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Extending the applicability of Newton's method using Wang's– Smale's α–theory

IOANNIS K. ARGYROS and SANTHOSH GEORGE

ABSTRACT. We improve semilocal convergence results for Newton's method by defining a more precise domain where the Newton iterate lies than in earlier studies using the Smale's α– theory. These improvements are obtained under the same computational cost. Numerical examples are also presented in this study to show that the earlier results cannot apply but the new results can apply to solve equations.

1. INTRODUCTION

Let B_1, B_2 be Banach spaces and $L(B_1, B_2)$ be the space of all bounded linear operators from B_1 to B_2. Throughout this paper $U(x,r)$ and $\bar{U}(x,r)$ stand, respectively for the open and closed balls in B_1 with center x and radius $r > 0$. In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0,$$

where F is a Fréchet-differentiable operator defined on a convex subset Ω of B_1.

Newton's method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \qquad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where x_0 is an initial point is undoubtedly the most popular iterative process for generating a sequence $\{x_n\}$ approximating x^*. Here, $F'(x)$ denotes the Fréchet-derivative of F for each $x \in \Omega$.

In this paper we extend the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition 2.5) for some $\gamma_0 \le \gamma$. Moreover, we define a more precise domain where the iterates lie. This way we obtain tighter upper bounds on the norms of $\| F'(x)^{-1} F'(x_0) \|$ for each $x \in \overline{U}(x_0, R)$ (see (2.7) and (2.8)) leading to weaker sufficient convergence conditions and a tighter convergence analysis than in earlier studies such as [5, 9, 10]. The approach of introducing center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [2, 3, 7].

Let $\gamma > 0$ be a parameter $x_0 \in B_1$ and $\Omega \subseteq B_1$. Wang in his work [10] on approximate zeros of Smale (cf. [9]) used the γ-Lipschitz condition at x_0

$$\| F'(x_0)^{-1} F''(x) \| \le \frac{2\gamma}{(1 - \gamma \| x - x_0 \|)^3} \ \text{ for each } x \in \Omega, \tag{1.2}$$

where $F'(x_0)^{-1} \in L(B_2, B_1)$ to show the following semi-local convergence result for Newton's method.

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Corresponding author: Ioannis K. Argyros; iargyros@cameron.edu

The most famous semi-local convergence result is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [2, 3]. Another important theorem inaugurated by Smale at the International Conference of Mathematics (cf. [9]), where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for analytic function, depending on the information at the initial point. Wang [10] generalized Smale's result by introducing the γ-condition (see (1.2)). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [6, Chapter 3.3] (see also [1, 8, 9, 10]).

Theorem 1.1. *Let $F : \Omega \subseteq B_1 \longrightarrow B_2$ be a twice Fréchet-differentiable operator. Suppose condition (1.2) holds,*

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta \text{ for some } \eta \ge 0,$$

and for

$$\alpha = \gamma\eta$$

$$\alpha \le 3 - 2\sqrt{2}. \tag{1.3}$$

Then, sequence $\{x_n\}$ generated by Newton's method (1.1) is well defined, remains in $U(x_0, t^*)$ for each $n = 0, 1, \ldots$ and converges to a unique solution $x^* \in \bar{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover, the following error estimates hold:

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le \frac{1}{(1 - \gamma\|x - x_0\|)^2} - 1, \tag{1.4}$$

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n$$

and

$$\|x_{n+1} - x^*\| \le t^* - t_n, \; t^* = \lim_{n\longrightarrow\infty} t_n,$$

where

$$t^* = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \le (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}$$

and the scalar sequence $\{t_n\}$ is defi ned by

$$\begin{aligned} t_0 = 0,\, t_1 &= \eta, \\ t_{n+1} &= t_n + \frac{\gamma(t_n - t_{n-1})^2}{(2 - \frac{1}{(1-\gamma t_n)^2})(1 - \gamma t_n)(1 - \gamma t_{n-1})^2} \\ &= t_n - \frac{\varphi(t_n)}{\varphi'(t_n)} \end{aligned}$$

for each $n = 1, 2, \ldots,$ where

$$\varphi(t) = \eta - t + \frac{\gamma t^2}{1 - \gamma t}.$$

The point t^* is the smallest positive zero of function $\varphi(t)$ which exists under the hypothesis (1.3). Moreover, it is worth noticing that condition (1.2) implies (1.4) but not necessarily vice versa, even if F is a twice Fréchet differentiable operator.

