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Normality degrees of finite groups

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ABSTRACT. In this paper we introduce and study the concept of normality degree of a finite group G. This quantity measures the probability of a random subgroup of G to be normal. Explicit formulas are obtained for some particular classes of finite groups. Several limits of normality degrees are also computed.

1. INTRODUCTION

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects which have been studied is the probability that two elements of a finite group G commute. It is called the *commutativity degree* of G, and has been investigated in many papers, as [3], [5]–[9] or [11]. Inspired by this concept, in [18] we introduced a similar notion for the subgroups of G, called the *subgroup commutativity degree* of G. This quantity is defined by

$$\begin{aligned} sd(G) &= \frac{1}{|L(G)|^2} \left|\{(H,K) \in L(G)^2 \mid HK = KH\}\right| = \\ &= \frac{1}{|L(G)|^2} \left|\{(H,K) \in L(G)^2 \mid HK \in L(G)\}\right| \end{aligned}$$

(where $L(G)$ denotes the subgroup lattice of G) and it measures the probability that two subgroups of G commute, or equivalently the probability that the product of two subgroups of G be a subgroup of G (recall also the natural generalization of $sd(G)$, namely the *relative subgroup commutativity degree* of a subgroup of G, introduced and studied in [20]).

A remarkable modular sublattice of $L(G)$ is the normal subgroup lattice $N(G)$, which consists of all normal subgroups of G. Note that for an arbitrary finite group G computing the number of subgroups, as well as the number of normal subgroups, is a difficult work. These numbers are in general unknown, excepting for few particular classes of finite groups.

In the following we introduce the quantity

$$ndeg(G) = \frac{|N(G)|}{|L(G)|},$$

which will be called the *normality degree* of G. Clearly, it constitutes a significant probabilistic aspect on subgroup lattices of finite groups, by measuring the probability of a random subgroup of such a group to be normal. The normality degree is closely connected to a special type of an action of a finite group on a lattice, introduced and studied in [15]. Recall that, given a finite group G acting on a lattice $(L, \wedge, \vee)$, we say that L is a

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G-$lattice$ if the following two equalities hold

$$g \circ (l \wedge l') = (g \circ l) \wedge (g \circ l'),$$
$$g \circ (l \vee l') = (g \circ l) \vee (g \circ l'),$$

for all $g \in G$ and $l, l' \in L$, that is the action $\circ$ of G on L is compatible with the binary operations $\wedge$ and $\vee$ of L. For a finite G-lattice L, the set $Fix_G(L) = \{l \in L \mid g \circ l = l, \forall\, g \in G\}$ forms a G-sublattice of L and the quantity

$$(*) \qquad \frac{|Fix_G(L)|}{|L|}$$

measures the probability of an element of L to be fixed with respect to $\circ$. Moreover, if we assume that both the initial element and the final element of L are contained in $Fix_G(L)$, then the map $f_L : L \longrightarrow L$ defined by $f_L(l) = \bigwedge_{g \in G} g \circ l$, for any $l \in L$, is isotone. Therefore the set $Fix(f_L) = \{l \in L \mid f_L(l) = l\}$ is also a G-sublattice of L, according to the well-known fixed-point theorem of complete lattices. Again, a specific quantity associated to L, namely

$$(**) \qquad \frac{|Fix(f_L)|}{|L|}\,,$$

can be seen as a probabilistic aspect on L, more precisely it measures the probability of an element of L to be a fixed point of f_L. One of the most important examples of a G-lattice is constituted by the subgroup lattice $L(G)$ associated to G. In this case the action of G on $L(G)$ is defined by $g \circ H = H^g$, for all $(g, H) \in G \times L(G)$, and $f_{L(G)}$ maps every subgroup $H \in L(G)$ into its core in G. Then both G-sublattices $Fix_G(L(G))$ and $Fix(f_{L(G)})$ of $L(G)$ will coincide with the normal subgroup lattice $N(G)$. In other words, both quantities $(*)$ and $(**)$ are equal to the normality degree $ndeg(G)$ of G. Hence $ndeg(G)$ measures the probability of a random subgroup of G to be a fixed point of $L(G)$ relative to the above canonical action of G on $L(G)$, and also to be a fixed point of the map $f_{L(G)}$.

