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On the Voronovskaja-type formula for the Bleimann, Butzer and Hahn bivariate operators

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ABSTRACT. In this paper we present two new alternative ways for the proof of Voronovskaja-type formula of the Bleimann, Butzer and Hahn bivariate operators, using the close connection between the recalled operators and Bernstein bivariate operators, respectively Stancu bivariate operators.

1. INTRODUCTION

Let $C[0,1] = \{f \in \mathbb{R}^{[0,1]} : f \text{ continuous on } [0,1]\}$ be the space of real-valued functions continuous on $[0,1]$. For any positive integer m, the classical Bernstein operators [15] are defined by

$$B_m(g;s) = \sum_{k=0}^{m} p_{m,k}(s) g\left(\tfrac{k}{m}\right), \tag{1.1}$$

for any function $g \in C[0,1]$, any $s \in [0,1]$, where $p_{m,k}(s) = \binom{m}{k} s^k (1-s)^{m-k}$ are the fundamental Bernstein polynomials. Let α, β be two real parameters satisfying the condition $0 \leq \alpha \leq \beta$. The Stancu operators [32] are defined by

$$P_m^{(\alpha,\beta)}(g;s) = \sum_{k=0}^{m} p_{m,k}(s) g\left(\tfrac{k+\alpha}{m+\beta}\right) = \sum_{k=0}^{m} \tbinom{m}{k} s^k (1-s)^{m-k} g\left(\tfrac{k+\alpha}{m+\beta}\right), \tag{1.2}$$

for any positive integer m, any $g \in C[0,1]$ and any $s \in [0,1]$. If $\alpha = \beta = 0$ the operators (1.2) become the classical Bernstein operators (1.1). For $\alpha = 0$ and $\beta = 1$ we get a particular case of operators (1.2) given by

$$P_m(g;s) = \sum_{k=0}^{m} p_{m,k}(s) g\left(\tfrac{k}{m+1}\right) = \sum_{k=0}^{m} \tbinom{m}{k} s^k (1-s)^{m-k} g\left(\tfrac{k}{m+1}\right). \tag{1.3}$$

Consider the space $C[0,+\infty) = \left\{f \in \mathbb{R}^{[0,+\infty)} \mid f \text{ continuous on } [0,+\infty)\right\}$. The Bleimann, Butzer and Hahn operators [16] are defined by

$$L_m(f;x) = \tfrac{1}{(1+x)^m} \sum_{k=0}^{m} \tbinom{m}{k} x^k f\left(\tfrac{k}{m+1-k}\right), \tag{1.4}$$

for any positive integer m, any function $f \in C[0,+\infty)$ and any $x \in [0,+\infty)$. In what follows, for simplicity, the Bleimann, Butzer and Hahn operators (1.4) will be called "BBH operators". A difficult problem concerning the BBH operators is to find the domain of convergence. Totik [33] studied the uniform approximation properties of these operators

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when f belongs to the class of continuous functions, that have finite limits at infinity. Jayasri and Sitaraman [21] considered the class

$$C_{P_N}[0,+\infty)=\left\{f\in C[0,+\infty)\,:\,f(x)=O\left((1+x)^N\right),\ x\to+\infty\right\},$$

where N is a fixed positive integer. They proved that if f belongs to $C_{P_N}[0,+\infty)$, then for each $x\geq 0$, $\lim\limits_{m\to\infty}L_m(f;x)=f(x)$. In [18], Hermann defined a class of functions in the following way

$$\mathcal{H}=\{f\in C[0,+\infty)\,:\,\log\left(|f(x)|+1\right)=o(x)\text{ as }x\to+\infty\}$$

and proved that if f belongs to $\mathcal{H}$ then, for each $x\geq 0$, $\lim\limits_{m\to\infty}L_m(f;x)=f(x)$. Moreover, if for some $\alpha>0$, $f(x)=e^{\alpha x}$, then $\lim\limits_{m\to\infty}L_m(f;x)=+\infty$, provided that x is sufficiently large. He also stated the following

Conjecture 1.1. [18] *If $f\in C[0,+\infty)$ and L_mf converges pointwise to f on $[0,+\infty)$, then $f\in\mathcal{H}$.*

