CARPATHIAN J. MATH. 33 (2017), No. 1, 01 - 08
Online version at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Stability in non-autonomous periodic systems with grazing stationary impacts

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ABSTRACT. This paper examines impulsive non-autonomous periodic systems whose surfaces of discontinuity and impact functions are not depending on the time variable. The $W-$map which alters the system with variable moments of impulses to that with fixed moments and facilitates the investigations, is presented. A particular linearizion system with two compartments is utilized to analyze stability of a grazing periodic solution. A significant way to keep down a singularity in linearizion is demonstrated. A concise review on sufficient conditions for the linearizion and stability is presented. An example is given to actualize the theoretical results.

1. INTRODUCTION

Grazing occurs whenever the solution of the impacting system meets the surface of discontinuity with zero velocity [8] or tangentially [3, 7, 10]. Our aim in this paper is to demonstrate regular behavior around the grazing solutions of the system as a different than the existing results which consider the complexity such as bifurcation and chaos around the grazing [5, 8, 10]. For this reason, by considering the geometry of the near solution with respect to the surfaces of discontinuity a special linearizion system is obtained around a periodic solution. To consider the application results, one can take into account the paper [9], where such systems can be applied for the analysis of the neural networks, where grazing is seen as a boundary between the firing and non-firing stages.

2. THE GRAZING SOLUTIONS

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively. Consider the open connected and bounded set $G \in \mathbb{R}^n$. Let $\Phi : G \to \mathbb{R}$ be a function, differentiable up to second order. $S = \Phi^{-1}(0)$ is a closed subset of G.

Let $x(\theta-)$ and $x(\theta+)$ be the left and right limits of a function $x(t)$ at the moment θ, respectively. Define $\Delta x(\theta) := x(\theta+) - x(\theta-)$ as the jump operator for $x(t)$ such that $x(\theta) \in S$ and $t = \theta$ is a moment when the solution meets the surface of discontinuity.

In this paper, we take into account the following system

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{x\in S} &= I(x), \end{aligned} \tag{2.1}$$

where $(t, x) \in \mathbb{R} \times G$, continuous function $f(t, x)$ is continuously differentiable with respect to x up to second order and $\Gamma = \{(t, x)|\Phi(x) = 0\} \subseteq \mathbb{R} \times S$ is the surface of discontinuity. The system is with stationary impulse conditions, since the differentable function $I(x)$ and the surface S do not depend on time.

For the convenience in notation, let us separate the differential equation part of the system (2.1)

Received: 05.02.2016. In revised form: 29.06.2016. Accepted: 20.07.2016
2010 *Mathematics Subject Classification.* 34A37, 37C75.
Key words and phrases. *Grazing impulse, stationary impact, linearization of a non-autonomous system with grazing impacts.*
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$$y' = f(t, y). \tag{2.2}$$

Assume that a solution $x_0(t) = x(t, t_0, x_0)$, of (2.1) has discontinuities at moments $t = \theta_i,\ i \in \mathbb{Z}$. Set the gradient vector of Φ with respect to x as $\nabla\Phi(x)$. The normal vector of Γ at a meeting moment, $t = \theta_i$, of the solution $x_0(t)$ can be determined as $\overrightarrow{n} = (0, \nabla\Phi(x_0(\theta_i))) \in \mathbb{R}^{n+1}$. For tangency, vectors $\overrightarrow{n}$ and $(1, f(\theta_i, x_0(\theta_i)))$ should be perpendicular. That is, $\langle \nabla\Phi(x_0(\theta_i)), f(\theta_i, x_0(\theta_i))\rangle = 0$, where $\langle , \rangle$ is the usual dot product.

In what follows, let $\|\cdot\|$ be the Euclidean norm, that is for a vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, the norm is equal to $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.