All the previous remarks give a strong motivation to study the normality degree of finite groups. In our paper a first step of this study is made.

The paper is organized as follows. Some basic properties and results on normality degree are presented in Section 2. Section 3 deals with normality degrees for two special classes of finite groups: semidirect products of finite cyclic groups and finite p-groups possessing a cyclic maximal subgroup. An interesting density result of normality degree is proved in Section 4. In the final section several conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on lattices (respectively on groups) can be found in [2] (respectively in [4] and [14]). For subgroup lattice concepts we refer the reader to [12], [15] and [16].

2. Basic properties of normality degree

Let G be a finite group. First of all, remark that the normality degree $ndeg(G)$ satisfies the following relation

$$0 < ndeg(G) \leq 1.$$

Moreover, we have $ndeg(G) = 1$ if and only if all subgroups of G are normal, that is G is a Dedekind group. As we have seen in [18], the normality degree and the subgroup commutativity degree of G are connected by the inequality

$$(2.1) \qquad ndeg(G) \leq sd(G).$$

Clearly, this becomes an equality if and only if for every subgroup H of G the set $C(H)$ consisting of all subgroups of G which commute with H coincides with $N(G)$. Since H itself is contained in $C(H)$, it must be normal and so G is a Dedekind group. Hence the following result holds.

Proposition 2.1. *Let G be a finite group. Then the following conditions are equivalent:*

a) $ndeg(G) = 1$.
b) $ndeg(G) = sd(G)$.
c) *G is a Dedekind group.*

Next, let $\mathcal{S}$ be a set of representatives for the conjugacy classes of subgroups of G with at least two elements. Then

$$|L(G)| = |N(G)| + \sum_{H \in \mathcal{S}} (G : N_G(H)),$$

which implies that

$$ndeg(G) = \frac{|N(G)|}{|N(G)| + \sum_{H \in \mathcal{S}} (G : N_G(H))} \tag{2.2}$$

This equality can be used to calculate the normality degree of finite groups whose conjugacy classes of subgroups are completely determined. The simplest examples are constituted by the symmetric groups S_3 and S_4, for which one obtains

$$ndeg(S_3) = \frac{1}{2} \text{ and } ndeg(S_4) = \frac{2}{15}\,.$$

In particular, if G is a finite p-group, (2) leads us to an inequality satisfied by $ndeg(G)$, namely

$$ndeg(G) \leq \frac{|N(G)|}{|N(G)| + p\,|\mathcal{S}|}\,.$$

In many situations computing the normality degree of a finite group is reduced to computing the number of all its subgroups. One of them is constituted by finite groups with few normal subgroups, as the symmetric groups.

Example 2.1. The following equality holds

$$ndeg(S_n) = \frac{3}{|L(S_n)|}\,, \text{ for all } n \geq 5.$$

Mention that we also have

$$ndeg(S_n \times S_n) = \frac{10}{|L(S_n \times S_n)|}\,, \text{ for all } n \geq 5,$$

and a formula for $ndeg(S_n^k)$ which depends only on $|L(S_n^k)|$ can be easily inferred, according to [10].

In the following assume that G and G' are two finite groups. It is obvious that if $G \cong G'$, then $ndeg(G) = ndeg(G')$. In particular, we infer that any two conjugate subgroups of a finite group have the same normality degree. The above conclusion is not true in the case when G and G' are only lattice-isomorphic, as shows the next example.

Example 2.2. Let G be the finite elementary abelian 3-group $\mathbb{Z}_3^n$ (where $n \geq 2$) and G' be a semidirect product of an elementary abelian normal subgroup A of order 3^{n-1} by the group $B \cong \mathbb{Z}_2$ which induces a nontrivial power automorphism on A. Then both G and G' are contained in the class $P(n, 3)$ (see [12], page 49) and so they are L-isomorphic. We have $ndeg(G) = 1$, because G is abelian. On the other hand, in Section 2 of [18] we have proved that $sd(G') < 1$. This implies that $ndeg(G') < 1$, in view of (1). Hence $ndeg(G) \neq ndeg(G')$.

