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Quantitative estimates in uniform and pointwise approximation by Bernstein-Durrmeyer-Choquet operators

SORIN G. GAL and SORIN TRIFA

ABSTRACT. For the qualitative results of uniform and pointwise approximation obtained in [8], we present here general quantitative estimates in terms of the modulus of continuity and of a K-functional, in approximation by the generalized multivariate Bernstein-Durrmeyer operator $M_{n,\Gamma_{n,x}}$, written in terms of Choquet integrals with respect to a family of monotone and submodular set functions, $\Gamma_{n,x}$, on the standard d-dimensional simplex. If $d = 1$ and the Choquet integrals are taken with respect to some concrete possibility measures, the estimate in terms of the modulus of continuity is detailed. Examples improving the estimates given by the classical operators also are presented.

1. INTRODUCTION

In the very recent paper [8], the first author have jointly obtained qualitative uniform and pointwise approximation results in approximation by the multivariate Bernstein-Durrmeyer linear operator defined in terms of the nonlinear Choquet integral with respect to a nonnegative, monotone, normalized and submodular set function $\mu : \mathcal{B}_{S^d} \to \mathbb{R}_+$, given by

$$M_{n,\mu}(f)(x)$$

$$(1.1)\qquad = \sum_{|\alpha|=n} \frac{(C)\int_{S^d} f(t)B_\alpha(t)d\mu(t)}{(C)\int_{S^d} B_\alpha(t)d\mu(t)} \cdot B_\alpha(x) := \sum_{|\alpha|=n} c(\alpha,\mu)\cdot B_\alpha(x),\ x\in S^d,\ n\in\mathbb{N},$$

where $\mathcal{B}_{S^d}$ denotes the sigma algebra of all Borel measurable subsets in the power set $\mathcal{P}(S^d)$ and f is supposed to be Choquet μ-integrable on the standard simplex

$$S^d = \{(x_1, ..., x_d); 0 \le x_1, ..., x_d \le 1,\ 0 \le x_1 + ... + x_d \le 1\}.$$

Note that in (1.1), it is used the notation

$$B_\alpha(x) = \frac{n!}{\alpha_0!\cdot\alpha_1!\cdot ...\cdot\alpha_d!}(1-x_1-x_2-...-x_d)^{\alpha_0}\cdot x_1^{\alpha_1}\cdot ...\cdot x_d^{\alpha_d}$$

$$=: \frac{n!}{\alpha_0!\cdot\alpha_1!\cdot ...\cdot\alpha_d!}\cdot P_\alpha(x),$$

where $\alpha = (\alpha_0, \alpha_1, ..., \alpha_d)$, $\alpha_j \in \mathbb{N}\bigcup\{0\}$, $j = 0, ..., d$, $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_d = n$.

The results in [8] generalize the results in [1]-[2], where μ is a nonnegative and bounded Borel measure on $\mathcal{B}_{S^d}$.

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Corresponding author: Sorin G. Gal; galso@uoradea.ro

The main goal of this paper is to obtain quantitative pointwise and uniform estimates in terms of the modulus of continuity and of a K-functional, in approximation by the more general multivariate Bernstein-Durrmeyer polynomial operators defined by

$$M_{n,\Gamma_{n,x}}(f)(x) = \sum_{|\alpha|=n} c(\alpha, \mu_{n,\alpha,x}) \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N}, \tag{1.2}$$

where

$$c(\alpha, \mu_{n,\alpha,x}) = \frac{(C)\int_{S^d} f(t)B_\alpha(t)d\mu_{n,\alpha,x}(t)}{(C)\int_{S^d} B_\alpha(t)d\mu_{n,\alpha,x}(t)} = \frac{(C)\int_{S^d} f(t)P_\alpha(t)d\mu_{n,\alpha,x}(t)}{(C)\int_{S^d} P_\alpha(t)d\mu_{n,\alpha,x}(t)}$$

and for every $n \in \mathbb{N}$ and $x \in S^d$, $\Gamma_{n,x} = (\mu_{n,\alpha,x})_{|\alpha|=n}$ is a family of bounded, monotone, submodular and strictly positive set functions on $\mathcal{B}_{S^d}$.

If the family $\Gamma_{n,x}$ reduces to one bounded, monotone, submodular and strictly positive set function (i.e. $\mu_{n,\alpha,x} = \mu$ for all n, x and α), then the operator given by (1.2) reduces to the operator considered in [8].

The plan of the paper is as follows. Section 2 contains some preliminaries on possibility theory and on Choquet integral. In Section 3, pointwise and uniform quantitative estimates in terms of the modulus of continuity and a K-functional for the approximation by the operators $M_{n,\Gamma_{n,x}}(f)(x)$ defined by (1.2) are obtained. In Section 4, in the particular case when $d = 1$ and the Choquet integrals are taken with respect to some concrete possibility measures, the pointwise estimate in terms of the modulus of continuity is detailed. Section 5 contains some concrete examples of Bernstein-Durrmeyer-Choquet operators improving the classical error estimates.

