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Existence of non-trivial complex unit neighborhoods

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ABSTRACT. First, we briefly mention the basic definitions and results on unit neighborhoods of zero. Next, we show the existence of certain non-trivial complex unit neighborhoods of zero. We expose a generalization of the construction method used on the mentioned particular case. Since this construction may not lead to a unit neighborhood of zero, we develop some necessary conditions. Finally, we describe our heuristic use of Wolfram Mathematica to prove the existence of non-trivial complex unit neighborhoods.

1. INTRODUCTION AND NOTATION

1.1. **Notation.** All rings will be considered to be non-zero, associative and unital, unless otherwise explicitly stated. If X is a topological space and A is a subset of X, then int (A), cl (A), and bd (A) will denote the topological interior, the topological closure, and the topological boundary of A, respectively. Let R be a ring and $A, B \subset R$, then $AB = \{ab : a \in A, b \in B\}$ and $A^2 = AA$. A absolute semi-value over a ring R is a non-negative function $|\cdot| : R \to \mathbb{R}$ such as $|ab| = |a||b|$, $|0| = 0$ and $|a + b| \leq |a| + |b|$. Given R an absolute semi-valued ring, we denote the open ball, the closed ball and the sphere of center $x \in R$ and radius $r > 0$ as $\mathsf{U}_R(x, r)$, $\mathsf{B}_R(x, r)$ and $\mathsf{S}_R(x, r)$ respectively. When $x = 0$ and $r = 1$, we will simply write U_R, B_R and S_R respectively.

1.2. **Introduction.** Topological rings are still an unexplored structure in some aspects. The classification of absolute-valued division real algebras started by A. Hurwitz, developed by A. A. Albert and finished by Urbanik and Wright [8] was an important motivation to the study of general topological rings. On 1969 the first exhaustive introduction to topological rings [2] is published by Arnautov, Mikhalev and Glavatsky. Twenty years later, Warner continues this foundational work publishing [10] and [11], which offer a rich structural perspective of topological rings. Despite of this, there are still open research lines on specific aspects of topological rings [1], [3], [5] and [9]. This article focuses on studying certain subsets of topological rings called unit neighborhoods of zero. Let us recall the concept of open units and closed units (introduced in [6]) upon which most of our results in this paper rely.

Definition 1.1 (Garcia-Pacheco and Piniella, 2015; [6])**.** Let R be a topological ring.

- A regular open neighborhood U of 0 is said to be an open unit provided that U is symmetric for the addition, idempotent for the multiplication, and $1 \in \mathrm{cl}(U)$.
- A regular closed neighborhood B of 0 is said to be a closed unit provided that int (B) is an open unit.

Recall that: A is regular open [closed] if $\mathrm{int}(\mathrm{cl}(A)) = A$ [$\mathrm{cl}(\mathrm{int}(A)) = A$]; A is symmetric for the addition if $A = -A$; A is idempotent for the multiplication if $A = A^2$.

Unit neighborhoods are an attempt to generalize the "ball" concept to general topological rings. A good generalization is given by Warner on [10, Chapter 19], but it only can

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be applied to division rings. Thus, special interest is given to propositions that relate unit neighborhoods to balls. In [6], real unit neighborhoods were fully characterized.

Theorem 1.1 (Garcia-Pacheco and Piniella, 2015; [6]). $\mathsf{U}_{\mathbb{R}}$ [$\mathsf{B}_{\mathbb{R}}$] *is the only non-trivial open* [*closed*] *unit neighborhood of* 0 *in* $\mathbb{R}$*, respectively.*

Connection and boundedness was explored for absolute semi-valued rings through the next Proposition.

Theorem 1.2 (Garcia-Pacheco and Piniella, 2015; [6]). *Let R be an absolute semi-valued ring such that $|1| = 1$, the open balls are connected and $\operatorname{int}(\mathsf{B}_R) = \mathsf{U}_R$. Consider a non-trivial open unit U of R. If $U \cap \mathcal{U}(R)$ is dense in U, then $U \subseteq \mathsf{U}_R$ and U is connected.*

However, the classification of the open units and the closed units of $\mathbb{C}$ was not complete. The following Theorem was proven.