An answer to the Hermann's conjecture was given by Abel and Ivan in two different papers. In [4], the authors determined the exact domain of convergence of L_mf for the exponential function $f(x)=a^x$, $(a>1)$ by showing that

$$\lim_{m\to\infty}L_m\left(a^t;x\right)=a^x\ \text{ iff }\ x\in\left[0,\tfrac{1}{a-1}\right).$$

As a by-product of their results, they also confirmed that $f\in\mathcal{H}$ is a sufficient condition for the pointwise convergence of L_mf to f and proved that Hermann's conjecture is true only for monotone functions. In [5], they gave a negative answer to Hermann's conjecture by constructing a counterexample function f satisfying $\lim\limits_{m\to\infty}L_m(f;x)=f(x)$ pointwise on $[0,+\infty)$, which is not an element of Hermann's class $\mathcal{H}$. Another class of functions

$$\mathcal{F}=\left\{f\in C[0,+\infty)\,:\,\text{for each }A>0,\ f(x)=O\left(e^{Ax}\right)\text{ as }x\to+\infty\right\}$$

was introduced by Jayasri and Sitaraman [22] , which proved that L_mf defines a pointwise approximation process on $\mathcal{F}$. In [16] is established that there exists a connection between Bernstein operators B_m and BBH operators L_m, basically given by the rational transformation $h(u)=u/(1+u)$, $u\in[0,+\infty)$ and its inverse $h^{-1}(v)=v/(1-v)$, $v\in[0,1)$. Several authors have tried to find different relationships between B_m and L_m. For instance, Mercer [23], respectively Abel [1] established that

$$L_m(f;x)=\sum_{k=0}^{m}\tbinom{m}{k}y^k(1-y)^{m-k}F\left(\tfrac{k}{m+1}\right),\tag{1.5}$$

where $y=x/(1+x)$ and $F(y)=f(y/(1-y))$, $y\in[0,1)$. In [12], the equality (1.5) was rewritten in the form

$$L_m\left(f;\tfrac{y}{1-y}\right)=P_m(F;y),\ \ y\in[0,1),\tag{1.6}$$

where $f\in C_*[0,+\infty)=\{f\in C[0,+\infty)\mid f(x)=o(x),\ (x\to\infty)\}$ and P_m denotes the Stancu type operator (1.3). Using the Voronovskaja-type formula for Stancu operators [32], it was derived the appropriate Voronovskaja-type formula for BBH operators, given by

$$\lim_{m\to\infty}m\cdot\left(L_m(f;x)-f(x)\right)=\frac{1}{2}x(1+x)^2f''(x),\tag{1.7}$$

where $f\in C_*[0,+\infty)$ and the second order derivative f'' is continuous in a neighborhood of $x\in[0,+\infty)$. The first Voronovskaja-type result for L_m was given by Totik [33], where the factor 2^{-1} is missing. Further proofs of formula (1.7) were established by Adell and

Badia [6], respectively Mercer [23]. In order to obtain several new results, respectively new properties of the BBH operators, Ivan [19] had an excellent idea to show the close connection between BBH operators and Bernstein operators, expressed by the formula

$$L_m(f;x) = (1+x)B_{m+1}\left(\tilde{f}; \tfrac{x}{x+1}\right), \; x \in [0,+\infty), \tag{1.8}$$

where $f \in \mathbb{R}^{[0,+\infty)}$ and $\tilde{f} \in \mathbb{R}^{[0,1]}$ is defined by

$$\tilde{f}(t) = \begin{cases} (1-t)f\left(\frac{t}{1-t}\right), & t \in [0,1), \\ 0, & t = 1. \end{cases} \tag{1.9}$$