2. Preliminaries

Firstly, we present a few known concepts in possibility theory useful for the next considerations. For details, see, e.g., [6].

Definition 2.1. For the non-empty set Ω, denote by $\mathcal{P}(\Omega)$ the family of all subsets of Ω.

(i) A function $\lambda : \Omega \to [0, 1]$ with the property $\sup\{\lambda(s); s \in \Omega\} = 1$, is called possibility distribution on Ω.

(ii) $P : \mathcal{P}(\Omega) \to [0, 1]$ is called possibility measure, if it satisfies the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property easily implies that $P(A) \leq P(B)$ and that $P(A \bigcup B) \leq P(A) + P(B)$.

Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0, 1]$, $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, $A \subset \Omega$. Also, if $f : \Omega \to \mathbb{R}_+$, then the possibilistic integral of f on $A \subset \Omega$ with respect to P_λ is defined by $(Pos)\int_A f dP_\lambda = \sup\{f(t) \cdot \lambda(t); t \in A\}$ (see, e.g., [6], Chapter 1).

Some known concepts and results concerning the Choquet integral can be summarized by the following.

Definition 2.2. Suppose $\Omega \neq \emptyset$ and let $\mathcal{C}$ be a σ-algebra of subsets in Ω.

(i) (see, e.g., [10], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$. Also, μ is called bounded if $\mu(\Omega) < +\infty$ and submodular if

$$\mu(A \bigcup B) + \mu(A \bigcap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C}.$$

(ii) (see, e.g., [10], p. 233, or [4]) If μ is a monotone set function, normalized on $\mathcal{C}$ and if $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable (i.e., for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$), then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu\left(F_\beta(f)\bigcap A\right) d\beta + \int_{-\infty}^0 \left[\mu\left(F_\beta(f)\bigcap A\right) - \mu(A)\right] d\beta,$$

with $F_\beta(f) = \{\omega \in \Omega; f(\omega) \geq \beta\}$. If $f \geq 0$ on A, then above we get $\int_{-\infty}^0 = 0$.

The function f will be called Choquet integrable on A if $(C)\int_A f d\mu \in \mathbb{R}$.

In what follows, we list some known properties of the Choquet integral.

Remark 2.1. If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold :

(i) For all $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (if $f \geq 0$ then see, e.g., [10], Theorem 11.2, (5), p. 228 and if f is of arbitrary sign, then see, e.g., [5], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [10], pp. 232-233, or [5], p. 65) $(C)\int_A (f + c)d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$.

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded, we have (see, e.g., [5], p. 75, Theorem 6.3)

$$(C)\int_A (f + g)d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu.$$

(iii) If $f \leq g$ on A then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [10], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f, g are of arbitrary sign).

(iv) Let $f \geq 0$. If $A \subset B$ then $(C)\int_A f d\mu \leq (C)\int_B f d\mu$. In addition, if μ is finitely subadditive, then $(C)\int_{A\bigcup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu$.

(v) It is immediate that $(C)\int_A 1 \cdot d\mu(t) = \mu(A)$.

(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0, \gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\emptyset) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions (see, e.g., [5], pp. 16-17, Example 2.1). For example, we can take $\gamma(t) = \sqrt{t}$.

If the above γ function is increasing, concave and satisfies only $\gamma(0) = 0$, then for any bounded Borel measure $m \leq 1$, $\mu(A) = \gamma(m(A))$ gives a simple example of bounded, monotone and submodular set function.

Note that any possibility measure μ is normalized, monotone and submodular. Indeed, the axiom $\mu(A\bigcup B) = \max\{\mu(A), \mu(B)\}$ implies the monotonicity, while the property $\mu(A\bigcap B) \leq \min\{\mu(A), \mu(B)\}$ implies the submodularity.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the usual Lebesgue type integral (see, e.g., [5], p. 62, or [10], p. 226).

3. Pointwise and uniform estimate for general Bernstein-Durrmeyer-Choquet operators

Recall that $\mu : \mathcal{B}_{S^d} \to [0, +\infty)$ is said strictly positive if for every open set $A \subset \mathbb{R}^n$ with $A \cap S^d \neq \emptyset$, we have $\mu(A \cap S^d) > 0$.