By a direct calculation we obtain

$$ndeg(S_3 \times \mathbb{Z}_2) = \frac{7}{16} \neq \frac{1}{2} = ndeg(S_3)ndeg(\mathbb{Z}_2)$$

and therefore in general we don't have $ndeg(G \times G') = ndeg(G)ndeg(G')$. Clearly, a sufficient condition in order to this equality holds is

$$\gcd(|G|, |G'|) = 1,$$

that is G and G' are of coprime orders. This remark can be extended to an arbitrary finite direct product.

Proposition 2.2. *Let G_i, $i = 1, 2, ..., k$, be a family of finite groups having coprime orders. Then*

$$ndeg(\prod_{i=1}^{k} G_i) = \prod_{i=1}^{k} ndeg(G_i).$$

The following immediate consequence of Proposition 2.2 shows that computing the normality degree of a finite nilpotent group is reduced to finite p-groups.

Corollary 2.1. *If G is a finite nilpotent group and G_i, $i = 1, 2, ..., k$, are the Sylow subgroups of G, then*

$$ndeg(G) = \prod_{i=1}^{k} ndeg(G_i).$$

3. Normality degrees for some classes of finite groups

In this section we determine explicitly the normality degree of several finite groups. The most significant results are obtained for the class of finite dihedral groups and for the class of finite p-groups possessing a cyclic maximal subgroup.

3.1. **The normality degree of some semidirect products of finite groups.** Let p be a prime, $n \geq 2$ be an integer such that $p \nmid n$ and $f : \mathbb{Z}_p \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ be a group homomorphism. Put $\hat{k}_0 = f(\bar{1})(\hat{1})$ and suppose that $k_0 \neq 1$. Then we have $\gcd(k_0, n) = 1$ and

$$f(\bar{x})(\hat{y}) = k_0^x \hat{y}, \text{ for any } \bar{x} \in \mathbb{Z}_p,\ \hat{y} \in \mathbb{Z}_n.$$

Denote by G be the semidirect product of $\mathbb{Z}_p$ and $\mathbb{Z}_n$ with respect to f. Recall that the operation of G is defined by

$$(\bar{x}_1, \hat{y}_1) \cdot (\bar{x}_2, \hat{y}_2) = (\bar{x}_1 + \bar{x}_2, k_0^{x_2} \hat{y}_1 + \hat{y}_2), \text{ for all } (\bar{x}_1, \hat{y}_1), (\bar{x}_2, \hat{y}_2) \in G.$$

It is well-known that the maps

$$\sigma_1 : \mathbb{Z}_p \longrightarrow G, \quad \sigma_1(\bar{x}) = (\bar{x}, \hat{0}), \quad \text{for any } \bar{x} \in \mathbb{Z}_p,$$
$$\sigma_2 : \mathbb{Z}_n \longrightarrow G, \quad \sigma_2(\hat{y}) = (\bar{0}, \hat{y}), \quad \text{for any } \hat{y} \in \mathbb{Z}_n,$$

are injective group homomorphisms. Moreover, if $H = \sigma_1(\mathbb{Z}_p)$ and $K = \sigma_2(\mathbb{Z}_n)$, then H is a subgroup of G and K is a normal subgroup of G, which satisfy

$$G = HK,\ H \cap K = \{(\bar{0}, \hat{0})\}.$$

In the following our goal is to compute explicitly the normality degree of G. First of all, we shall give a precise description of $L(G)$ (for more details, see Section 3.2 of [15]). Let G_1 be a subgroup of G. Then $\mid G_1 \mid$ is a divisor of pn.

In the case when $p \nmid |G_1|$ we shall prove that $G_1 \subseteq K$. Indeed, if we assume that $G_1 \nsubseteq K$, then we have $K \subset G_1K \subseteq G$ and so the index $(G_1K : K)$ of K in G_1K is ≥ 2. Since $p = (G : K) = (G : G_1K)(G_1K : K)$ is prime, one obtains $(G_1K : K) = p$ and $(G : G_1K) = 1$, i.e. $G_1K = G$. It results

$$G_1/G_1 \cap K \cong G_1K/K = G/K,$$

which shows that $|G_1/G_1 \cap K| = p$ and therefore $p \mid |G_1|$, a contradiction. Hence

$$G_1 \in L(K) = L(\sigma_2(\mathbb{Z}_n)) = \sigma_2(L(\mathbb{Z}_n)). \tag{3.3}$$

In the case when $p \mid |G_1|$ at least a subgroup of order p is contained in G_1. Let $\{H_1 = H, H_2, ..., H_{n_p}\}$ be the set of all Sylow p-subgroups of G, where $n_p = (G : N_G(H))$. By a direct calculation, the normalizer $N_G(H)$ of H in G can be easily determined.