Consider a function $H(t, x) := \langle \nabla\Phi(x), f(t, x)\rangle$, with $(t, x) \in \mathbb{R}\times S$. A point $(\theta_i, x_0(\theta_i))$ is a *grazing point* and θ_i a *grazing moment* for the solution $x_0(t)$ of (2.1) if $H(\theta_i, x_0(\theta_i)) = 0$ and $I(x_0(\theta_i)) = 0$. The solution $x_0(t)$ of (2.1) is *grazing* if it has a grazing point $(\theta_i, x_0(\theta_i))$. The moment θ_i is the grazing moment of the solution $x_0(t)$. A point $(\theta_i, x_0(\theta_i))$ is a *transversal point* and θ_i a transversal moment for $x_0(t)$ if $H(\theta_i, x_0(\theta_i)) \neq 0$.

In what follows, the following condition will be needed.

(H1) For each grazing point $(\theta_i, x_0(\theta_i))$ there is a number $\delta > 0$ such that $H(t, x) \neq 0$ and $J(x) \notin S$ if $0 < |t - \theta_i| < \delta$ and $0 < \|x - x_0(\theta_i)\| < \delta$.

It is also clear that function $H(t, x) \neq 0$ near a transversal point.

Consider a solution $x(t) = x(t, \theta_i, x_0 + \Delta x)$ of (2.1) with a small $\|\Delta x\|$. Due to the geometrical reasons caused by the tangency at the grazing point, this solution may not intersect the surface of discontinuity near $(\theta_i, x_0(\theta_i))$. So, there are two different behaviors of it with respect to the surface of discontinuity, they are:

(N1) The solution $x(t)$ intersects the surface of discontinuity Γ at a moment near to θ_i.

(N2) There is no intersection moment of $x(t)$ close to θ_i.

2.1. **B-equivalence to a system with fixed moments of impulses.** Consider the solution $x_0(t) : \mathcal{I} \to \mathbb{R}^n$, $\mathcal{I} \subseteq \mathbb{R}$, of (2.1). Assume that all discontinuity points θ_i of $x_0(t)$, $i \in \mathcal{A}$, are interior points of $\mathcal{I}$, where $\mathcal{A}$ is an interval in $\mathbb{Z}$. There exists a positive number r, such that r-neighborhoods $G_i(r)$ of $(\theta_i, x_0(\theta_i))$ do not intersect each other. Fix $i \in \mathcal{A}$ and let $\xi(t) = x(t, \theta_i, x)$, $(\theta_i, x) \in G_i(r)$, be a solution of (2.2), which satisfies $(N1)$, and $\tau_i = \tau_i(x)$ the meeting time of $\xi(t)$ with S and $\psi(t) = x(t, \tau_i, \xi(\tau_i) + J(\xi(\tau_i)))$ another solution of (2.2). Denote $W_i(x) = \psi(\theta_i) - x$ and one can define the map $W_i(x)$ as

$$W_i(x) = \int_{\theta_i}^{\tau_i} f(s, \xi(s))ds + J(x + \int_{\theta_i}^{\tau_i} f(s, \xi(s))ds) + \int_{\tau_i}^{\theta_i} f(s, \psi(s))ds. \tag{2.3}$$

It is a map of an intersection of the plane $t = \theta_i$ with $G_i(r)$ into the plane $t = \theta_i$. Let us present the following system of differential equations with impulses at fixed moments,

$$\begin{aligned} y' &= f(t, y), \\ \Delta y|_{t=\theta_i} &= W_i(y(\theta_i)), \end{aligned} \tag{2.4}$$

where f is the same as the function in system (2.4) and the map W_i, $i \in \mathcal{A}$, is defined by equation (2.3) if $x(t)$ satisfies condition $(N1)$. Otherwise, if a solution $x(t)$ satisfies $(N2)$, then we assume that it admits the discontinuity moment θ_i with zero jump such that $W_i(x(\theta_i)) = 0$. Let us introduce the sets $F_r = \{(t, x) | t \in \mathcal{I}, \|x - x_0(t)\| < r\}$, and $G_i^+(r)$, $i \in \mathcal{A}$, an $r-$ neighborhood of the point $(\theta_i, x_0(\theta_i+))$. Write $G^r = F_1 \cup (\cup_{i\in\mathcal{A}} G_i(r)) \cup (\cup_{i\in\mathcal{A}} G_i^+(r))$. Take r sufficiently small so that $G^r \subset \mathbb{R} \times G$. Denote by $G(h)$ an h-neighborhood of $x_0(0)$. Systems (2.1) and (2.4) are $B-$equivalent in G^r [1], i.e. solutions of the system with the same initial data coincide on their common domains except possibly intervals near θ_i. The complete description of the equivalence is given in [1].