The support of μ is defined by

$$supp(\mu) = \{x \in S^d; \mu(N_x) > 0 \text{ for every open neighborhood } N_x \in \mathcal{B}_{S^d} \text{ of } x\}.$$

Note that the strict positivity of μ, evidently implies the condition $supp(\mu) \setminus \partial S^d \neq \emptyset$, which guarantees that $(C)\int_{S^d} B_\alpha(t)d\mu(t) > 0$, for all B_α.

Let us consider $C_+(S^d) = \{f : S^d \to \mathbb{R}_+; f \text{ is continuous on } S^d\}$, endowed with the norm $\|F\|_{C(S^d)} = \sup\{|F(x)|; x \in S^d\}$.

The main result of this section consists in the following general quantitative estimates in pointwise and uniform approximation.

Theorem 3.1. *For each fixed $n \in \mathbb{N}$ and $x \in S^d$, let $\Gamma_{n,x} = \{\mu_{n,\alpha,x}\}_{|\alpha|=n}$ be a family of bounded, monotone, submodular and strictly positive set functions on $\mathcal{B}_{S^d}$.*

(i) For every $f \in C_+(S^d)$, $x = (x_1, ..., x_d) \in S^d$, $n \in \mathbb{N}$, we have

$$|M_{n,\Gamma_{n,x}}(f)(x) - f(x)| \le 2\omega_1(f; M_{n,\Gamma_{n,x}}(\varphi_x)(x))_{S^d},$$

where $M_{n,\Gamma_{n,x}}(f)(x)$ is given by (1.2), $\|x\| = \sqrt{\sum_{i=1}^d x_i^2}$, $\varphi_x(t) = \|t - x\|$ and $\omega_1(f; \delta)_{S^d} = \sup\{|f(t) - f(x)|; t, x \in S^d, \|t - x\| \le \delta\}$.

(ii) Suppose that the family $\Gamma_{n,x}$ does not depend on x. Then, for any $f \in C_+(S^d)$ and $n \in \mathbb{N}$, we get

$$\|M_{n,\Gamma_n}(f) - f\|_{C(S^d)} \le 2K\left(f; \frac{\Delta_n}{2}\right),$$

where $\Delta_n = \sum_{i=1}^d \|M_{n,\Gamma_n}(|\varphi_{e_i} - x_i\mathbf{1}|)\|_{C(S^d)}$,

$$K(f; t) = \inf_{g \in C_+^1(S^d)} \{\|f - g\|_{C(S^d)} + t\|\nabla g\|_{C(S^d)}\},$$

$C_+^1(S^d)$ is the subspace of all functions $g \in C_+(S^d)$ with continuous partial derivatives $\partial_i g$, $i \in \{1, ..., d\}$ and $\|\nabla g\|_{C(S^d)} = \max_{i=\{1,...,d\}}\{\|\partial_i g\|_{C(S^d)}\}$, $\varphi_{e_i}(x) = x_i$, $i \in \{1, ..., d\}$, $x = (x_1, ..., x_d)$, $\mathbf{1}(x) = 1$, for all $x \in S^d$.

Proof. (i) For $x \in S^d$, $n \in \mathbb{N}$ and $|\alpha| = n$ arbitrary fixed, let us consider $T_{n,\alpha,x} : C_+(S^d) \to \mathbb{R}_+$ defined by

$$T_{n,\alpha,x}(f) = (C)\int_{S^d} f(t)P_\alpha(t)d\mu_{n,\alpha,x}(t), f \in C_+(S^d).$$

Since each $\mu_{n,\alpha,x}$ is monotone and submodular, by the proof and statement of Lemma 3.1 in [8] (implied in fact by the previous Remark 2.1, (i), (ii), (iii)), it follows that each $T_{n,\alpha,x}$ is positively homogeneous, sublinear and monotonically increasing and satisfies $|T_{n,\alpha,x}(f) - T_{n,\alpha,x}(g)| \le T_{n,\alpha,x}(|f - g|)$.

This immediately implies that $M_{n,\Gamma_{n,x}}$ keeps the same properties and as a consequence, it follows

$$|M_{n,\Gamma_{n,x}}(f)(x) - M_{n,\Gamma_{n,x}}(g)(x)| \le M_{n,\Gamma_{n,x}}(|f - g|)(x), \tag{3.3}$$

$M_{n,\Gamma_{n,x}}(\lambda f) = \lambda M_{n,\Gamma_{n,x}}(f)$, $M_{n,\Gamma_{n,x}}(f + g) \le M_{n,\Gamma_{n,x}}(f) + M_{n,\Gamma_{n,x}}(g)$ and that $f \le g$ on S^d implies $M_{n,\Gamma_{n,x}}(f) \le M_{n,\Gamma_{n,x}}(g)$ on S^d, for all $\lambda \ge 0$, $f, g \in C_+(S^d)$, $n \in \mathbb{N}$, $|\alpha| = n$, $x \in S^d$.