Using the relations (1.8) and (1.9) some of the best known properties of the Bernstein operators can be directly transferred to the BBH operators. Developing the above ideas, in [3] the authors got new approximation properties of the BBH operators. Agratini [7] introduced and studied a generalization of the operators (1.4), while in [2] Abel considered BBH bivariate operators and studied some of their approximation properties. In [27] and [29] were constructed Bézier type curves, respectively surfaces for the BBH operators. Recent results concerning the Voronovskaja type theorem for certain linear positive operators were obtained in [9], [10], [11], [13], [14], [17], [24], [25], [26], [30]. Altin, Doğru and Özarslan [8] considered BBH bivariate operators, defined for any $f \in \mathbb{R}^{[0,+\infty)\times[0,+\infty)}$ by

$$L_{m,n}(f;x,y) = \tfrac{1}{(1+x)^m} \cdot \tfrac{1}{(1+y)^n} \sum_{k=0}^{m}\sum_{j=0}^{n} \tbinom{m}{k}\tbinom{n}{j} x^k y^j f\left(\tfrac{k}{m+1-k}, \tfrac{j}{n+1-j}\right) \tag{1.10}$$

and established some of their Korovkin type approximation properties. Independently of them, Pop [28] reconsidered the BBH bivariate operators (1.10) and proved the following Voronovskaja-type formula, given by

Theorem 1.1. *Suppose that $f \in \mathbb{R}^{[0,+\infty)\times[0,+\infty)}$ and the second order partial derivatives f''_{x^2}, f''_{y^2} are continuous in a neighborhood of $(x,y) \in [0,+\infty)\times[0,+\infty)$. The following Voronovskaja-type formula holds*

$$\lim_{m\to\infty} m \cdot (L_{m,m}(f;x,y) - f(x,y)) = \frac{1}{2}x(1+x)^2 f''_{x^2}(x,y) + \frac{1}{2}y(1+y)^2 f''_{y^2}(x,y). \tag{1.11}$$

Remark 1.1. The proof of Theorem 1.1 is classical, based on the Taylor expansion of the function f in a neighborhood of (x,y).

The aim of the present paper is to derive the Voronovskaja-type formula (1.11) using the close connection between Bernstein bivariate operators, respectively Stancu bivariate operators and BBH bivariate operators.

2. The close connection between Bernstein bivariate operators and BBH bivariate operators

The focus of this section is to obtain the Voronovskaja-type formula (1.11), using the close connection between Bernstein bivariate operators and BBH bivariate operators. We shall use the following notations:

$$[0,1]^2 = [0,1]\times[0,1], \; \mathbb{R}_+^2 = [0,+\infty)\times[0,+\infty),$$

$$C\left([0,1]^2\right) = \left\{ f \in \mathbb{R}^{[0,1]^2} \mid f \text{ continuous on } [0,1]^2 \right\},$$

$$C\left(\mathbb{R}_+^2\right) = \left\{ f \in \mathbb{R}^{\mathbb{R}_+^2} \mid f \text{ continuous on } \mathbb{R}_+^2 \right\},$$

$$C_*\left(\mathbb{R}_+^2\right) = \left\{ f \in C\left(\mathbb{R}_+^2\right) \mid f(x,y) = o(xy), \; (x\to\infty \text{ or } y\to\infty) \right\}.$$

We recall that, for any positive integers m, n, any $f \in C\left([0,1]^2\right)$ and any $(x,y) \in [0,1]^2$, the classical Bernstein bivariate operators are defined by

$$B_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)f\left(\tfrac{k}{m}, \tfrac{j}{n}\right). \tag{2.12}$$

It is also known, see [31], the following Voronovskaja-type formula for the Bernstein bivariate operators (2.12).

Theorem 2.2. *Suppose that $f \in C\left([0,1]^2\right)$ and the second order partial derivatives f''_{x^2}, f''_{y^2} are continuous in a neighborhood of $(x,y) \in [0,1]^2$. The following Voronovskaja-type formula holds*

$$\lim_{m\to\infty} m \cdot \left(B_{m,m}(f;x,y) - f(x,y)\right) = \frac{1}{2}x(1-x)f''_{x^2}(x,y) + \frac{1}{2}y(1-y)\,f''_{y^2}(x,y). \tag{2.13}$$

Following the idea of Ivan [19], [20] for the univariate case, we associate to $f \in C_*\left(\mathbb{R}_+^2\right)$ the function $\tilde{f} \in C\left([0,1]^2\right)$, defined by

$$\tilde{f}(s,t) = \begin{cases} (1-s)(1-t)f\left(\frac{s}{1-s}, \frac{t}{1-t}\right), & (s,t) \in [0,1) \times [0,1), \\ 0, & s = 1 \text{ or } t = 1. \end{cases} \tag{2.14}$$

Now, we can prove the following

Lemma 2.1. *For any $f \in C\left(\mathbb{R}_+^2\right)$ the following formula*

$$L_{m,n}(f;x,y) = (1+x)(1+y)B_{m+1,n+1}\left(\tilde{f}; \tfrac{x}{1+x}, \tfrac{y}{1+y}\right),\ (x,y) \in \mathbb{R}_+^2 \tag{2.15}$$

holds true.