3. LINEARIZATION AROUND GRAZING SOLUTIONS

Consider a grazing solution $x_0(t) = x(t, 0, x_0),\ x_0 \in G$, of (2.1) which was introduced in the last section. We will demonstrate that one can write the variational system for the solution as follows:

$$\begin{aligned} u' &= A(t)u, \\ \Delta u|_{t=\theta_i} &= B_i u(\theta_i), \end{aligned} \tag{3.5}$$

where the matrix $A(t) \in \mathbb{R}^{n\times n}$ of the form $A(t) = \frac{\partial f(t,x_0(t))}{\partial x}$. We call the second equation in (3.5) as the *linearization at a moment of discontinuity or at a point of discontinuity*. It is different for transversal and grazing points. However, the first differential equation in (3.5) is common for all type of solutions. The matrices B_i will be described in the remaining part of the paper for each type.

Linearization at the transversal point is analyzed completely in Chapter 6, [1]. The $B-$ equivalent system (2.4) is involved in the analysis, since the solution $x_0(t)$ satisfies also the equation (2.4) at all moments of time, and near solutions do the same for all moments except small neighborhoods of the discontinuity moment, θ_i. Thus, it is easy to see that the system of variations around $x_0(t)$ for (2.1) and (2.4) are identical. Let $x_0(\theta_i)$ be a transversal point. We consider the reduced B-equivalent system and use the functions $\tau_i(x)$ and $W_i(x)$, defined by equation (2.3), are presented in Subsection 2.1 for linearization. Differentiating $\Phi(x(\tau_i(x))) = 0$, we have

$$\frac{\partial \tau_i(x_0(\theta_i))}{\partial x_{0j}} = -\frac{\langle \Phi_x(x_0(\theta_i)), \frac{\partial x_0(\theta_i)}{\partial x_{0j}}\rangle}{\langle \Phi_x(x_0(\theta_i)), f(\theta_i, x_0(\theta_i))\rangle}, j = 1, \ldots, n. \tag{3.6}$$

The Jacobian $W_{ix}(x_0(\theta_i)) = [\frac{\partial W_i(x_0(\theta_i))}{\partial x_{01}}, \frac{\partial W_i(x_0(\theta_i))}{\partial x_{02}}, \ldots, \frac{\partial W_i(x_0(\theta_i))}{\partial x_{0n}}]$ is evaluated by the following expression

$$\frac{\partial W_i(x_0(\theta_i))}{\partial x_{0j}} = (f(\theta_i, x_0(\theta_i)) - f(\theta_i, x_0(\theta_i) + I(x_0(\theta_i))))\frac{\partial \tau_i}{\partial x_{0j}} + \frac{\partial I}{\partial x_0}(e_j + f(\theta_i, x_0(\theta_i))\frac{\partial \tau_i}{\partial x_{0j}}), \tag{3.7}$$

where $e_j = (\underbrace{0, \ldots, 1}_{j}, 0, \ldots, 0),\ j = 1, 2, \ldots, n$. Next, considering the second equation in (2.4) and using mean value theorem, we obtain that $\Delta(x(\theta_i) - x_0(\theta_i)) = W_i(x(\theta_i) - x_0(\theta_i)) = W_{ix}(x_0(\theta_i))(x(\theta_i) - x_0(\theta_i)) + O(\|x(\theta_i) - x_0(\theta_i)\|)$. From that, it is seen that the linearization at a transversal moment is determined with the matrix $B_i = W_{ix}(x_0(\theta_i))$.