Denoting $e_0(t) = 1$ for all $t \in S^d$, since obviously $M_{n,\Gamma_{n,x}}(e_0)(x) = 1$ for all $x \in S^d$ and taking into account the properties in Remark 2.1, (i) and (3.3), for any fixed x we obtain

$$\begin{aligned} |M_{n,\Gamma_{n,x}}(f)(x) - f(x)| &= |M_{n,\Gamma_{n,x}}(f(t))(x) - M_{n,\Gamma_{n,x}}(f(x))(x)| \\ &\le M_{n,\Gamma_{n,x}}(|f(t) - f(x)|)(x). \end{aligned} \tag{3.4}$$

But taking into account the properties of the modulus of continuity, for all $t, x \in S^d$ and $\delta > 0$, we get

$$|f(t) - f(x)| \le \omega_1(f; \|t - x\|)_{S^d} \le \left[\frac{1}{\delta}\|t - x\| + 1\right]\omega_1(f; \delta)_{S^d}. \tag{3.5}$$

Now, from (3.4) and applying $M_{n,\Gamma_{n,x}}$ to (3.5), by the properties of $M_{n,\Gamma_{n,x}}$ mentioned after the inequality (3.3), we immediately get

$$|M_{n,\Gamma_{n,x}}(f)(x) - f(x)| \leq \left[\frac{1}{\delta} M_{n,\Gamma_{n,x}}(\varphi_x)(x) + 1\right] \omega_1(f;\delta)_{S^d}.$$

Choosing here $\delta = M_{n,\Gamma_{n,x}}(\varphi_x)(x)$, we obtain the desired estimate.

(ii) Let $f, g \in C_+(S^d)$. We have

$$\begin{aligned} & f(x) - M_{n,\Gamma_n}(f)(x) \\ &= f(x) - g(x) + M_{n,\Gamma_n}(g)(x) - M_{n,\Gamma_n}(f)(x) + g(x) - M_{n,\Gamma_n}(g)(x), \end{aligned}$$

which, by using (3.3) too, implies

$$\begin{aligned} & |f(x) - M_{n,\Gamma_n}(f)(x)| \\ &\leq |f(x) - g(x)| + |M_{n,\Gamma_n}(g)(x) - M_{n,\Gamma_n}(f)(x)| + |g(x) - M_{n,\Gamma_n}(g)(x)| \\ &\leq |f(x) - g(x)| + M_{n,\Gamma_n}(|g - f|)(x) + |g(x) - M_{n,\Gamma_n}(g)(x)| \\ &\leq 2\|f - g\|_{C(S^d)} + |g(x) - M_{n,\Gamma_n}(g)(x)|. \end{aligned}$$

By following the lines in the proof of Theorem 4.5 in [3], since from the lines after relation (3.3) in the above point (i), the operator M_{n,Γ_n} is monotone and subadditive, for all $g \in C^1_+(S^d)$, $x \in S^d$, we immediately get

$$\begin{aligned} & |g(x) - M_{n,\Gamma_n}(g)(x)| \\ &\leq M_{n,\Gamma_n}(|g - g(x)\mathbf{1}|)(x) \leq \|\nabla g\|_{C(S^d)} \cdot M_{n,\Gamma_n}\left(\sum_{i=1}^{d} |\varphi_{e_i} - x_i \mathbf{1}|\right)(x) \\ &\leq \|\nabla g\|_{C(S^d)} \cdot \sum_{i=1}^{d} M_{n,\Gamma_n}\left(|\varphi_{e_i} - x_i\mathbf{1}|\right)(x) \leq \|\nabla g\|_{C(S^d)} \cdot \Delta_n. \end{aligned}$$

Concluding, it follows

$$\|f - M_{n,\Gamma_n}(f)\|_{C(S^d)} \leq 2\left[\|f - g\|_{C(S^d)} + \frac{\Delta_n}{2}\|\nabla g\|_{C(S^d)}\right],$$

which immediately implies the required estimate in (ii). □

Remark 3.2. The positivity of function f in Theorem 3.1, (i), (ii) is necessary because of the positive homogeneity of the Choquet integral used in its proof. However, if f is of arbitrary sign and lower bounded on S^d with $f(x) - m \geq 0$, for all $x \in S^d$, then the statement of Theorem 3.1, (i), (ii) can be restated for the slightly modified Bernstein-Durrmeyer operator defined by

$$M^*_{n,\Gamma_{n,x}}(f)(x) = M_{n,\Gamma_{n,x}}(f - m)(x) + m.$$

Indeed, in the case of Theorem 3.1, (i), this is immediate from $\omega_1(f - m;\delta)_{S^d} = \omega_1(f;\delta)_{S^d}$ and from $M^*_{n,\Gamma_{n,x}}(f)(x) - f(x) = M_{n,\Gamma_{n,x}}(f - m)(x) - (f(x) - m)$. Note that in the case of Theorem 3.1, (ii), since we may consider here that $m < 0$, we immediately get the relations