Proof. Taking the definition of the BBH bivariate operators (1.10) into account, it follows

$$L_{m,n}\left(f; \tfrac{s}{1-s}, \tfrac{t}{1-t}\right) = \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{m}{k}\binom{n}{j} s^k(1-s)^{m-k}t^j(1-t)^{n-j} f\left(\tfrac{k}{m+1-k}, \tfrac{j}{n+1-j}\right) =$$

$$= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{m+1}{k}\binom{n+1}{j}\left(1 - \tfrac{k}{m+1}\right)\left(1 - \tfrac{j}{n+1}\right) s^k(1-s)^{m-k}t^j(1-t)^{n-j} \cdot$$

$$\cdot f\left(\frac{\frac{k}{m+1}}{1-\frac{k}{m+1}}, \frac{\frac{j}{n+1}}{1-\frac{j}{n+1}}\right) =$$

$$= \tfrac{1}{(1-s)(1-t)} \sum_{k=0}^{m+1}\sum_{j=0}^{n+1} \binom{m+1}{k}\binom{n+1}{j} s^k(1-s)^{m+1-k}t^j(1-t)^{n+1-j}\tilde{f}\left(\tfrac{k}{m+1}, \tfrac{j}{n+1}\right) =$$

$$= \tfrac{1}{(1-s)(1-t)} B_{m+1,n+1}\left(\tilde{f}; s, t\right),\ (s,t) \in [0,1) \times [0,1).$$

Denoting $x = \frac{s}{1-s}$, $y = \frac{t}{1-t}$, for $(s,t) \in [0,1) \times [0,1)$ it follows $(x,y) \in \mathbb{R}_+^2$ and we get the relation (2.15). □

The connection between the second order partial derivatives of f and $\tilde{f}$ follows by straightforward calculation and is contained in the following

Lemma 2.2. *If the function $\tilde{f}$ has the second order partial derivatives $\tilde{f}''_{s^2}$, $\tilde{f}''_{t^2}$ at $(s,t) \in [0,1) \times [0,1)$, the following equalities hold*

$$\tilde{f}''_{s^2}(s,t) = \frac{1-t}{(1-s)^3}\, f''_{s^2}\left(\tfrac{s}{1-s}, \tfrac{t}{1-t}\right), \tag{2.16}$$

$$\tilde{f}''_{t^2}(s,t) = \frac{1-s}{(1-t)^3}\, f''_{t^2}\left(\tfrac{s}{1-s}\,,\,\tfrac{t}{1-t}\right). \tag{2.17}$$

Theorem 2.3. *If the function $f \in C_*\left(\mathbb{R}_+^2\right)$ and the second order partial derivatives f''_{x^2}, f''_{y^2} are continuous in a neighborhood of $(x,y) \in \mathbb{R}_+^2$, then the Voronovskaja-type formula* (1.11) *holds.*

Proof. Taking the relations (2.15) and (2.14) into account, it follows

$$L_{m,m}(f;x,y) - f(x,y) =$$
$$= (1+x)(1+y)\left(B_{m+1,m+1}\left(\tilde{f};\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) - \tilde{f}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right)\right).$$

Applying the limit on the above equality, we get

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y) - f(x,y)\right) =$$
$$= (1+x)(1+y)\lim_{m\to\infty}\tfrac{m(m+1)}{m+1}\left(B_{m+1,m+1}\left(\tilde{f};\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) - \tilde{f}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right)\right) =$$
$$= (1+x)(1+y)\lim_{m\to\infty}(m+1)\left(B_{m+1,m+1}\left(\tilde{f};\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) - \tilde{f}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right)\right).$$

Using the result from Theorem 2.2, it follows

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y) - f(x,y)\right) = \tag{2.18}$$
$$= (1+x)(1+y)\left(\frac{x}{2(1+x)^2}\,\tilde{f}''_{x^2}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) + \frac{y}{2(1+y)^2}\,\tilde{f}''_{y^2}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right)\right).$$