Theorem 1.3 (Garcia-Pacheco and Piniella, 2015; [6]). $\mathsf{U}_{\mathbb{C}}$ [$\mathsf{B}_{\mathbb{C}}$] *is the only non-trivial open* [*closed*] *unit neighborhood of* 0 *in* $\mathbb{C}$ *satisfying* $\mathsf{S}_{\mathbb{C}} \subset cl(\mathsf{U}_{\mathbb{C}})$ [$\mathsf{S}_{\mathbb{C}} \subset \mathsf{B}_{\mathbb{C}}$].

One can notice that there is a misprint in [6], since the last hypothesis was not written at all, although it was used implicitly in the last step of the proof. This Theorem is not completely satisfactory, since it does not examine all unit neighborhoods, but a reduced subset of them. Thus, we devote this paper to deepen our knowledge on non-trivial complex unit neighborhoods.

2. Construction of non-trivial complex unit neighborhoods

In order to construct our non-trivial examples, we need a previous Lemma.

Lemma 2.1. *Let $r > 0$ and $f : [-\pi, \pi] \to \mathbb{R}$ be defined as $f(t) = 1 - \dfrac{2(1-r)}{\pi}\sqrt{(\pi - |t|)|t|}$ and $x, y \in [-\pi, \pi]$. Then f verifies*

(1) $f(x) = f(-x)$
(2) $f(x - \pi) = f(x)$ *if* $x \geq 0$
(3) $f(x + \pi) = f(x)$ *if* $x \leq 0$
(4) $\sqrt{(\pi - |t|)|t|}$ *attains its maximum at $\pm\pi/2$ and its minimum at 0 and $\pm\pi$.*
(5) *f attains its minimum at $\pm\pi/2$ and its maximum at 0 and $\pm\pi$.*
(6) *If* $|x + y| \leq \pi$, $f(x + y) \geq f(x)f(y)$
(7) *If* $x + y > \pi$, $f(x + y - \pi) \geq f(x)f(y)$
(8) *If* $x + y < -\pi$, $f(x + y + \pi) \geq f(x)f(y)$

Proof. (1) is immediate. For (2), notice that $x + y - \pi < 0$. Then

$$\sqrt{(\pi - |x + y - \pi|)(|x + y - \pi|)} = \sqrt{(x + y)(\pi - (x + y))} = \sqrt{|x + y|(\pi - |x + y|)}$$

which proves $f(x + y - \pi) = f(x + y)$. (3) is a direct application of (2). (4) and (5) are elementary real analysis exercises. In order to prove (6), we will first consider $x, y \geq 0$. We can compute

$$\begin{aligned}
&f(x + y) - f(x)f(y) = \\
&= 1 - \frac{2(1-r)}{\pi}\sqrt{(\pi - (x + y))(x + y)} - \\
&- \left(1 - \frac{2(1-r)}{\pi}\sqrt{(\pi - x)x}\right)\left(1 - \frac{2(1-r)}{\pi}\sqrt{(\pi - y)y}\right) = \\
&= 1 - \frac{2(1-r)}{\pi}\sqrt{(\pi - (x + y))(x + y)} - 1 + \frac{2(1-r)}{\pi}\sqrt{(\pi - x)x} +
\end{aligned}$$

$$+\frac{2(1-r)}{\pi}\sqrt{(\pi-y)y}-\left(\frac{2(1-r)}{\pi}\right)^2\sqrt{(\pi-x)x(\pi-y)y}=$$

$$=\left(\frac{2(1-r)}{\pi^2}\right)\left(\overbrace{\pi\sqrt{(\pi-x)x}+\pi\sqrt{(\pi-y)y}}^{=A}+\overbrace{2r\sqrt{(\pi-x)x|(\pi-y)y}}^{=C}-\right.$$

$$\left.-\overbrace{\left(\pi\sqrt{(\pi-(x+y))(x+y)}+2\sqrt{(\pi-x)x(\pi-y|)y}\right)}^{=B}\right)$$