By means of these discussions, one can conclude that the matrix B_i in (3.5) can be in the following form

$$B_i = \begin{cases} O_n & \text{if} \quad (N1) \text{ is valid,} \\ W_{ix} & \text{if} \quad (N2) \text{ is valid,} \end{cases} \tag{3.8}$$

where O_n denotes the $n \times n$ zero matrix. There may appear singularity in W_{ix} for the linearization at a grazing point. To overcome the singularity, the following conditions will be needed.

- $(A1)$ The map $W_i(x)$ in (2.3) is differentiable if $x = x_0(\theta_i)$.
- $(A2)$ The inequality $\tau_i(x) < \theta_{i+1} - \theta_i - \epsilon$ is true for some positive ϵ on a set of points near $x_0(\theta_i)$, which satisfy condition $(N1)$.

Denote by $\bar{x}(t), j = 1, 2, \ldots, n$, a solution of (2.4) such that $\bar{x}(t_0) = x_0 + \Delta x$, $\Delta x = (\xi_1, \xi_2, \ldots, \xi_n)$, and let η_j be the moments of discontinuity of $\bar{x}(t)$.

The following conditions are required in what follows.

(A) For all $t \in \mathcal{I} \setminus \cup_{i \in \mathcal{A}} \widehat{(\eta_i, \theta_i]}$, the following equality is satisfied

$$\bar{x}(t) - x_0(t) = \sum_{i=1}^{n} u_i(t)\xi_i + O(\|\Delta x\|), \tag{3.9}$$

where $u_i(t) \in PC(\mathcal{I}, \theta)$ and $\mathcal{I}$ is a finite subset of $\mathbb{R}$.

(B) There exist constants $\nu_{ij}, j \in \mathcal{A}$, such that

$$\eta_j - \theta_j = \sum_{i=1}^{n} \nu_{ij}\xi_i + O(\|\Delta x\|); \tag{3.10}$$

(C) The discontinuity moment η_j of the near solution $\bar{x}(t)$ approaches to the discontinuity moment $\theta_j, j \in \mathcal{A}$, of grazing one as ξ tends to zero.

If $x_0(\theta_i)$ is a transversal point and the conditions (A) and (B) are valid, then the solution $\bar{x}(t)$ has a linearization with respect to solution $x_0(t)$. If the point $x_0(\theta_i)$ is grazing, the condition (B) may not be true. So, to obtain a linearization at a grazing point, the conditions (A) and (B) or (A) and (C) should be validated. The solution $x_0(t)$ is differentiable with respect to the initial value x_0 on $\mathcal{I}, t_0 \in \mathcal{I}$, if for each solution $\bar{x}(t)$ with sufficiently small Δx the linearization exists. The functions $u_i(t)$ and ν_{ij} depend on Δx and uniformly bounded on a neighborhood of x_0.

The systems (2.1) and (2.4) are $B-$equivalent, that is why it is acceptable to linearize system (2.4) instead of system (2.1) around $x_0(t) = x(t, t_0, x_0)$, which is a solution of both systems. Thus, by applying linearization to (2.4), the system (3.5) is obtained. Additionally, the linearization matrix B_i in (3.5) for the grazing point also has to be defined by the formula (3.8), where W_{ix} exists by condition $(A1)$.

4. Stability of grazing periodic solutions

Assume additionally that $f(t,x)$ in (2.1) is $T-$ periodic in time, i.e. $f(t+T,x) = f(t,x)$, for $T > 0$, with a grazing $T-$periodic solution $\Psi(t) : \mathbb{R}_+ \to D$ and $\theta_i,\ i \in \mathbb{Z}$, be the points of discontinuity satisfy $\theta_{i+p} = \theta_i + T, p$ is a natural number. In what follows, we assume the validity of the next condition.

(A3) For each $\Delta x \in \mathbb{R}^n$, the variational system for the near solution $x(t) = x(t, t_0, x_0 + \Delta x)$ to $\Psi(t)$ is one of the following m periodic homogeneous linear impulsive systems

$$\begin{aligned} u' &= A(t)u, \\ \Delta u|_{t=\theta_i} &= B_i^{(j)}u, \end{aligned} \tag{4.11}$$

such that $B_{i+p}^{(j)} = B_i^{(j)}, i \in \mathbb{Z},\ j = 1, \ldots, m$, where the number m cannot be larger than 2^k, k denotes the number of grazing point in the interval $[0, T]$.

