)$, $B = ((2k+2)n,2ln)$, $C = ((2k+2)n,(2l+2)n)$. For every $u \in \mathbb{Z}$, $(2k+1)n \le u \le (2k+2)n$, the sequence $G_u = ((u,y)|\ 2ln \le y \le u+2(l-k)n)$ is a $\mathcal{B}^2$-path in G^2 (contained in T), so that G_u is a $\mathcal{B}^2$-connected set. Similarly, for every $v \in \mathbb{Z}$, $2ln \le v \le (2l+1)n$, the sequence $H_v = ((x,v)|\ v+2(k-l)n \le x \le (2k+2)n)$ is a $\mathcal{B}^2$-path in G^2 (contained in T), so that H_v is a $\mathcal{B}^2$-connected set. We clearly have $T = \bigcup\{G_u;\ (2k+1)n \le u \le (2k+2)n\} \cup \bigcup\{H_v;\ 2ln \le v \le (2l+1)n\}$. It may easily be seen that $G_u \cap H_v \neq \emptyset$ whenever $(2k+1)n \le u \le (2k+2)n$ and $2ln \le v \le (2l+1)n$. For every natural number $i < 2n+2$, we put

$$S_i = \begin{cases} G_{(2k+1)n+\frac{i}{2}} & \text{if } i \text{ is even,} \\ H_{2ln+\frac{i-1}{2}} & \text{if } i \text{ is odd.} \end{cases}$$

Then $(S_i|\ i < 2n+2)$ is a sequence with the property that its members with even indices form the sequence $(G_u|\ (2k+1)n \le u \le (2k+2)n)$ and those with odd indices form the sequence $(H_v|\ 2ln \le v \le (2l+1)n)$. Hence, $\bigcup\{S_i|\ i < 2n+2\} = \bigcup\{G_u;\ (2k+1)n \le u \le (2k+2)n\} \cup \bigcup\{H_v;\ 2ln \le v \le (2l+1)n\}$ and every pair of consecutive members of $(S_i|\ i < 2n+2)$ has a non-empty intersection. Thus, since $T = \bigcup\{S_i|\ i < 2n+2\}$, T is $\mathcal{B}^2$-connected. For each of the other three n-fundamental triangles given by z, the proof is analogous, and the same is true also for every set obtained from an n-fundamental triangle (given by z) by subtracting some of its edges.

We will say that a (finite or infinite) sequence S of n-fundamental triangles is a tiling sequence if the members of S are pairwise different and every member of S, excluding the first one, has an edge in common with at least one of its predecessors. Given a tiling sequence S of n-fundamental triangles, we denote by S' the sequence obtained from S by subtracting, from every member of the sequence, all its edges that are not shared with any other member of the sequence. By the firs part of the proof, for every tiling sequence S of n-fundamental triangles, the set $\bigcup\{T;\ T \in S\}$ is $\mathcal{B}^2$-connected and the same is true for the set $\bigcup\{T;\ T \in S'\}$.

Let J be an n-fundamental circle in the graph $G(\mathcal{B}^2)$. Then J constitutes the border of a polygon $S_F \subseteq \mathbb{Z}^2$ consisting of n-fundamental triangles. More precisely, S_F is the union of some n-fundamental triangles such that any pair of them is disjoint or meets in just one edge in common. Let U be a tiling sequence of the n-fundamental triangles contained in S_F. Since S_F is finite, U is finite, too, and we have $S_F = \bigcup\{T;\ T \in U\}$. As every n-fundamental triangle $T \in U$ is $\mathcal{B}^2$-connected, so is S_F. Similarly, U' is a finite sequence with $S_F - J = \bigcup\{T;\ T \in U'\}$ and, since every member of U' is connected (by the first part of the proof), $S_F - J$ is connected, too.

Further, let V be a tiling sequence of n-fundamental triangles which are not contained in S_F. Since the complement of S_F in $\mathbb{Z}^2$ is infinite, V is infinite, too. Put $S_I = \bigcup\{T;\ T \in V\}$. As every n-fundamental triangle $T \in V$ is $\mathcal{B}^2$-connected, so is S_I. Similarly, V' is a finite sequence with $S_I - J = \bigcup\{T;\ T \in V'\}$ and, since every member of V' is connected (by the first part of the proof), $S_I - J$ is connected, too.