Lemma 3.1. *The following equality holds*

$$N_G(H) = \{\ (\bar{x}, \hat{y}) \in G \mid \bar{x} \in \mathbb{Z}_p,\ \hat{y} \in \langle \frac{\hat{n}}{d} \rangle\},$$

where $d = \gcd(k_0 - 1, n)$.

Then $n_p = \dfrac{n}{d} = \dfrac{n}{\gcd(k_0 - 1, n)}$ *. For every* $i \in \{1, 2, ..., n_p\}$ *there exists* $z_i \in G$ $(z_1 = (\bar{0}, \hat{0}))$ *such that* $H_i = H^{z_i}$*. One obtains*

$$G = G^{z_i} = (HK)^{z_i} = H^{z_i}K^{z_i} = H_iK,\ i = 1, 2, ..., n_p\,.$$

Suppose that $H_i \subseteq G_1$ *for some* $i \in \{1, 2, ..., n_p\}$*. It results* $G_1 = G_1 \cap G = G_1 \cap (H_iK) = H_i(G_1 \cap K)$ *and thus* G_1 *is contained in the set*

$$\mathcal{A} = \{H^{z_i}\sigma_2(\langle \frac{\hat{n}}{k} \rangle) \mid k|n,\ i = \overline{1, n_p}\} = \{(H\sigma_2(\langle \frac{\hat{n}}{k} \rangle))^{z_i} \mid k|n,\ i = \overline{1, n_p}\}. \tag{3.4}$$

In order to determine the number of elements of $\mathcal{A}$*, we need to compute the normalizer in G of such an element.*

Lemma 3.2. *If k is a divisor of n, then*

$$N_G(H\sigma_2(\langle \frac{\hat{n}}{k} \rangle)) = \{\ (\bar{x}, \hat{y}) \in G \mid \bar{x} \in \mathbb{Z}_p,\ \hat{y} \in \langle \frac{\hat{n}}{\varepsilon(k)} \rangle\},$$

where $\varepsilon(k) = \gcd(k(k_0 - 1), n)$.

From Lemma 3.2 we easily infer that

$$|\mathcal{A}| = \sum_{k|n} \frac{\varepsilon(k)}{\gcd(k_0 - 1, n)} = \sum_{k|n} \gcd(k, \frac{n}{\gcd(k_0 - 1, n)}). \tag{3.5}$$

Now, by using the relations (3.3), (3.4) *and* (3.5)*, we are able to describe the subgroup structure of G.*

Proposition 3.3. *The subgroup lattice $L(G)$ of the above semidirect product G is given by the following equality:*

$$L(G) = \sigma_2(L(\mathbb{Z}_n)) \cup \mathcal{A}.$$

Moreover, the total number of subgroups of G is

$$|L(G)| = \tau(n) + \sum_{k|n} \gcd(k, \frac{n}{\gcd(k_0-1, n)}),$$

where $\tau(n)$ denotes the number of all divisors of n.

Clearly, the normal subgroups of G are all subgroups contained in K and G itself, that is

$$N(G) = \sigma_2(L(\mathbb{Z}_n)) \cup \{G\},$$

and therefore

$$|N(G)| = \tau(n) + 1.$$

Hence we have proved the following theorem.