$$\begin{aligned} K(f - m; t) &= \inf_{g \in C^1_+(S^d)} \{\|f - (g + m)\|_{C(S^d)} + t\|\nabla g\|_{C(S^d)}\} \\ &= \inf_{g \in C^1_+(S^d)} \{\|f - (g + m)\|_{C(S^d)} + t\|\nabla (g + m)\|_{C(S^d)}\} \\ &= \inf_{h \in C^1(S^d),\, h \geq m} \{\|f - h\|_{C(S^d)} + t\|\nabla h\|_{C(S^d)}\}. \end{aligned}$$

4. Particular Bernstein-Durrmeyer-Choquet operators

Since the estimates in Theorem 3.1, (i), (ii) are of very general and abstract form, involving the apparently difficult to be calculated Choquet integrals, it is of interest to obtain concrete expressions for the orders of approximation.

In this sense, we will apply Theorem 3.1, (i) for $d = 1$ and for some special choices of the submodular set functions.

Thus, we will consider the case of the measures of possibility. Denoting $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, let us define $\lambda_{n,k}(t) = \frac{p_{n,k}(t)}{k^k n^{-n}(n-k)^{n-k}\binom{n}{k}} = \frac{t^k(1-t)^{n-k}}{k^k n^{-n}(n-k)^{n-k}}$, $k = 0, ..., n$. Here, by convention we consider $0^0 = 1$, so that the cases $k = 0$ and $k = n$ have sense.

By considering the root $\frac{k}{n}$ of $p'_{n,k}(x)$, it is easy to see that $\max\{p_{n,k}(t); t \in [0,1]\} = k^k n^{-n}(n-k)^{n-k}\binom{n}{k}$, which implies that each $\lambda_{n,k}$ is a possibility distribution on $[0,1]$. Denoting by $P_{\lambda_{n,k}}$ the possibility measure induced by $\lambda_{n,k}$ and $\Gamma_{n,x} := \Gamma_n = \{P_{\lambda_{n,k}}\}_{k=0}^{n}$ (i.e. Γ is independent of x), the nonlinear Bernstein-Durrmeyer polynomial operators given by (1.2), in terms of the Choquet integrals with respect to the set functions in Γ_n, will become

$$D_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_0^1 f(t)t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}{(C)\int_0^1 t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}. \tag{4.6}$$

It is easy to see that any possibility measure $P_{\lambda_{n,k}}$ is bounded, monotone, submodular and strictly positive, $n \in \mathbb{N}$, $k = 0, 1, ..., n$, so that we are under the hypothesis of Theorem 3.1.

We can state the following result.

Theorem 4.2. *If $D_{n,\Gamma_n}(f)(x)$ is given by (4.6), then for every $f \in C_+([0,1])$, $x \in [0,1]$ and $n \in \mathbb{N}$, $n \geq 2$, we have*

$$|D_{n,\Gamma_n}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{(1+\sqrt{2})\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{n}} + \frac{1}{n}\right)_{[0,1]}.$$

For its proof, we need the following auxiliary result.

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $x \in [0,1]$. Denoting*

$$A_{n,k}(x) := \sup\{|t-x|t^k(1-t)^{n-k}; t \in [0,1]\} =$$

$$\max\{\sup\{(t-x)t^k(1-t)^{n-k}; t \in [x,1]\}, \sup\{(x-t)t^k(1-t)^{n-k}; t \in [0,x]\}\},$$

with the convention $0^0 = 1$, for all $k = 0, ..., n$ we have

$$A_{n,k}(x) = \max\{(t_2 - x)t_2^k(1-t_2)^{n-k}, (x-t_1)t_1^k(1-t_1)^{n-k}\},$$

with t_1, t_2 given by

$$t_1 = \frac{nx+k+1-\sqrt{\Delta}}{2(n+1)},\ t_2 = \frac{nx+k+1+\sqrt{\Delta}}{2(n+1)}, \tag{4.7}$$

where

$$\Delta = (nx+k+1)^2 - 4kx(n+1) = n^2\left[(x+(k+1)/n)^2 - 4x\frac{k}{n}\cdot\frac{n+1}{n}\right]$$

$$= (nx-k)^2 + 2x(n-k) + 2k(1-x) + 1 \geq 1.$$

Proof. Let us denote $H_{n,k}(t) = t^k(1-t)^{n-k}|t-x|$, with $k \in \{0, ..., n\}$. We have two cases : (i) $1 \le k \le n-1$ and (ii) $k = 0$ or $k = n$.