Notice that if $x=0$ or $y=0$ (which, because of $|x+y|\leq\pi$, is equivalent to $y=\pi$ or $x=\pi$) then $f(x+y)-f(x)f(y)$. Thus we can consider $x,y>0$. It is clear that $C>0$. We are going to compute A^2 and B^2 to compare both quantities

$$A^2=\left(\pi\sqrt{(\pi-x)x}+\pi\sqrt{(\pi-y)y}\right)^2=$$

$$=\overbrace{\pi^2(\pi-x)x+\pi^2(\pi-y)y}^{=A_1}+\overbrace{2\pi^2\sqrt{(\pi-x)x(\pi-y)y}}^{=A_2}$$

$$B^2=\left(\pi\sqrt{(\pi-(x+y))(x+y)}+2\sqrt{(\pi-x)x(\pi-y)y}\right)^2=$$

$$=\overbrace{\pi^2(\pi-(x+y))(x+y)+4(\pi-x)x(\pi-y)y}^{=B_1}+$$

$$+\overbrace{4\pi\sqrt{(\pi-(x+y))(x+y)(\pi-x)x(\pi-y)y}}^{=B_2}$$

Notice that B_1 can be expressed as

$$B_1=\pi^2(\pi-(x+y))(x+y)+4(\pi-x)x(\pi-y)y=\pi^2(\pi-x)x+\pi^2y(\pi-y-2x)+4(\pi-x)x(\pi-y)y$$

Considering this and $x,y>0$, we can calculate

$$A_1-B_1=\cancel{\pi^2(\pi-x)x}+\pi^2(\pi-y)y-\cancel{\pi^2(\pi-x)x}-\pi^2y(\pi-y-2x)-4(\pi-x)x(\pi-y)y=$$

$$=\pi^2(\pi-y)y-\pi^2(\pi-y-2x)y-4(\pi-x)x(\pi-y)y=2\pi^2xy-4(\pi-x)x(\pi-y)y$$

Now we can compute

$$A^2-B^2=(A_1-B_1)+(A_2-B_2)=2\pi^2\left(\sqrt{(\pi-x)x(\pi-y)y}+xy\right)-$$

$$-4\left(\pi\sqrt{(\pi-(x+y))(x+y)(\pi-x)x(\pi-y)y}+(\pi-x)x(\pi-y)y\right)=$$

$$=2\pi^2\left(\sqrt{(\pi-x)x(\pi-y)y}+xy\right)-$$

$$-4\sqrt{(\pi-x)x(\pi-y)y}\left(\pi\sqrt{(\pi-(x+y))(x+y)}+\sqrt{(\pi-x)x(\pi-y)y}\right)$$

$$\frac{A^2-B^2}{2\sqrt{(\pi-x)x(\pi-y)y}}=\pi^2\left(1+\frac{\sqrt{xy}}{\sqrt{(\pi-x)(\pi-y)}}\right)-$$

$$-2\left(\pi\sqrt{(\pi-(x+y))(x+y)}+\sqrt{(\pi-x)x(\pi-y)y}\right)$$

From (4) we have $\frac{\pi}{2} \geq \sqrt{(\pi - x)x}$. On one hand, $\pi^2 \geq 2\pi\sqrt{(\pi - (x+y))(x+y)}$. On the other,

$$\pi^2 \frac{\sqrt{xy}}{\sqrt{(\pi - x)(\pi - y)}} \geq \frac{\pi}{2}\sqrt{xy} \geq \sqrt{(\pi - x)x(\pi - y)y}$$

This shows that $A^2 \geq B^2$. Since $A > 0$, it implies $A \geq B$, finishing the proof for $x, y \geq 0$.

If $x, y \leq 0$, one can obtain easily $f(x+y) = f((-x) + (-y)) \geq f(-x)f(-y) = f(x)f(y)$ using (1) and the positive case.

If $x \geq 0$, $y \leq 0$. We distinguish two cases. If $x + y \geq 0$, then using (2) and the preivous case, we get

$$f(x+y) = f(x+y-\pi) = f((x-\pi) + y) \geq f(x-\pi)f(y) = f(x)f(y)$$

If $x + y \leq 0$, using (3) instead of (2) we get

$$f(x+y) = f(x+y+\pi) = f(x + (y+\pi)) \geq f(x)f(y+\pi) = f(x)f(y)$$

which completes the proof of (6).