Theorem 3.1. *The normality degree of the above semidirect product G is given by the following equality:*

$$ndeg(G) = \frac{\tau(n)+1}{\tau(n) + \sum_{k|n} \gcd(k, \frac{n}{\gcd(k_0-1, n)})}. \tag{3.6}$$

Remark 3.1. Let $r = \frac{n}{\gcd(k_0-1, n)}$. Then $1 \leq (k, r) \leq k, r$, for all divisors k of n. So, by (3.6) we infer that $ndeg(G)$ satisfies the following inequalities:

$$ndeg(G) \leq \frac{\tau(n)+1}{2\tau(n)}, \tag{3.7}$$

$$ndeg(G) \geq \frac{\tau(n)+1}{\tau(n)+\sigma(n)}, \tag{3.8}$$

$$ndeg(G) \geq \frac{\tau(n)+1}{\tau(n)(r+1)} > \frac{1}{r+1}. \tag{3.9}$$

Next, let us assume that $p = 2$ and $k_0 = n - 1$. Then the group G studied above is the dihedral group D_{2n}. Recall that D_{2n} is the symmetry group of a regular polygon with n sides and it has the order $2n$. The most convenient abstract description of D_{2n} is obtained by using its generators: a rotation x of order n and a reflection y of order 2. Under these notations, we have

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1} \rangle.$$

Since n is odd, it results $\gcd(k_0 - 1, n) = \gcd(n-2, n) = 1$ and so

$$\sum_{k|n} \gcd(k, \frac{n}{\gcd(k_0-1, n)}) = \sum_{k|n} \gcd(k, n) = \sigma(n),$$

where $\sigma(n)$ denotes the sum of all divisors of n. Thus, (3.6) leads us to

$$ndeg(D_{2n}) = \frac{\tau(n)+1}{\tau(n)+\sigma(n)}, \tag{3.10}$$

that is (3.8) becomes an equality for $G = D_{2n}$ with n odd.

A similar formula can be also obtained for even positive integers n. In this case it is well-known that we have

$$N(D_{2n}) = L(\langle x\rangle) \cup \{D_{2n}, \langle x^2, y\rangle, \langle x^2, xy\rangle\}$$

and therefore

$$ndeg(D_{2n}) = \frac{\tau(n)+3}{\tau(n)+\sigma(n)} \tag{3.11}$$

Hence (3.10) and (3.11) imply the following result.

Corollary 3.2. *The normality degree of the dihedral group D_{2n} is given by the following equality:*

$$ndeg(D_{2n}) = \begin{cases} \dfrac{\tau(n)+1}{\tau(n)+\sigma(n)}, & \text{if } n \text{ is odd} \\[2ex] \dfrac{\tau(n)+3}{\tau(n)+\sigma(n)}, & \text{if } n \text{ is even} \end{cases} \tag{3.12}$$

Remark 3.2. A simple arithmetic exercise shows that $\tau(n)+2 \le \sigma(n)$, for all odd positive integers $n \neq 1$, and $\tau(n)+6 \le \sigma(n)$, for all even positive integers $n \neq 2, 4$. These inequalities give us an upper bound for the normality degree of D_{2n}, namely

$$ndeg(D_{2n}) \le \frac{1}{2}\,,$$

for all $n \neq 2, 4$. Mention also that we have $ndeg(D_{2n}) = \dfrac{1}{2}$ if and only if $n = 3$, that is in the class of finite dihedral groups only D_6 (which is isomorphic to S_3) has the normality degree $\dfrac{1}{2}$.

In the end of this subsection, we note that the normality degrees of other semidirect products of finite groups can be also computed. Such an example is constituted by ZM-groups, that is finite groups with all Sylow subgroups cyclic. It is well-known (see [4], I) that a ZM-group possesses a presentation of type

$$\mathrm{ZM}(m,n,r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r\rangle\,,$$

where the triple (m, n, r) satisfies the conditions

$$\gcd(m,n) = \gcd(m, r-1) = 1 \text{ and } r^n \equiv 1 \pmod m.$$

The subgroups of $\mathrm{ZM}(m,n,r)$ have been computed in [1]:

$$|L(\mathrm{ZM}(m,n,r))| = \sum_{m_1|m}\sum_{n_1|n} \gcd(m_1, \frac{r^n-1}{r^{n_1}-1})\,,$$

while the number of normal subgroups of $\mathrm{ZM}(m,n,r)$ has been determined in [21]:

$$|N(\mathrm{ZM}(m,n,r))| = \sum_{n_1|n} \tau(\gcd(m, r^{n_1}-1))\,.$$

In this way, one obtains

$$ndeg(\mathrm{ZM}(m,n,r)) = \frac{\displaystyle\sum_{n_1|n} \tau(\gcd(m, r^{n_1}-1))}{\displaystyle\sum_{m_1|m}\sum_{n_1|n} \gcd(m_1, \frac{r^n-1}{r^{n_1}-1})}\,. \tag{3.13}$$