Case (i). For $t \in [x, 1]$ we obtain $H_{n,k}(t) = (t-x)t^k(1-t)^{n-k}$ and from $H'_{n,k}(t) = t^{k-1}(1-t)^{n-k-1}[-t^2(n+1)+t(nx+k+1)-kx] = 0$, it follows $-t^2(n+1)+t(nx+k+1)-kx = 0$, which has the discriminant

$$\Delta = (nx+k+1)^2 - 4kx(n+1) = (nx-k)^2 + 2x(n-k) + 2k(1-x) + 1 \ge 1.$$

Therefore, the quadratic equation has two real distinct solutions $t_1 < t_2$

$$t_1 = \frac{nx+k+1-\sqrt{\Delta}}{2(n+1)}, \; t_2 = \frac{nx+k+1+\sqrt{\Delta}}{2(n+1)},$$

where by simple calculation we derive $0 \le t_1 < t_2 \le 1$. Also, since $H_{n,k}(0) = H_{n,k}(x) = H_{n,k}(1) = 0$ and $H_{n,k}(t) \ge 0$ for $t \in [x, 1]$, simple graphical reasonings show that the only possibility is $0 \le t_1 \le x \le t_2 \le 1$, with $t = t_2$ maximum point on $[x, 1]$ for $H_{n,k}(t)$.

Similarly, for $t \in [0, x]$, since $H_{n,k}(t) = (x-t)t^k(1-t)^{n-k}$, using the above reasonings we obtain $H'_{n,k}(t) = t^{k-1}(1-t)^{n-k-1}[t^2(n+1) - t(nx+k+1) + kx]$ and that t_1 is a maximum point of $H_{n,k}(t)$ on $[0, x]$.

In conclusion, with t_1, t_2 given by (4.7), we get

$$A_{n,k}(x) = \max\{(t_2-x)t_2^k(1-t_2)^{n-k}, \; (x-t_1)t_1^k(1-t_1)^{n-k}\}.$$

Case (ii). Suppose first that $k = 0$. By the calculation from the case (i), for $t \in [x, 1]$ we get $0 = t_1 \le x \le t_2 = \frac{nx+1}{n+1} \le 1$, $H_{n,0}(t) \ge 0$ and $H_{n,0}(x) = H_{n,0}(1) = 0$, which by similar graphical reasonings leads to the fact that the maximum of $H_{n,0}(t)$ on $[x, 1]$ is $H_{n,0}(t_2) = (t_2-x)(1-t_2)^n$. Therefore, we recapture the case (i) with the convention that $0^0 = 1$. Similarly, for $t \in [0, x]$, we get that the maximum of $H_{n,0}(t)$ is $H_{n,0}(t_1) = (x-t_1)(1-t_1)^n$

The subcase $k = n$ is similar, which proves the lemma. □

Proof of Theorem 4.2. According to Theorem 3.1, (i) we have to estimate

$$D_{n,\Gamma_n}(\varphi_x)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_0^1 |t-x|t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}{(C)\int_0^1 t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}.$$

First of all, by Definition 2.2, (ii), we get

$$(C)\int_0^1 t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t) = \int_0^{+\infty} P_{\lambda_{n,k}}(\{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\})d\beta$$

$$= \int_0^1 P_{\lambda_{n,k}}(\{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\})d\beta$$

$$= \int_0^1 \sup\{\lambda_{n,k}(s); s \in \{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\}\}d\beta$$

$$= \frac{1}{k^k n^{-n}(n-k)^{n-k}} \cdot \int_0^1 \sup\{s^k(1-s)^{n-k}; s \in \{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\}\}d\beta.$$

For simplicity, denote $E_{n,k} = k^k n^{-n}(n-k)^{n-k}$, where again we take $0^0 = 1$. Since for $\beta > E_{n,k}$ we have $\{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\} = \emptyset$ and since we can take $\sup\{s^k(1-s)^{n-k}; s \in \emptyset\} = 0$, it follows

$$(C)\int_0^1 t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)$$

$$= \frac{1}{E_{n,k}} \cdot \int_0^{E_{n,k}} \sup\{s^k(1-s)^{n-k}; s \in \{t \in [0,1]; t^k(1-t)^{n-k} \ge \beta\}\}d\beta$$

$$= \frac{1}{E_{n,k}} \cdot \int_0^{E_{n,k}} E_{n,k} d\beta = E_{n,k}. \tag{4.8}$$