To prove (7), suppose $x+y > \pi$. Without loss of generality we can assume $x > 0$. Using (2) we get

$$f(x+y-\pi) = f((x-\pi) + y) \geq f(x-\pi)(y) = f(x)f(y)$$

Showing (8) is an analogous use of (3).

□

Now we can prove the main result of this paper.

Theorem 2.4. *For every $0 < r < 1$ there exists a non-trivial complex open unit neighborhood of zero containing $\mathsf{U}_{\mathbb{C}}(0, r)$.*

Proof. Define for every $0 < r < 1$

$$f : [-\pi, \pi] \to \mathbb{R} \qquad f(t) = 1 - \frac{2(1-r)}{\pi}\sqrt{(\pi - |t|)|t|}$$

as in the previous Lemma. We define for every chosen r the following set

$$U_r = \{z \in \mathbb{C} \,:\, |z| < f(\arg(z))\}$$

Clearly $1 \in \mathrm{cl}(U_r)$ because $[0,1) \subset U_r$. Notice that $\arg(-z) = \arg(z) - \pi$ if $\arg(z) > 0$ and $\arg(-z) = \arg(z) + \pi$ if $\arg(z) < 0$. By Lemma 2.1 (2), (3) this implies $f(\arg(z)) = f(\arg(-z))$. Since $|z| = |-z|$, it is direct that $U_r = -U_r$ by its own definition. To prove that $U_r \subset U_r^2$, consider $z \in U_r$. Since U_r is open, we can find an open ball centered in z and fully contained in U_r. This implies the existence of some $\varepsilon > 0$ such as $(1+\varepsilon)z \in U_r$. Since $[0,1) \subset U_r$, one has

$$z = (1+\varepsilon)z\left(\frac{1}{1+\varepsilon}\right) \subset U_r^2$$

To show that U_r is regular open, first observe that $\mathrm{cl}(U_r) = B_r := \{z \in \mathbb{C} \,:\, |z| \leq f(\arg(z))\}$. Indeed, every z with $|z| = f(\arg(z))$ is an accumulation point of the open segment joining 0 and z, which is clearly in U_r. This shows $B_r \subset \mathrm{cl}(U_r)$. But B_r is closed since f is continuous and $B_r = f^{-1}([0,1])$; so the reciprocal inclussion also holds. Also notice that $\mathrm{int}(B_r) = U_r$, since if $z \in B_r$ and $|z| = f(\arg(z))$, then $(1+1/n)z \notin B_r$ for every $n \in \mathbb{N}$. Joining both identities we get $\mathrm{int}(\mathrm{cl}(U_r)) = \mathrm{int}(B_r) = U_r$ proving U_r regularity.

Finally, to show that $U_r^2 \subset U_r$, consider $z, z' \in U_r$. Then $f(\arg(zz'))$ is equal to $f(\arg(z) + \arg(z'))$ or $f(\arg(z) + \arg(z') - 2\pi)$ or $f(\arg(z) + \arg(z') + 2\pi)$ depending on if $\arg(z) + \arg(z') \in [-\pi, \pi]$, if $\arg(z) + \arg(z') > \pi$ or if $\arg(z) + \arg(z') < \pi$ respectively. Applying Lemma 2.1 (6), (7), (8) respectively and then (2) or (3) if necessary, it is obtained that $f(\arg(zz')) \geq f(\arg(z))f(\arg(z')) \geq |z||z'| = |zz'|$ and thus $zz' \in U_r$.

□

Of course, a non-trivial closed unit can be constructed using this U_r.

Corollary 2.1. *For every* $0 < r < 1$ *there exists a non-trivial complex closed unit neighborhood of zero containing* $\mathsf{B}_{\mathbb{C}}(0, r)$.