Finally, remark that, by taking $n = 2$ and $r = m - 1$ with m odd in (3.13), the previous formula(3.10) is obtained. This is not a surprise, according to the group isomorphism $\mathrm{ZM}(m, 2, m-1) \cong D_{2m}$.

3.2. **The normality degree of finite p-groups possessing a cyclic maximal subgroup.** Let p be a prime, $n \geq 3$ be an integer and denote by $\mathcal{G}$ the class consisting of all finite p-groups of order p^n having a maximal subgroup which is cyclic. Obviously, $\mathcal{G}$ contains finite abelian p-groups of type $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ whose normality degree is 1, but some finite nonabelian p-groups belong to $\mathcal{G}$, too. They are exhaustively described in Theorem 4.1, [14], II: a nonabelian group is contained in $\mathcal{G}$ if and only if it is isomorphic to

- $M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ y^{-1}xy = x^{p^{n-2}+1} \rangle$

when p is odd, or to one of the following groups

- $M(2^n)$ $(n \geq 4)$,
- the dihedral group D_{2^n},
- the generalized quaternion group

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1,\ yxy^{-1} = x^{2^{n-1}-1} \rangle,$$

- the quasi-dihedral group

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{n-2}-1} \rangle\ (n \geq 4)$$

when $p = 2$.

In the following the normality degrees of the above p-groups will be determined. We recall first the explicit formulas for the total number of subgroups of these groups, found in [18].

Lemma 3.3. *The following equalities hold:*

a) $|L(M(p^n))| = (1+p)n + 1 - p\,,$
b) $|L(D_{2^n})| = 2^n + n - 1\,,$
c) $|L(Q_{2^n})| = 2^{n-1} + n - 1\,,$
d) $|L(S_{2^n})| = 3 \cdot 2^{n-2} + n - 1\,.$

In order to compute the normality degree of the nonabelian p-groups that belong to $\mathcal{G}$, we need to know the number of their normal subgroups. Our reasoning is founded on the following simple remark: such a group G possesses a unique normal subgroup of order p, namely $\langle x^q \rangle$ (where $q = p^{n-2}$ and x denotes a generator of a cyclic maximal subgroup of G). We infer that there exists a bijection between the set of nontrivial normal subgroups of G and $N(G/\langle x^q \rangle)$, that is

$$|N(G)| = 1 + |N(G/\langle x^q \rangle)|\,. \tag{3.14}$$

For $G = M(p^n)$, the minimal normal subgroup $\langle x^q \rangle$ is in fact the commutator subgroup $D(M(p^n))$ of $M(p^n)$ and we have

$$M(p^n)/D(M(p^n)) \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}\ .$$

Since $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ is abelian, the number of its normal subgroups coincides with the number of all its subgroups. Put $x_n = |L(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}})|$. This number can be easily determined by using the following lemma, established in [17] *(see also* [19]*).*

Lemma 3.4. *For every $0 \le \alpha \le \alpha_1 + \alpha_2$, the number of all subgroups of order $p^{\alpha_1+\alpha_2-\alpha}$ in the finite abelian p-group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ $(\alpha_1 \le \alpha_2)$ is*

$$\begin{cases} \dfrac{p^{\alpha+1}-1}{p-1}\,, & \text{if} \quad 0 \le \alpha \le \alpha_1 \\ \dfrac{p^{\alpha_1+1}-1}{p-1}\,, & \text{if} \quad \alpha_1 \le \alpha \le \alpha_2 \\ \dfrac{p^{\alpha_1+\alpha_2-\alpha+1}-1}{p-1}\,, & \text{if} \quad \alpha_2 \le \alpha \le \alpha_1+\alpha_2. \end{cases}$$