On the other hand, denoting $A_{n,k}(x) = \sup\{|t-x|t^k(1-t)^{n-k}; t \in [0,1]\}$, by Remark 2.1, (iii), (v) and by Lemma 4.1, for $t_1 < t_2$ in (4.7) we obtain

$$(C)\int_0^1 |t-x|t^k(1-t)^{n-k} dP_{\lambda_{n,k}}(t) \le (C)\int_0^1 A_{n,k}(x) dP_{\lambda_{n,k}}(t)$$

$$= A_{n,k}(x)(C)\int_0^1 1 dP_{\lambda_{n,k}}(t) = \max\{(t_2-x)t_2^k(1-t_2)^{n-k}, (x-t_1)t_1^k(1-t_1)^{n-k}\}$$

$$\le (t_2-x)t_2^k(1-t_2)^{n-k} + (x-t_1)t_1^k(1-t_1)^{n-k}.$$

Since $\frac{t_2^k(1-t_2)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} \le 1$, $\frac{t_1^k(1-t_1)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} \le 1$ and by Lemma 4.1 we get

$$\frac{A_{n,k}(x)}{k^k n^{-n}(n-k)^{n-k}} \le (t_2-x)\cdot\frac{t_2^k(1-t_2)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} + (x-t_1)\cdot\frac{t_1^k(1-t_1)^{n-k}}{k^k n^{-n}(n-k)^{n-k}}$$

$$\le t_2 - t_1 = \frac{\sqrt{\Delta}}{n+1} \le \frac{\sqrt{(nx-k)^2 + 2x(n-k) + 2k(1-x) + 1}}{n}$$

$$\le \sqrt{(x-k/n)^2 + 2x/n + (2k/n)\cdot(1-x)/n + 1/n^2}$$

$$\le |x-k/n| + \sqrt{2x}/\sqrt{n} + (\sqrt{2k}/\sqrt{n})\cdot\sqrt{(1-x)/n} + 1/n,$$

this immediately implies

$$D_{n,\Gamma_n}(\varphi_x)(x) \le \sum_{k=0}^n p_{n,k}(x)(|x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k/n}\cdot\sqrt{(1-x)/n} + 1/n)$$

$$\le \frac{\sqrt{x(1-x)}}{\sqrt{n}} + \frac{\sqrt{2x}}{\sqrt{n}} + \frac{\sqrt{2}\sqrt{x(1-x)}}{\sqrt{n}} + \frac{1}{n} = \frac{(1+\sqrt{2})\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{n}} + \frac{1}{n}.$$

Above we have used the well-known estimate $\sum_{k=0}^n p_{n,k}(x)|x-k/n| \le \frac{\sqrt{x(1-x)}}{\sqrt{n}}$ and the Cauchy-Schwarz inequality for Bernstein polynomials, $B_n(f)(x) \le \sqrt{B_n(f^2)(x)}$, applied for $f(t) = \sqrt{t}$.

Finally, applying Theorem 3.1, (i) the proof of Theorem 4.2 follows. □

5. Examples improving the classical estimates

This section contains some concrete examples improving the classical estimates.

Example 5.1. Since the Bernstein-Durrmeyer-Choquet operators in this paper can be defined with respect to a family of Borel or Choquet measures, combined in various ways, this fact offers a very high flexibility and generality, allowing to construct operators having even better approximation properties. As a first example, it is clear that $B_n(f)(x)$ can also be viewed as the Bernstein-Durrmeyer operators in the case when Γ_n is composed by the Dirac measures $\delta_{k/n}$, $k = 0, ..., n$. With this occasion, we note that since the Dirac measures are not strictly positive, it is clear that the strict positivity of the set functions in Theorem 3.1 is not always necessary.

Example 5.2. In formula (4.6), let us replace the family Γ_n of measures of possibilities $P_{\lambda_{n,k}}$, $k = 0, ..., n$, by the family consisting in the Dirac measures $\delta_{k/n}$, $k = 0, 1, ..., n-1$, (which are Borel measures and therefore with the corresponding Choquet integrals reducing to the classical ones) together with a monotone, submodular, strictly positive set function μ. Then, denoting by $B_n(f)(x)$ the classical Bernstein operators, for D_{n,Γ_n} in (4.6) we get

$$D_{n,\Gamma_n}(f)(x) - f(x) = \left[\sum_{k=0}^{n-1} p_{n,k}(x) f\left(\frac{k}{n}\right) + x^n \cdot \frac{(C)\int_0^1 f(t)t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)}\right] - f(x)$$

$$= B_n(f)(x) - f(x) + x^n \left[\frac{(C)\int_0^1 f(t)t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)} - f(1)\right].$$

Suppose now that $f \geq 0$ is strictly increasing and strictly convex on $[0,1]$ and, for example, that $\mu(A) = \sqrt{m(A)}$ or $\mu(A) = \sin[m(A)]$, with m the Lebesgue measure. The strict convexity implies $B_n(f)(x) - f(x) > 0$ for all $x \in (0,1)$ and the property of f to be strictly increasing easily implies

$$\frac{(C)\int_0^1 f(t)t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)} - f(1) < \frac{f(1) \cdot (C)\int_0^1 t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)} - f(1) = 0.$$