2. SEMI-LOCAL CONVERGENCE OF NEWTON'S METHOD

We need the definitions:

Definition 2.1. Let $F : \Omega \longrightarrow B_2$ be a twice Fréchet-differentiable operator. Operator F satisfies the center γ_0-Lipschitz condition at x_0 if for each $x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \frac{1}{(1-\gamma_0\|x-x_0\|)^2} - 1. \tag{2.5}$$

In view of (2.5) and the choice of γ_0, we have that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \frac{1}{(1-\gamma_0\|x-x_0\|)^2} - 1 < 1. \tag{2.6}$$

Then, by (2.6) and the Banach lemma on invertible operators [2, 3, 8] $F'(x_0)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(x_0)\| \leq (2 - \frac{1}{(1-\gamma_0\|x-x_0\|)^2})^{-1}. \tag{2.7}$$

The corresponding result using (1.4) is

$$\|F'(x)^{-1}F'(x_0)\| \leq (2 - \frac{1}{(1-\gamma\|x-x_0\|)^2})^{-1}. \tag{2.8}$$

Estimate (2.7) is more precise than (2.8), since $\gamma_0 \leq \gamma$ leading to a more precise majorizing sequence which in turn leads to the advantages already stated. Set $\Omega_0 = U(x_0, (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma_0})$.

Definition 2.2. Let $F : \Omega \longrightarrow B_2$ be a twice Fréchet-differentiable operator. Operator F satisfies the β-Lipschitz condition at x_0 on $\Omega \cap \Omega_0$, if

$$\|F'(x_0)^{-1}F''(x)\| \leq \frac{2\beta}{(1-\beta\|x-x_0\|)^3} \tag{2.9}$$

for each $x \in \Omega \cap \Omega_0$.

Clearly, we have that

$$\beta \leq \gamma,$$

since $\Omega \cap \Omega_0 \subseteq \Omega$. Define scalar sequences $\{r_n\}$ and $\{s_n\}$ by

$$r_0 = s_0 = 0, \;\; r_1 = s_1 = \eta,$$
$$r_{n+1} = r_n + \frac{\beta(r_n - r_{n-1})^2}{(2 - \frac{1}{(1-\gamma_0 r_n)^2})(1-\beta r_n)(1-\beta r_{n-1})^2},$$
$$s_{n+1} = s_n + \frac{\beta(s_n - s_{n-1})^2}{(2 - \frac{1}{(1-\gamma s_n)^2})(1-\beta s_n)(1-\beta s_{n-1})^2}.$$

Then, we can show the following semilocal convergence result for Newton's method (1.1).

Theorem 2.2. *Let $F : \Omega \longrightarrow B_2$ be a twice-Fréchet differentiable operator and let x_0 be such that $F'(x_0)^{-1} \in L(B_2, B_1)$. Suppose operator F is center γ_0-Lipschitz on Ω_0 and β-Lipschitz on $\Omega \cap \Omega_0$ with $\gamma_0 \leq \beta$,*

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$\alpha_0 = \beta\gamma \leq 3 - 2\sqrt{2}. \tag{2.10}$$

Then, sequence $\{x_n\}$ generated by Newton's method (1.1) is well defined, remains in $U(x_0, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^* \in \bar{U}(x_0, r^*)$ of equation $F(x) = 0$, where

$$r^* = \frac{1+\alpha_0 - \sqrt{(1+\alpha_0)^2 - 8\alpha_0}}{4\beta} \leq (1 - \frac{1}{\sqrt{2}})\frac{1}{\beta}.$$

Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n$$

and

$$\|x^* - x_n\| \leq r^* - r_n, \; r^* = \lim_{n \longrightarrow \infty} r_n.$$

Proof. Follow the proof of Wang's theorem in [10] and simply notice that the iterates x_n lie in $\Omega \cap \Omega_0$ which is a more precise location than Ω used in [10]. □

A simple inductive argument leads to:

Proposition 2.1. *Suppose that the hypotheses of Theorem 1.1 and Theorem 2.2 hold. Then, the following hold*

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n \leq s_{n+1} - s_n \leq t_{n+1} - t_n \tag{2.11}$$

$$r^* = \lim_{n \longrightarrow \infty} r_n \leq s^* \leq t^*$$

and

$$\alpha \leq 3 - 2\sqrt{2} \Longrightarrow \alpha_0 \leq 3 - 2\sqrt{2}$$

but not necessarily vice versa, unless, if $\beta = \gamma$.