In particular, the total number of subgroups of $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ is

$$\frac{1}{(p-1)^2}\left[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+1}-(\alpha_1+\alpha_2+3)p+(\alpha_1+\alpha_2+1)\right].$$

By taking $\alpha_1 = 1$ and $\alpha_2 = n-2$, one obtains

$$x_n = \frac{1}{(p-1)^2}\left[(n-2)p^3-(n-4)p^2-(n+2)p+n\right] = (1+p)n-2p\,.$$

So, (3.14) *becomes*

$$|N(M(p^n))| = 1 + x_n = (1+p)n+1-2p\,.$$

For every $G \in \{D_{2^n}, Q_{2^n}, S_{2^n}\}$ the minimal normal subgroup $\langle x^q \rangle$ coincides with the center $Z(G)$ of G and we have

$$G/Z(G) \cong D_{2^{n-1}},$$

therefore

$$|N(G)| = 1 + |N(D_{2^{n-1}})|.$$

Let $G = D_{2^{n-1}}$ in the above equality and set $y_n = |N(D_{2^{n-1}})|$. Then the integer sequence $(y_n)_{n\in\mathbb{N}^*}$ satisfies the recurrence relation $y_n = 1+y_{n-1}$, which shows that $y_n = n+3$, for any $n \in \mathbb{N}^*$. *Thus*

$$|N(G)| = 1 + y_{n-1} = y_n = n+3\,,$$

for all above 2-groups G. Hence we have proved the following result.

Lemma 3.5. *The following equalities hold:*

a) $|N(M(p^n))| = (1+p)n+1-2p\,,$
b) $|N(G)| = n+3\,$, *for all* $G \in \{D_{2^n}, Q_{2^n}, S_{2^n}\}$.

Now, it is clear that Lemmas 3.3 and 3.5 imply the next theorem.

Theorem 3.2. *The normality degrees of the nonabelian groups in the class $\mathcal{G}$ are given by the following equalities:*

1) $ndeg(M(p^n)) = \dfrac{(1+p)n+1-2p}{(1+p)n+1-p}\,,$
2) $ndeg(D_{2^n}) = \dfrac{n+3}{2^n+n-1}\,,$
3) $ndeg(Q_{2^n}) = \dfrac{n+3}{2^{n-1}+n-1}\,,$
4) $ndeg(S_{2^n}) = \dfrac{n+3}{3\cdot 2^{n-2}+n-1}\,.$

Remark 3.3. The normality degree of the dihedral group D_{2^n} can also be directly computed by using Corollary 3.2:

$$ndeg(D_{2^n}) = \frac{\tau(2^{n-1})+3}{\tau(2^{n-1})+\sigma(2^{n-1})} = \frac{n+3}{2^n+n-1}\ .$$

The following limits are immediate from Theorem 3.2

Corollary 3.3. *We have:*

a) $\lim\limits_{n\to\infty} ndeg(M(p^n)) = 1$, *for any fixed prime* p.
b) $\lim\limits_{n\to\infty} ndeg(G) = 0$, *for all* $G \in \{D_{2^n}, Q_{2^n}, S_{2^n}\}$.

We end this section by mentioning that the normality degree of any finite nilpotent group whose Sylow subgroups belong to $\mathcal{G}$ can explicitly be calculated, in view of Corollary 2.5.

4. A density result of normality degree

As we have seen in Section 3, there are some sequences of finite groups $(G_n)_{n\in\mathbb{N}}$ satisfying $\lim\limits_{n\to\infty} ndeg(G_n) \in \{0,1\}$. In this section our purpose is to extend this result by proving that each x in the interval [0,1] is the limit of a certain sequence of normality degrees of finite groups.

First of all, we shall prove the above result for rational numbers in [0,1].

Theorem 4.3. *For every* $x \in [0,1]\cap\mathbb{Q}$ *there exists a sequence* $(G_n)_{n\in\mathbb{N}}$ *of finite groups such that* $\lim\limits_{n\to\infty} ndeg(G_n) = x$.