Proof. Just consider $\mathrm{cl}(U_r)$, whose interior is U_r as proven in Theorem 2.4 and, consequently, is a closed unit neighborhood of zero. □

3. Generalized construction and necessary conditions

We are interested in generalizing this set construction method using arbitrary functions in order to construct more examples of closed unit neighborhoods different from those we know. Consider $F \in \mathcal{C}^1(0, \pi/2)$. We can define

$$f_0(F, r, x) = 1 - \frac{1-r}{F(\pi/2)} F(x)$$

which verifies $f_0(F, r, 0) = 1$ and $f_0(F, r, \pi/2) = r$. Notice that the function $f(x)$ used in Lemma 2.1 and Theorem 2.4 is equal to $f_0(F, r, x)$ using $F(x) = \sqrt{(\pi - |x|)|x|}$. However, the domain of $f_0(F, r, \cdot)$ is smaller. This problem is easily solved expanding symmetrically:

$$f(F, r, x) = \begin{cases} f(F, r, x) = f_0(F, r, |x|) & \text{if} \quad x \in [-\pi/2, \pi/2] \\ f(F, r, x) = f_0(F, r, \pi - |x|) & \text{if} \quad x \in [-\pi, -\pi/2] \cup [\pi/2, \pi] \end{cases}$$

Using this f, one can define for each $F \in \mathcal{C}^1(0, \pi/2)$ the set

$$U_{F,r} = \{z \in \mathbb{C} \, : \, |z| < f(F, r, \arg(z))\}$$

Notice that the set U_r defined on Theorem 2.4 can be constructed this way choosing $F(x) = \sqrt{(\pi - x)x}$. It is simple to see that $U_{F,r}$ is not an open unit neighborhood for every $F \in \mathcal{C}^1(0, \pi/2)$, as the following example shows.

Example 3.1. Choose $F \in \mathcal{C}^1(0, \pi/2)$ such as $f(F, r, \pi/4) > \sqrt{r}$. By definition of $U_{F,r}$, there exists $\varepsilon > 0$ such as $x = (\sqrt{r} + \varepsilon)e^{\pi/4} \in U_{F,r}$. Then

$$|x^2| = (\sqrt{r} + \varepsilon)^2 = r + 2\sqrt{r}\varepsilon + \varepsilon^2 > r = f(F, r, \pi/2) = f(F, r, \arg(x^2))$$

and thus $x^2 \notin U_{F,r}$. Consequently, $U_{F,r}^2 \not\subset U_{F,r}$ and $U_{F,r}$ is not an open unit neighborhood of zero.

We present necessary conditions which have to be impossed to F for $U_{F,r}$ to be an open unit neighborhood of zero. The first demands F to be bounded by 1.

Proposition 3.1. *Let* $F \in \mathcal{C}^1(0, \pi/2)$, $0 < r < 1$. *If there exists* $x \in [0, \pi/2]$ *such as* $f(F, r, x) > 1$, *then* $U_{F,r}$ *is not an open unit neighborhood of zero.*

Proof. In that case, there exists $z \in U_{F,r}$ such as $|z| > 1$. Lemma 1.2 makes $U_{F,r} = \mathbb{C}$, which contradicts $U_{F,r}$ own definition. □

The next proposition conditions the behaviour of F next to 1 (and, by additive symmetry, -1) in terms of its first derivative.

Proposition 3.2. *Let* $F \in \mathcal{C}^1(0, \pi/2)$, $0 < r < 1$. *If* $f'_+(F, r, 0)$ *is finite then* $U_{F,r}$ *is not an open unit neighborhood of zero.*

Proof. If $f'_+(F,r,0)$ is finite, its rightside Taylor series can be written as

$$f(F,r,x) = 1 + f'_+(F,r,0)x + O_+(x^2).$$

Notice also that, $f'_+(F,r,0) \leq 0$; indeed, otherwise there would exist $x \in U_{F,r}$ such as $|x| > 1$, which leads us to contradiction using the same argument on Proposition 3.1. Consider $z \in \mathbb{C}$ such as $z \in \mathrm{bd}\,(U_{F,r})$, that is, $|z| = f(F,r,\arg(z))$, and $\arg(z) > 0$. Notice that

$$\lim_{\arg(z)\to 0^+} |z^2| = \lim_{\arg(z)\to 0^+} f(F,r,\arg(z))^2 = (1 + f'_+(F,r,0)\arg(z))^2 =$$
$$= 1 + 2f'_+(F,r,0)\arg(z) + \left(f'_+(F,r,0)\arg(z)\right)^2 > 1 + 2f'_+(F,r,0)\arg(z) =$$
$$= \lim_{\arg(z)\to 0^+} f(F,r,2\arg(z)) = \lim_{\arg(z)\to 0^+} f\left(F,r,\arg\left(z^2\right)\right)$$