Proposition 2.1 justifies the claim about the advantages made at the introduction.

Notice also that sequence $\{s_n\}$ converges to s^* under condition (2.9) but not necessarily under (2.10). Moreover, the local convergence given in [4] which improved the local convergence in [4, 10] can also be improved using the new idea. Indeed, by replacing x_0 by x^*, we get that the convergence radii R_0 and R^* given, respectively by Theorem 1.1 and Theorem 2.2 are $R_0 = (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}$ and $R_1 = (1 - \frac{1}{\sqrt{2}})\frac{1}{\beta}$ so $R_0 \leq R_1$, since $\beta \leq \gamma$.

Next, we present an example involving a nonlinear integral equation of Chandrasekhar-type [3].

Example 2.1. Let $B_1 = B_2 = C[0, 1]$ be equipped with the max-norm. Let $\Omega = U(0, r)$ for some $r > 2$. Define F on Ω by

$$F(x)(s) = x(s) - y(s) - \lambda \int_0^1 k(s, t)x^3(t)dt \text{ for each } x \in C[0, 1] \text{ and each } s \in [0, 1],$$

where $y \in C[0, 1]$ is given, λ is a real parameter and the kernel k is the Green's function defined by

$$k(s, t) = \begin{cases} t(1 - s), & t \leq s \\ s(1 - t), & s < t. \end{cases}$$

Then, the Fréchet-derivative of F is defined by

$$(F'(x)(w))(s) = w(s) - 3\lambda \int_0^1 k(s, t)x^2(t)w(t)dt \text{ for each } w \in C[0, 1] \text{ and each } s \in [0, 1].$$

Let us choose in particular, $x_0(s) = y(s) = 1$ and $|\lambda| < \frac{8}{3}$. Then, we have that (see e.g. [3, Chapter 1]) $\|I - F'(x_0)\| < \frac{3}{8}|\mu|$, $F'(x_0)^{-1} \in L(B_2, B_1)$, $\|F'(x_0)^{-1}\| \leq \frac{8}{8-3|\lambda|}$, $\eta = \frac{|\lambda|}{8-3|\lambda|}$, $\beta = \gamma_0 = \frac{12|\lambda|}{8-3|\lambda|}$ and $\gamma = \frac{6r|\lambda|}{8-3|\lambda|}$. Notice that

$$\gamma_0 < \gamma, \tag{2.12}$$

since $r > 2$. Therefore, in view of (2.12) our results in this paper improves the corresponding ones using (1.3) [9, 10].

Example 2.2. Let $B_1 = B_2 = \mathbb{R}, p \in (0,1), x_0 = 1$, $\Omega = U(x_0, \frac{1}{2-p})$ and define function on Ω by

$$F(x) = x^3 - p.$$

Define $\Omega^* = U(x_0, 1-p)$. Then, we have

$$\Omega^* \subseteq \Omega, \text{ if } \ p \in [0.381966, 1).$$

We restrict function F on Ω^*. Let $L_0 = 3-p$ and $L = 2(2-p)$. Then, Argyros showed in [2, 3] that for each $x, y \in \Omega^*$

$$|F'(x_0)^{-1}(F'(x) - F'(x_0))| \leq L_0|x - x_0| \tag{2.13}$$

$$|F'(x_0)^{-1}(F'(x) - F'(y))| \leq L|x - y|. \tag{2.14}$$

In view of (1.2) and (2.14), we have $L \leq 2\gamma$, so we choose $\gamma = 2-p$. Then, since $\eta = \frac{1}{3}(1-p)$, condition (2.9) is satisfied, if

$$0.6255179 \leq p < 1. \tag{2.15}$$

We must have

$$U(x_0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}) \subseteq U(x_0, 1-p),$$

which is true for

$$0 < p \leq 0.7631871. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$0.6255179 < p \leq 0.7631871. \tag{2.17}$$