Proof. For $x = 0$ and $x = 1$ our statement is already verified in the previous section, by taking $G_n = D_{2^n}$ (or $G_n = Q_{2^n}$, or $G_n = S_{2^n}$) and $G_n = M(p^n)$, respectively. Let $x \in (0,1)\cap\mathbb{Q}$. Then $x = \dfrac{a}{b}$, where $a, b \in \mathbb{N}^*$ and $a < b$. Denote by $(p_n)_{n\in\mathbb{N}}$ the sequence of the prime numbers and choose the disjoint strictly increasing subsequences $(k_n^1), (k_n^2), ..., (k_n^{b-a})$ of $\mathbb{N}$. We also consider $G_i = M(p_{k_n^i}^{a+i+1})$, $i = 1, 2, ..., b-a$. Then the normality degree of G_i is given by

$$ndeg(G_i) = \frac{(a+i-1)p_{k_n^i} + a+i+2}{(a+i)p_{k_n^i} + a+i+2}$$

and we have

$$\lim_{n\to\infty} ndeg(G_i) = \frac{a+i-1}{a+i}\ ,$$

for all $i = \overline{1, b-a}$. Let $G = \prod_{i=1}^{b-a} G_i$. From Corollary 2.5 it results

$$ndeg(G) = \prod_{i=1}^{b-a} ndeg(G_i).$$

Hence

$$\lim_{n\to\infty} ndeg(G_n) = \prod_{i=1}^{b-a} \lim_{n\to\infty} ndeg(G_i) = \prod_{i=1}^{b-a} \frac{a+i-1}{a+i} = \frac{a}{b} = x,$$

which completes our proof. □

Since the set $[0,1]\cap\mathbb{Q}$ is dense in [0,1], by Theorem 5.1 we infer the following corollary.

Corollary 4.4. *For every $x \in [0,1]$ there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of finite groups such that* $\lim_{n\to\infty} ndeg(G_n) = x$.

Let $a, b \in \mathbb{N}^*$ with $a < b$. In general, there is no finite group G satisfying both equalities

$$|N(G)| = a \text{ and } |L(G)| = b.$$

The above system has no solution G even in the particular case when $b = a+1$. In contrast with this statement, for several values of a we are able to construct finite groups G such that

$$ndeg(G) = \frac{a}{a+1}\,.$$

For example, we have $ndeg(S_3) = \dfrac{1}{2}$ and $ndeg(M(5^4)) = \dfrac{3}{4}$ (more generally, a fraction $\dfrac{a}{a+1}$ is the normality degree of a finite p-group of type $M(p^n)$ if and only if there is a prime q such that $q+1$ divides $a+3$). Inspired by these examples, we came up with the following conjecture.

Corollary 4.5. *For every $a \in \mathbb{N}^*$ there exists a finite group G such that* $ndeg(G) = \dfrac{a}{a+1}$.

Finally, notice that it is natural to generalize Conjecture 5.3 in the following manner.

Conjecture 4.1. *For every $x \in (0,1] \cap \mathbb{Q}$ there exists a finite group G such that $ndeg(G) = x$.*

5. Conclusions and further research

All our previous results show that the concept of normality degree introduced in this paper can constitute a significant aspect of probabilistic finite group theory. It is clear that the study started here can successfully be extended to other classes of finite groups. This will surely be the subject of some further research.

Two interesting conjectures on normality degree have been formulated in Section 4. Another open problems concerning this topic are the following:

Problem 5.1. Given a finite group G, a subgroup H of G and a normal subgroup N of G, which is the connection between $ndeg(G)$ and $ndeg(H)$, respectively between $ndeg(G)$ and $ndeg(G/N)$?

Problem 5.2. Give explicit formulas for the normality degrees of other classes of finite groups.

Problem 5.3. For a fixed $a \in (0,1)$, describe the structure of finite groups G satisfying

$$ndeg(G) = (\leq,\ \geq)\ a.$$

Problem 5.4. Study the properties of the map $ndeg$ from the class of finite groups to [0,1]. What can be said about two finite groups having the same normality degree?

Problem 5.5. As we have seen in Corollary 3.3, the following equalities hold

$$\lim_{n\to\infty} ndeg(D_{2^{n-1}}) = \lim_{n\to\infty} ndeg(Q_{2^{n-1}}) = \lim_{n\to\infty} ndeg(S_{2^{n-1}}) = 0.$$

Is this true for other "natural" classes of finite groups?

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