The collection of m systems (4.11) is *the variational system around the periodic grazing solution*. For each of these systems, we find the matrix of monodromy, $U_j(T)$ and denote corresponding Floquet multipliers by $\rho_i^{(j)}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Next, the following assumption is needed,

(A4) $|\rho_i^{(j)}| < 1, i = 1, \ldots, n$, for each $j = 1, \ldots, m$.

Theorem 4.1. *Under the assumption that conditions $(H1)$, $(A1) - (A4)$ are valid. Then, $T-$periodic solution $\Psi(t)$ of (2.1) is asymptotically stable, if the Lipschitz constant l is sufficiently small.*

Proof. Let $\theta_i, i \in \mathbb{Z}$, be the discontinuity moments of $\Psi(t)$. There exists a natural number p, such that $\theta_{i+p} = \theta_i + T$ for all $i \in \mathbb{Z}$. Because of conditions $(H1)$ and $B-$differentiability of $\Psi(t)$, there exists continuous dependence on initial data and consequently there exists a neighborhood of $(\theta_i, x_0(\theta_i))$ such that any solutions which starts in the set will have moments of discontinuity which constitute a $B-$ sequence [1] with difference between neighbors approximately equal to the distance between corresponding neighbor moments of discontinuity of the periodic solution $\Psi(t)$. For this reason, the variational system for $\Psi(t)$, can be determined through B-reduced system. □

On the basis of above discussion, the variational system takes the form

$$\begin{aligned} z' &= A(t)z + \phi(t, z), \\ \Delta z|_{t=\theta_i} &= B_i^{(j)} z + \psi_i(z), \quad j = 1, 2, \ldots, m, \end{aligned} \tag{4.12}$$

where $\phi(t, z) = [f(t, \Psi(t)+z) - f(t, \Psi(t))] - A(t)z$ and $\psi_i(z) = W_i(\Psi(\theta_i)+z) - W_i(\Psi(\theta_i)) - B_i^{(j)} z$, are continuous functions, and matrices $B_i^{(j)}$ satisfy condition $(A4)$. Denote $Y_j(t)$, $j = 1, 2, \ldots, m$ the fundamental matrix of (4.13) adjoint to (4.12) linear homogeneous system

$$\begin{aligned} y' &= A(t)y, \\ \Delta y|_{t=\theta_i} &= B_i^{(j)} y. \end{aligned} \tag{4.13}$$

Due to the conditions $(A3)$ and $(A4)$, there exist numbers $K > 0$ and $\gamma > 0$ such that for all $j = 1, 2, \ldots, m$, the following estimate holds

$$\|Y_j(t, s)\| \le K e^{-\gamma(t-s)}. \tag{4.14}$$

Any solution of (4.12) neighbor to the trivial one can be written as one of the following form

$$z(t, z_0) = Z(t, t_0) z_0 + \int_{t_0}^{t} Y_j(t, s)\phi(s, z(s, z_0))ds + \sum_{t_0 \le \theta_i < t} Y_j(t, \theta_i)\psi_i(z(\theta_i, z_0)). \tag{4.15}$$

The functions $\phi(t, z)$ and $\psi(t, z)$ satisfy the inequalities

$$\|\phi(t, z)\| \le l\|z\|, \ \|\psi_i(z)\| \le l\|z\|, \tag{4.16}$$

and for all $t > t_0$, $\|z\| < k$, $k > 0$. There exists positive θ such that the inequality $\theta_{i+1} - \theta_i > \theta$ is true. Using inequalities (4.14), (4.16), and Gronwall-Bellman Lemma [1], we obtain the following estimate $\|z(t, t_0)\| \le K e^{-(\gamma - Kl - \frac{1}{\theta}\ln(1+kl))(t-t_0)}\|z_0\|$. For sufficiently small l, it is true that $\gamma - Kl - \frac{1}{\theta}\ln(1 + kl) > 0$. The theorem is proved.