But, from Lemma 2.2 we have

$$\tilde{f}''_{x^2}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) = \frac{(1+x)^3}{1+y}\, f''_{x^2}(x,y),$$
$$\tilde{f}''_{y^2}\left(\tfrac{x}{1+x}\,,\,\tfrac{y}{1+y}\right) = \frac{(1+y)^3}{1+x}\, f''_{y^2}(x,y).$$

Putting the above results in (2.18), we get

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y) - f(x,y)\right) =$$
$$= (1+x)(1+y)\left(\frac{x}{2(1+x)^2}\,\frac{(1+x)^3}{1+y}\, f''_{x^2}(x,y) + \frac{y}{2(1+y)^2}\,\frac{(1+y)^3}{1+x}\, f''_{y^2}(x,y)\right) =$$
$$= \frac{1}{2}x(1+x)^2\, f''_{x^2}(x,y) + \frac{1}{2}y(1+y)^2 f''_{y^2}(x,y),$$

which is the desired Voronovskaja-type formula (1.11). □

3. The close connection between Stancu bivariate operators and BBH bivariate operators

The focus of this section is to obtain the Voronovskaja-type formula (1.11), using the close connection between Stancu bivariate operators and BBH bivariate operators. For any positive integers m, n, any $f \in C\left([0,1]^2\right)$ and any $(x,y) \in [0,1]^2$, the Stancu bivariate operators are defined by

$$P_{m,n}(f;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) f\left(\tfrac{k}{m+1}\,,\,\tfrac{j}{n+1}\right). \tag{3.19}$$

The following Voronovskaja-type formula for the Stancu bivariate operators (3.19) is well-known and can be found, for instance in [28].

Theorem 3.4. *Suppose that $f \in C\left([0,1]^2\right)$ and the second order partial derivatives f''_{x^2}, f''_{y^2} are continuous in a neighborhood of $(x,y) \in [0,1]^2$. The following Voronovskaja-type formula holds*

$$\lim_{m\to\infty} m \cdot (P_{m,m}(f;x,y) - f(x,y)) = \tag{3.20}$$
$$= -x\, f'_x(x,y) - y\, f'_y(x,y) + \frac{1}{2}x(1-x)\, f''_{x^2}(x,y) + \frac{1}{2}y(1-y)\, f''_{y^2}(x,y).$$

Following the idea in [12] for the univariate case, we associate to the function $f \in C_*\left(\mathbb{R}^2_+\right)$ the function $F \in C\left([0,1]^2\right)$, defined by

$$F(s,t) = \begin{cases} f\left(\frac{s}{1-s}, \frac{t}{1-t}\right), & (s,t) \in [0,1) \times [0,1), \\ 0, & s = 1 \text{ or } t = 1. \end{cases} \tag{3.21}$$

Now, we can prove the following

Lemma 3.3. *For any $f \in C\left(\mathbb{R}^2_+\right)$ the following formula*

$$L_{m,n}(f;x,y) = P_{m,n}\left(F; \frac{x}{x+1}, \frac{y}{y+1}\right),\ (x,y) \in \mathbb{R}^2_+ \tag{3.22}$$

holds true.

Proof. Taking the definition of the BBH bivariate operators (1.10) into account, it follows

$$L_{m,n}\left(f; \frac{s}{1-s}, \frac{t}{1-t}\right) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(s)\, p_{n,j}(t) f\left(\frac{k}{m+1-k}, \frac{j}{n+1-j}\right) =$$
$$= \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(s)\, p_{n,j}(t) f\left(\frac{\frac{k}{m+1}}{1-\frac{k}{m+1}}, \frac{\frac{j}{n+1}}{1-\frac{j}{n+1}}\right) =$$
$$= \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(s)\, p_{n,j}(t) F\left(\frac{k}{m+1}, \frac{j}{n+1}\right) = P_{m,n}(F;s,t),\ \ (s,t) \in [0,1) \times [0,1).$$

Denoting $x = \frac{s}{1-s}$, $y = \frac{t}{1-t}$ for $(s,t) \in [0,1) \times [0,1)$ it follows $(x,y) \in \mathbb{R}^2_+$ and we get the relation (3.22). □