Set $y = \gamma_0|x - x_0|$ and $L_0 = d\gamma_0,\ d > 0, \gamma_0 > 0$. Using (2.5) and (2.13), we must have

$$L_0|x - x_0| \leq \frac{1}{(1 - \gamma_0|x - x_0|)^2} - 1$$

or

$$d(1-y)^2 \leq 2 - y$$

or

$$dy^2 + (1-2d)y + d - 2 \leq 0. \tag{2.18}$$

Let e.g. $d = 2$, then $\gamma_0 = \frac{L_0}{2} = \frac{e-1}{2}$ and (2.18) becomes $(p-3)(p-1) \leq 3$ or $p(p-4) \leq 0$, which is true. We must show $(1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0} \leq 1 - p$ or $p^2 - 4p + 1 + \sqrt{2} \geq 0$, which is true for

$$0 < p \leq 0.7407199. \tag{2.19}$$

Notice that $\Omega_0 \subset \Omega$, since $(1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0} < \frac{1}{\gamma}$ or $p \leq 3 + \sqrt{2}$, which is true, so

$$\Omega \cap \Omega_0 = \Omega_0.$$

Then, for $x \in \Omega_0$

$$\begin{aligned} |F'(x_0)^{-1}F''(x)| &= 2|x| \leq 2(|x - x_0| + |x_0|) \\ &\leq 2((1 - \frac{1}{\sqrt{2}})\frac{2}{3-p} + 1) \end{aligned}$$

must be smaller than 2β, so we can choose

$$\beta = 1 + (1 - \frac{1}{\sqrt{2}})\frac{2}{3-p} = 1 + \frac{2 - \sqrt{2}}{3-p}.$$

Notice that $\beta < \gamma$, if (2.19) holds. We also have that $\gamma_0 < \beta$, if

$$\frac{3-p}{2} < 1 + \frac{2-\sqrt{2}}{3-p}$$

or if

$$p^2 - 4p - 1 + 2\sqrt{2} < 0$$

or, if

$$0.5263741 < p < 1. \tag{2.20}$$

We also must have

$$(1 - \frac{1}{\sqrt{2}})\frac{1}{\beta} \leq 1 - p$$

or

$$2p^2 + (\sqrt{2} - 10)p + 4 + \sqrt{2} \leq 0,$$

which is true for

$$p \leq 0.767996. \tag{2.21}$$

Then, notice that

$$1 - p \leq \frac{1}{\gamma},$$

if $p^2 - 3p + 1 \leq 0$, which is true for

$$0.381966 \leq p < 1.$$

Then, we have that $\alpha_0 \leq 3 - 2\sqrt{2} = q$, if $(1 + \frac{2-\sqrt{2}}{3-p})\frac{1}{3}(1-p) \leq q$ or if

$$p^2 + (\sqrt{2} - 6 + 3q)p + 5 - \sqrt{2} - 9q \leq 0,$$

which is true for

$$0.5857931 \leq p < 1. \tag{2.22}$$

In view of (2.19), (2.20), (2.21) and (2.22) we must have

$$0.5857931 \leq p \leq 0.7407199.$$

Define intervals I and I_1 by

$$I = [0.5857931, 0.6255179) \tag{2.23}$$

and

$$I_1 = (0.7407199, 0.7631871]. \tag{2.24}$$

In view of (2.17), (2.23) and (2.24), we see that for $p \in I$, Theorem 1.1 cannot guarantee the convergence of Newton's method (1.1) to $x^* = \sqrt[3]{p}$. However, Theorem 2.2 guarantees the convergence of Newton's method (1.1) to x^*. Notice that, if $p \in I_1$, then we can set $\beta = \gamma = \gamma_0$.

Next, we compare the error bounds. Choose $p = 0.65$. Then, we have the following comparison table, which shows that the new error bounds (see also (2.11)) are more precise than the ones in [10].

n	$t_{n+1} - t_n$	$s_{n+1} - s_n$	$r_{n+1} - r_n$
1	0.1167	0.1167	0.1167
2	0.0333	0.0337	0.0260
3	0.0088	0.0048	0.0017
4	0.0024	$1.0958e-04$	$7.8741e-06$
5	$6.6130e-04$	$5.7796e-08$	$1.6553e-10$

TABLE 1. Comparison table

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DEPARTMENT OF MATHEMATICAL SCIENCES
CAMERON UNIVERSITY
LAWTON, OK 73505, USA
E-mail address: iargyros@cameron.edu

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES
NIT KARNATAKA, INDIA-575 025
E-mail address: sgeorge@nitk.ac.in