Example 4.1. In this example, we consider the following non-autonomous system of differential equation with variable moments of impulses

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -0.002x_2 - x_1 - 1 - 0.002\sin(t), \\ x_3' &= -0.2x_3 - 0.1x_2, \\ \Delta x_2|_{x \in S} &= -(1 + 0.9x_2)x_2, \end{aligned} \tag{4.17}$$

where $x = (x_1, x_2, x_3)$ and the discontinuity surface S is written as $S = \{(x_1, x_2, x_3)|\Phi(x) = x_1 = 0\}$. It has a periodic solution $\Psi(t) = (-1 + \cos(t), -\sin(t), x_3(t))$, where $x_3(t)$ is the periodic solution of the third system in (4.17). Consider the function $H(t, x)$ at the point $(\theta_i, \Psi(\theta_i)) = (2\pi i, \Psi(2\pi i))$, $i \in \mathbb{Z}$. It is true that $H(2\pi i, \Psi(2\pi i)) = 0$, $J(\Psi(2\pi i)) = 0$ and

$H(t,x) \neq 0$ for some number $\delta > 0$, such that $|t-2\pi i| < \delta$ and $\|x-\Psi(2\pi i)\| < \delta$. This validates $(H1)$. Moreover, fix some $\theta_i = 2\pi i$ then we can say that $(\theta_i, \Psi(\theta_i)) = (2\pi i, \Psi(2\pi i)) = (2\pi i, 0, 0)$ is a grazing point. Since the point $(2\pi i, 0, 0)$, $i \in \mathbb{Z}$, belongs to $\Psi(t)$, then we can say that $\Psi(t)$ is a grazing periodic solution. Additionally, all points $(2\pi i, \Psi(2\pi i)),\ i \in \mathbb{Z}$ are grazing. In the system (4.17), we will take into first two equations together with the jump equation to analyze the asymptotic stability of the grazing cycle of it. Because, the equation without perturbation $-0.1x_2$, is itself asymptotically stable. We will conclude at the end whole system have asymptotically stable grazing periodic solution.

Next, let us continue with the linearization. Because the point $(2\pi i, 0, 0)$ is a grazing point, we will consider the linearization by applying formulas (3.6) and (3.8). By means of condition $(H1)$, it is true that the solutions intersect the surface of discontinuity transversely near the grazing one. Denote the grazing point by $x^* = (\Psi(0)) = (0,0)$. Assume that $x(t) = x(t, 0, x^* + \Delta x)$, $\Delta x = (\Delta x_1, \Delta x_2)$ is not a grazing solution. That is, the point $x^* + \Delta x$ is not a point of $\Psi(t)$. So, the meeting point $\bar{x} = (\bar{x}_1, \bar{x}_2) = (x_1(\xi, 0, (x^* + \Delta x)), x_2(\xi, 0, (x^* + \Delta x))$, is transversal one. Moreover, let ξ be the meeting moment with Γ. It is clear $\bar{x}_1 = 0$ and $\bar{x}_2 > 0$. Due to the transversality of $\bar{x}$, the first component $\frac{\partial \tau_i(\bar{x})}{\partial x_1}$ of the gradient $\nabla \tau_i(\bar{x})$ can be determined by formula (3.6) and it is $\frac{\partial \tau_i(\bar{x})}{\partial x_1} = -\frac{1}{\bar{x}_2}$. At the grazing point, the derivative is evaluated as $\frac{\partial \tau_i(x^*)}{\partial x_1} = -\infty$.