The connection between the first, respectively second partial derivatives of f and F follows by straightforward calculation and is contained in the following

Lemma 3.4. *If the function $F \in C\left([0,1]^2\right)$ has the first, respectively second order partial derivatives F'_s, F'_t, F''_{s^2}, F''_{t^2} at $(s,t) \in [0,1) \times [0,1)$, the following equalities hold*

$$F'_s(s,t) = \frac{1}{(1-s)^2}\, f'_s\left(\frac{s}{1-s}, \frac{t}{1-t}\right), \tag{3.23}$$

$$F'_t(s,t) = \frac{1}{(1-t)^2}\, f'_t\left(\frac{s}{1-s}, \frac{t}{1-t}\right), \tag{3.24}$$

$$F''_{s^2}(s,t) = \frac{2}{(1-s)^3}\, f'_s\left(\frac{s}{1-s}, \frac{t}{1-t}\right) + \frac{1}{(1-s)^4}\, f''_{s^2}\left(\frac{s}{1-s}, \frac{t}{1-t}\right), \tag{3.25}$$

$$F''_{t^2}(s,t) = \frac{2}{(1-t)^3}\, f'_t\left(\frac{s}{1-s}, \frac{t}{1-t}\right) + \frac{1}{(1-t)^4}\, f''_{t^2}\left(\frac{s}{1-s}, \frac{t}{1-t}\right). \tag{3.26}$$

Theorem 3.5. *If the function $f \in C_*\left(\mathbb{R}^2_+\right)$ and the second order partial derivatives f''_{x^2}, f''_{y^2} are continuous in a neighborhood of $(x,y) \in \mathbb{R}^2_+$, then the Voronovskaja-type formula* (1.11) *holds.*

Proof. Taking the relation (3.22) into account, it follows

$$L_{m,m}(f;x,y)-f(x,y)=P_{m,m}\left(F;\tfrac{x}{1+x},\tfrac{y}{1+y}\right)-F\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right).$$

Applying the limit on the above equality, we get

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y)-f(x,y)\right)=\lim_{m\to\infty} m\cdot\left(P_{m,m}\left(F;\tfrac{x}{1+x},\tfrac{y}{1+y}\right)-F\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)\right).$$

The results from Theorem 3.4 leads to

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y)-f(x,y)\right)= \tag{3.27}$$

$$=-\frac{x}{1+x}F'_x\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)-\frac{y}{1+y}F'_y\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)+\frac{x}{2(1+x)^2}F''_{x^2}\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)+$$
$$+\frac{y}{2(1+y)^2}F''_{y^2}\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right).$$

From Lemma 3.4 we have

$$F'_x\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)=(1+x)^2f'_x(x,y), \tag{3.28}$$

$$F'_y\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)=(1+y)^2f'_y(x,y), \tag{3.29}$$

$$F''_{x^2}\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)=2(1+x)^3f'_x(x,y)+(1+x)^4f''_{x^2}(x,y), \tag{3.30}$$

$$F''_{y^2}\left(\tfrac{x}{1+x},\tfrac{y}{1+y}\right)=2(1+y)^3f'_y(x,y)+(1+y)^4f''_{y^2}(x,y). \tag{3.31}$$

Taking (3.28)–(3.31) into account, from (3.27) it follows

$$\lim_{m\to\infty} m\cdot\left(L_{m,m}(f;x,y)-f(x,y)\right)=-\frac{x}{1+x}(1+x)^2f'_x(x,y)-\frac{y}{1+y}(1+y)^2f'_y(x,y)+$$
$$+\frac{x}{2(1+x)^2}\left(2(1+x)^3f'_x(x,y)+(1+x)^4f''_{x^2}(x,y)\right)+$$
$$+\frac{y}{2(1+y)^2}\left(2(1+y)^3f'_y(x,y)+(1+y)^4f''_{y^2}(x,y)\right)=$$
$$=\frac{1}{2}x(1+x)^2\,f''_{x^2}(x,y)+\frac{1}{2}y(1+y)^2\,f''_{y^2}(x,y),$$

which is the desired Voronovskaja-type formula (1.11). □

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