So, for $x \in (0,1)$, $D_{n,\Gamma_n}(f)(x)$ approximates better than $B_n(f)(x)$, since

$$|D_{n,\Gamma_n}(f)(x) - f(x)| < \max\left\{|B_n(f)(x) - f(x)|, x^n \left|\frac{(C)\int_0^1 f(t)t^n d\mu(t)}{(C)\int_0^1 t^n d\mu(t)} - f(1)\right|\right\}.$$

Example 5.3. In formula (4.6), let us replace the family of measures of possibilities $\Gamma_n = \{P_{\lambda_{n,k}}\}_{k=0}^n$, by the family $\Gamma_n = \{\nu_{n,0}, \nu_{n,n}, \mu_{n-2,k-1}, k = 1, ..., n-1\}$, where the set functions $\mu_{n-2,k-1}, k = 1, ..., n-1$ are the Lebesgue measure, $\nu_{n,0} = \delta_0$ (Dirac measure), $\nu_{n,n}$ is a monotone, submodular and strictly positive set function and define the genuine Bernstein-Durrmeyer-Choquet operators by

$$U_{n,\Gamma_n}(f)(x) = p_{n,0}(x) \cdot \frac{(C)\int_0^1 f(t)(1-t)^n d\nu_{n,0}}{(C)\int_0^1 (1-t)^n d\nu_{n,0}} + p_{n,n}(x) \cdot \frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}}{(C)\int_0^1 t^n d\nu_{n,n}}$$

$$+ \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \frac{(C)\int_0^1 f(t)p_{n-2,k-1}(t) d\mu_{n-2,k-1}(t)}{(C)\int_0^1 p_{n-2,k-1}(t) d\mu_{n-2,k-1}(t)}.$$

Denoting by $G_n(f)(x)$, the classical genuine Bernstein-Durmeyer operator (see, e.g., [9]), we immediately obtain

$$U_{n,\Gamma_n}(f)(x) - f(x) = G_n(f)(x) - f(x) + x^n \left[\frac{(C)\int_0^1 f(t)t^n d\nu_{n,n}(t)}{(C)\int_0^1 t^n d\nu_{n,n}(t)} - f(1)\right].$$

Since the strict convexity of f implies $G_n(f)(x) - f(x) > 0$ for all $x \in (0,1)$ (see, e.g., Lemma 2.1, (iv) in [9]), similar reasonings with those for the previous example show that if $f \geq 0$ is strictly convex and strictly increasing on $[0,1]$ (and, for example, $\nu_{n,n}(A) = \sqrt{m(A)}$ or $\nu_{n,n}(A) = \sin[m(A)]$), then $U_{n,\Gamma_n}(f)(x)$ approximates better f on $(0,1)$ than $G_n(f)(x)$.

Example 5.4. In [7], for the nonlinear Picard-Choquet operators we have obtained a general estimate similar to that for the classical Picard operators, while for functions of the form $f(x) = Me^{-Ax}$, $M, A > 0$, we got there essentially better estimates.

Remark 5.3. Since the formula for the operators $D_{n,\Gamma_n}(f)$ in (4.6) involves Choquet integrals with respect to possibilistic measures, it is natural to ask for a study of the convergence properties of $D_{n,\Gamma_n}(f)$, in the case when the Choquet integrals $(C)\int_0^1$ in (4.6) are replaced (all, or some of them only !) by possibilistic integrals $(Pos)\int_0^1$, as they are defined by Definition 2.1, (ii). For example, if all the Choquet integrals are replaced by possibilistic integrals, from (4.6) we easily get the new operators (which seem still to have good convergence properties !)

$$\overline{D}_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(Pos)\int_0^1 f(t)t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}{(Pos)\int_0^1 t^k(1-t)^{n-k}dP_{\lambda_{n,k}}(t)}$$

$$= \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{\sup\{f(t)[t^k(1-t)^{n-k}]^2; t \in [0,1]\}}{\sup\{[t^k(1-t)^{n-k}]^2; t \in [0,1]\}}.$$

A detailed study of the convergence properties for all these kinds of Bernstein-Durrmeyer-possibilisitc operators will be made elsewhere.

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Department of Mathematics and Computer Science
University of Oradea
Universităţii 1, 410087 Oradea, Romania
E-mail address: galso@uoradea.ro

Babeş-Bolyai University
Faculty of Mathematics and Informatics
M. Kogăniceanu 1, 400084 Cluj-Napoca, Romania
E-mail address: sorin.trifa@yahoo.com