To find a linearization, let $\xi(t) = x(t, \theta_i, x)$ be a solution of first two equations in (4.17) and $\tau_i = \tau_i(x)$ be meeting time of it, and $\psi(t) = x(t, \tau_i, \xi(\tau_i) + J(\xi(\tau_i))$ be an another solution of them. Taking derivative of the formula (2.3) with respect to $x(t)$, we get

$$\begin{aligned} \frac{\partial W_i(x)}{\partial x_1^0} &= \int_{\theta_i}^{\tau_i(x)} \frac{\partial f(s,x(s))}{\partial x} \frac{\partial x(s)}{\partial x_1^0} ds + f(s,x(s)) \frac{\partial \tau_i(x)}{\partial x_1^0} + J_x(x)(e_1 + f(s,x(s)) \frac{\partial \tau_i(x)}{\partial x_1^0}) \\ &+ f(s,\psi(s)) \frac{\partial \tau_i(x)}{\partial x_1^0} + \int_{\tau_i(x)}^{\theta_i} \frac{\partial f(s,\psi(s))}{\partial x} \frac{\partial x(s)}{\partial x_1^0} ds, \end{aligned} \tag{4.18}$$

where $e_1 = (1,0)^T$, T denotes the transpose of a matrix. Considering the formula (3.6) for the point $\bar{x} = (\bar{x}_1, \bar{x}_2)$, the derivative $\frac{\partial \tau_i(\bar{x})}{\partial x_1^0}$ can be evaluated as $\frac{\partial \tau_i(\bar{x})}{\partial x_1^0} = -\frac{1}{\bar{x}_2}$. Substituting $x = \bar{x}$ to (4.18), and considering (4.17) with it, we have

$$\frac{\partial W_i(\bar{x})}{\partial x_1^0} = \begin{bmatrix} R\bar{x}_2 - 1 \\ 0.002(1 - R\bar{x}_2) + 2R(0.002\bar{x}_2 - \bar{x}_1 - 1 - 0.002\sin(\xi)) \end{bmatrix}. \tag{4.19}$$

Similarly, differentiating (2.3) with $x(t)$ we get

$$\begin{aligned} \frac{\partial W_i(x)}{\partial x_2^0} &= \int_{\theta_i}^{\tau_i(x)} \frac{\partial f(s,x)}{\partial x} \frac{\partial x(s)}{\partial x_2^0} ds + f(s,x(s)) \frac{\partial \tau_i(x)}{\partial x_2^0} + J_x(x)(e_2 + f(s,x(s)) \frac{\partial \tau_i(x)}{\partial x_2^0}) \\ &+ f(s,\psi(s)) \frac{\partial \tau_i(x)}{\partial x_2^0} + \int_{\tau_i(x)}^{\theta_i} \frac{\partial f(s,\psi(s))}{\partial x} \frac{\partial x(s)}{\partial x_2^0} ds, \end{aligned} \tag{4.20}$$

where $e_2 = (0,1)^T$.

Evaluating $\frac{\partial \tau_i(\bar{x})}{\partial x_2^0}$ with (3.6) at the transversal point $\bar{x} = (\bar{x}_1, \bar{x}_2)$, then we get $\frac{\partial \tau_i(\bar{x})}{\partial x_2^0} = 0$. Considering the right hand side of (4.20) at the point $x = \bar{x}$, we obtain

$$\frac{\partial W_i(\bar{x})}{\partial x_2^0} = \begin{bmatrix} 0 \\ -2R\bar{x}_2 \end{bmatrix}. \tag{4.21}$$

Joining (4.19) and (4.21), the matrix $W_{ix}(\bar{x})$ can be obtained as

$$(4.22)\quad W_{ix}(\bar{x}) = \begin{bmatrix} R\bar{x}_2 - 1 & 0 \\ 0.002(1 - R\bar{x}_2) + 2R(0.002\bar{x}_2 - \bar{x}_1 - 1 - 0.002\sin(\xi)) & -2R\bar{x}_2 \end{bmatrix}.$$

Taking into account formula (4.22), we can say that the map $W_i(x)$ is differentiable at $x = x^*$. Thus, condition $(A1)$ is true. The meeting moment $\tau_i(x)$ can not been taken into account whenever it satisfies $(N1)$. So, to validate $(A2)$, we should only consider those which satisfies $(N2)$. To verify it, let us take into account a solution of the first equation in (4.17) which starts at the point $\bar{x} = (0, \bar{x}_2) \in \mathcal{S}$. The solution of (4.17) at $\bar{x}$ is the form

$$x(t, 0, \bar{x}) = \frac{\bar{x}_2}{\sqrt{1 - (0.001)^2}} \exp(0.001t)\sin(\sqrt{1 - (0.001)^2}t).$$

This solution meets the surface $\mathcal{S}$ at the moment $\bar{t} = \frac{2\pi}{\sqrt{1-(0.001)^2}}$, again. Thus, the meeting moment $\tau_i(x) = \bar{t} < 2\pi - \epsilon$, where ϵ is a small positive number and this verifies $(A2)$.