Thus, we can choose $z \in \mathrm{bd}\,(U_{F,r})$ enough close to 1 such as $z^2 \notin U_{F,r}$. Consequently, we can also find $z' \in U_{F,r}$ satisfying $(z')^2 \notin U_{F,r}$. □

Notice that Proposition 3.2 still holds if we write $f'_-(F,r,0)$ instead of $f'_+(F,r,0)$, since $f(F,r,\cdot)$ is symmetric. This can be used to describe some characteristics of those $U_{F,r}$ which in fact are open units, as the following Proposition shows.

Proposition 3.3. *If $U_{F,r}$ is an open unit neighborhood of zero, then it is non-convex.*

Proof. Since $U_{F,r}$ is an open neighborhood of zero, there exists $\rho > 0$ such as $\mathsf{B}_{\mathbb{C}}(0,\rho) \subset U_{F,r}$. In particular, $\rho i \in U_{F,r}$. Define $s(x) = \{z \in \mathbb{C} \,:\, z = tx + (1-t)\rho i\,,\, t \in [0,1]\}$. If $U_{F,r}$ were convex, $s(x) \subset U_{F,r}$ for every $x \in U_{F,r}$. Since $(0,1) \subset U_{F,r}$, we have $\bigcup_{x\in(0,1)} s(x) \subset U_{F,r}$. But this would imply that there were an open segment between $\rho i/2$ and 1 fully contained in $U_{F,r}$, which is in contradiction to Proposition 3.2. □

4. Heuristic approach using Wolfram Mathematica

While proving Theorem 2.4, we made use of Wolfram Mathematica in order to obtain an intuitive idea of the properties of the sets we were constructing. We show up the procedures since they have been simple but useful for proving the Theorem. Our aim was to develop a code which produces a random table of points of $U_{F,r}$ and represents all their possible products, so we can get an intuitive graphic idea of $U^2_{F,r}$. If the point cloud of $U^2_{F,r}$ is not included in $U_{F,r}$ then F does not generate any multiplicatively idempotent set and, consequently, any unit neighborhood of zero. Thus, we used the following code:

4.1. **Code.**

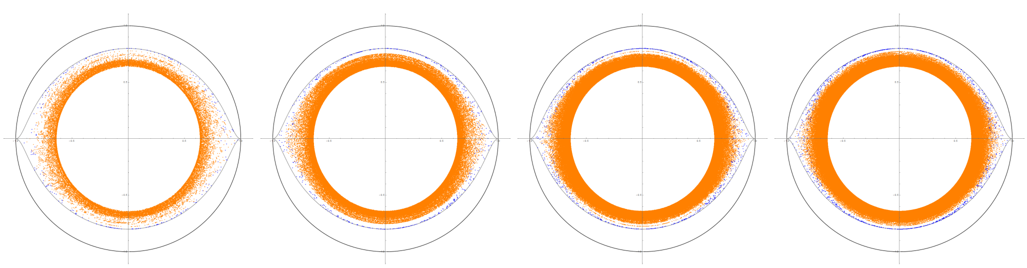
```
RandomTable[numpoints_, f_] := Module[{i, randomarg, randommod},
  randomarg = Table[RandomReal[{-Pi, Pi}], {i, 1, numpoints}];
  randommod =
   Table[RandomReal[{R, f[randomarg[[i]]]}], {i, 1, numpoints}];
  Table[{randommod[[i]] Cos[randomarg[[i]]],
    randommod[[i]] Sin[randomarg[[i]]]}, {i, 1, numpoints}]
  ]
AllProducts[table_] := Module[{i, j, products, numpoints},
  products = {};
  numpoints = Dimensions[table][[1]];
  For[i = 1, i <= numpoints, i++,
   For[j = i, j <= numpoints, j++,
     products = Join[products,
        {{
          table[[i, 1]] table[[j, 1]] - table[[i, 2]] table[[j, 2]],
          table[[i, 1]] table[[j, 2]] + table[[i, 2]] table[[j, 1]]
          }}
        ];
```

```
      ];
    ];
    products
    ]
  PlotMyTable[table_, f_] :=
   Show[ParametricPlot[{Cos[t], Sin[t]}, {t, -Pi, Pi},
      PlotStyle -> Black, ImageSize -> 600,
      AspectRatio -> 1]
    , ParametricPlot[{f[t] Cos[t], f[t] Sin[t]}, {t, -Pi, Pi},
      PlotStyle -> Gray]
    , ListPlot[table, PlotStyle -> {Orange, Blue}]
    ]
  RandomCone[numpoints_, f_, arg_, width_] :=
   Module[{i, randomarg, randommod},
    randomarg =
     Table[RandomReal[{arg - width, arg + width}], {i, 1, numpoints}];
    randommod =
     Table[RandomReal[{Min[f[arg - width], f[arg + width]],
        f[randomarg[[i]]]}], {i, 1, numpoints}];
    Table[{randommod[[i]] Cos[randomarg[[i]]],
      randommod[[i]] Sin[randomarg[[i]]]}, {i, 1, numpoints}]
    ]
  PlotMyCone[cone_, f_] :=
    Show[ParametricPlot[{Cos[t], Sin[t]}, {t, -Pi, Pi},
      PlotStyle -> Black, ImageSize -> 1000,
      AspectRatio -> 1, PlotRange -> {{0, 1}, {0, 1}}]
     , ParametricPlot[{f[t] Cos[t], f[t] Sin[t]}, {t, -Pi, Pi},
      PlotStyle -> Gray]
     , ListPlot[cone, PlotStyle -> {Orange, Blue}]
     ]
```

RandomTable generates a table of *numpoints* complex numbers inside $U_{F,r}$. Notice that it forces the points to be outside $\mathsf{U}_{\mathbb{C}}(0, R)$ where R is the maximum radium we can choose such as $\mathsf{U}_{\mathbb{C}}(0, R) \subset U_{F,r}$. This is made in order to get points closer to the boundary, which are the relevant ones since $\mathsf{B}_{\mathbb{C}}(0, 1)\mathsf{U}_{\mathbb{C}}(0, R) \subset \mathsf{U}_{\mathbb{C}}(0, R) \subset U_{F,r}$.*AllProducts* receives a table of complex numbers and gives the table of all possible products between them. For a more localizated point cloud, we define *RandomCone*, which is a version of *RandomTable* that concentrates the points inside an specified cone. This sacrifices global representation but improves precision mantaining the number of points used (and thus, computation time). *PlotMyTable* and *PlotMyCone* are just plotting functions.

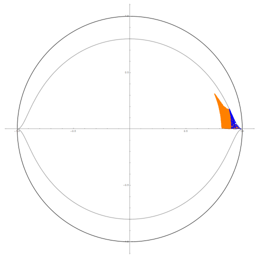
4.2. **Results.** The set we pretended to investigate was U_r as defined on Theorem 2.4. We chose $r = 0.8$ for our examples. We generated four different point clouds using initial tables of 250, 500, 750 and 1000 points respectively:

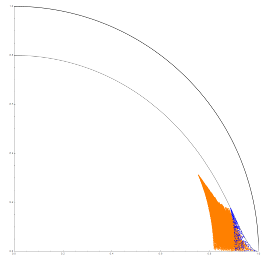
```
Table[PlotMyTable[AllProducts[RandomTable[250 i, f]], f]}, {i, 1, 4}]
```



Blue corresponds to initial data while orange corresponds to the products. In order to obtain localized information, we generated a cone with a 500 point initial table :

```
PlotMyCone[AllProducts[RandomCone[500, f, Pi/32, Pi/32]], f]
```





We concluded that U_r behaviour was very likely to satisfy multiplicative idempotency. Since the algebraic manipulation required to prove Lemma 2.1 was cumbersome, this first intuition was really helpful as it encouraged us to continue on our work.

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