It may easily be seen that every $\mathcal{B}^2$-walk $C = (z_i|\ i \le k)$, $k > 0$ a natural number, in the 8-adjacency graph G^2 on $\mathbb{Z}^2$ connecting a point of $S_F - J$ with a point of $S_I - J$ meets J (i.e., meets an edge of an n-fundamental triangle which is contained in J). Therefore, the set $\mathbb{Z}^2 - J = (S_F - J) \cup (S_I - J)$ is not $\mathcal{B}^2$-connected in $(\mathbb{Z}^2, u_{\mathcal{B}^2})$. We have shown that $S_F - J$ and $S_I - J$ are $\mathcal{B}^2$-components of the induced subgraph $\mathbb{Z}^2 - J$ of $G(\mathcal{B}^2)$, so that the condition (2) in Definition 3.1 is satisfied. Thus, J is a $\mathcal{B}^2$-Jordan curve in $G(\mathcal{B}^2)$. Clearly, $S_F - J$ is finite, $S_I - J$ is infinite, and both S_F and S_I are $\mathcal{B}^2$-connected. The proof is completed. $\square$

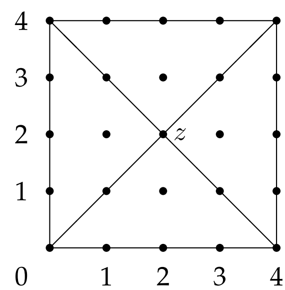
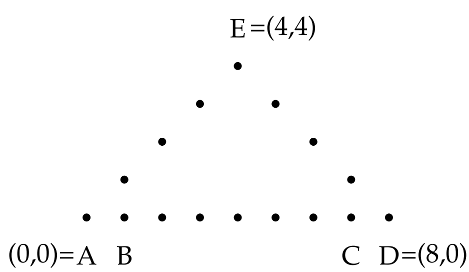


FIGURE 3. The four types of 2-fundamental triangles.

The following example illustrates the results attained and shows their possible applications in digital image processing.

Example 3.1. Consider the following (digital picture of a) triangle:



The triangle is a 2-fundamental circle in $G(\mathcal{B}^2)$, hence a $\mathcal{B}^2$-Jordan curve in $G(\mathcal{B}^2)$ by Theorem 3.2. But, for $\mathcal{B} \subseteq \mathcal{P}_1(\mathbb{Z}_2)$, the triangle is not a $\mathcal{B}^2$-Jordan curve in $G(\mathcal{B}^2)$ because it does not satisfy the condition (1) in Definition 3.1. In other words, the triangle is not a Jordan curve in the Khalimsky topology on $\mathbb{Z}^2$ (because Jordan curves in the Khalimsky topology on $\mathbb{Z}^2$ may never turn at the acute angle $\frac{\pi}{4}$). In order that this triangle be a Jordan curve in the Khalimsky topology on $\mathbb{Z}^2$, we have to delete the points A,B,C and D. But this will lead to a substantial deformation of the triangle.

4. Conclusion and future work

We introduced and studied a connectedness in simple graphs induced by sets of walks of the same lengths. We discussed a graph on the digital plane $\mathbb{Z}^2$ with certain sets of walks of the same lengths and showed that the connectedness induced by any of these sets of walks allows for a digital analogue of the Jordan curve theorem. We even demonstrated that, for the considered sets of walks of the same lengths greater than 1, the connectedness provides a richer variety of Jordan curves than the Khalimsky topology on the digital plane. Since Jordan curves represent boundaries of objects in digital images, this result indicates that the graph introduced provides a convenient structure on the digital plane $\mathbb{Z}^2$ for solving the problems of digital image processing that are closely related to boundaries such as pattern recognition, boundary detection, contour filling, etc.

The graph discussed is obtained as the strong product of two copies of the 2-adjacency graph on the digital line $\mathbb{Z}$ with a certain set of walks. It would be a natural continuation of our investigations to study graphs (with walk sets) that are obtained as strong products on m copies of the 2-adjacency graph for an arbitrary natural number $m > 2$. But such general investigations would be quite laborious and, therefore, in the forthcoming research, we will start with the most simple and applicable case of $m = 3$. Our goal will be to show that the walk set in the graph on the digital plane $\mathbb{Z}^3$ obtained as the strong product of three copies of the 2-adjacency graph on the digital line $\mathbb{Z}$ induces a connectedness that satisfies the Jordan-Brouwer theorem (i.e., the 3-dimensional analogue of the Jordan curve theorem).

Acknowledgement. This work was supported by the Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4Innovations excellence in science - LQ1602".

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