We apply it when $x \to x^*$, as well as $\xi \to 2\pi$, where 2π is the first grazing discontinuity point of periodic solution $\Psi(t)$, then we obtain that

$$(4.23)\qquad W_{ix}(x^*) = \begin{bmatrix} -1 & 0 \\ 0.002 - 2R & 0 \end{bmatrix}.$$

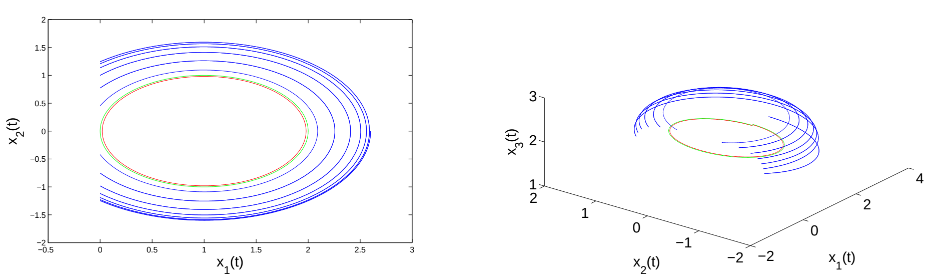
Consequently, the function $W_i(x)$ is differentiable at the grazing point $x = x^*$ and $(A1)$ is valid.

On the basis of the above discussion, we can say that the variational system consists of $m = 2$ linear homogenous subsystems:

$$(4.24)\qquad \begin{aligned} z' &= A(t)z, \\ \Delta z|_{t=\theta_i} &= B^{(j)}z,\ j = 1, 2, \end{aligned}$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -0.002 \end{bmatrix}$, $\theta_i = 2\pi i$, $B^{(1)} = O_2$ and $B^{(2)} = \begin{bmatrix} -1 & 0 \\ 0.002 - 2R & 0 \end{bmatrix}$.

One can check easily that system (4.24) is a $(2\pi, 1)-$periodic system this validates $(A3)$. The multipliers of (4.24) are $\rho_1^{(1)} = 0.9844$, $\rho_2^{(1)} = 0.9844$, $\rho_1^{(2)} = 0.9844$, $\rho_2^{(2)} = 0.098$. Thus, the condition $(A4)$ is valid. Consequently, by means of above assertion, it is easy to say that the periodic solution $\Psi(t)$ is asymptotically stable.



(A) The components $x_1(t), x_2(t)$ and $x_3(t)$ of the system with initial values $x(0) = (2.6, 0, 1.9)$, $x(0) = (2, 0, 1.9)$ and $x(0) = (1.98, 0, 1.9)$, in blue, green and red, respectively.

(B) The green is the periodic solution $\Psi(t)$ of the system, the blue and red curves are the phase portraits of the solutions of the system with initial values $x_0 = (2.6, 0, 1.9)$ and $x(0) = (1.98, 0, 1.9)$.

In Fig. 1a, the three dimensional components are depicted with initial values $x(0) = (2.6, 0, 1.9)$, $x(0) = (2, 0, 1.9)$ and $x(0) = (1.98, 0, 1.9)$, in blue, green and red, respectively. It is hard to see the behavior of the solution in three dimensional space. For this reason, we consider the projection of Fig. 1a to the two dimensional space $x_1 - x_2$ which is depicted in Fig. 1b. In Fig. 1b, one can see that the blue curves approaches the grazing cycle $\Psi(t)$ which is drawn in green as time increases and the inside red solution is approaching the grazing cycle as well.

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