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New Suzuki-Berinde type fixed point results

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ABSTRACT. The aim of this article is to improve the results of Piri et al. [Fixed Point Theory and Applications 2014, 2014:210] by introducing new types of contractions say Suzuki-Berinde type F-contractions and Suzuki type rational F-contractions. We also establish a common fixed point theorem for a sequence of multivalued mappings. An example is also given to support our main results.

1. INTRODUCTION

In the theory of metric spaces, Banach fixed point theorem [4] is an important tool which guarantees the existence and uniqueness of fixed points of certain self mappings. It actually provides a constructive method to find fixed points. This theorem was first stated by a Polish Mathematician Stefan Banach in 1922.

There are many generalizations of Banach fixed point theorem in the literature. One of the most interesting generalizations is characterization of metric completeness and it was first given by Suzuki [18] in 2008. Further, Berinde [5, 6] studied many kinds of contraction mappings and gave the concept of almost contraction in following way.

Definition 1.1. [6] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be generalized almost contraction if there exists a constant $\lambda \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

Recently, in 2012 Wardowski [19] introduced new type of contractions called F-contractions and extended the contractive condition on such mappings. He defined F-contraction as follows:

Definition 1.2. [19] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a $F-$contraction if there exists $\tau > 0$ such that for $x, y \in X$;

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) \tag{1.1}$$

where, $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

- (F_1) F is strictly increasing;
- (F_2) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Consistent with Wordowski [19], we denote by Δ_F the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(F_1) - (F_3)$ conditions.

Very recently, Secelean [17] proved the following lemma and replaced condition (F_2) by an equivalent but a more simple condition $(F_{2'})$.

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Lemma 1.1. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:*

(a) if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$;
(b) if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.
He replaced (F_2) with the following condition.
$(F_{2'})$ $\inf F = -\infty$
or, also, by
$(F_{2''})$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

On the other hand, Piri et al. [16] utilized much simple condition $(F_3^{'})$ instead of (F_3) in F-contraction and established some new fixed point theorems regarding this condition.
$(F_{3'})$ F is continuous on $(0, \infty)$.
We denote by $\digamma$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F_1), $(F_2^{'})$ and $(F_3^{'})$. For more details in this direction, we refer the reader to[1, 2, 7, 9, 10, 11, 12, 13, 14, 15].

In this paper, we first generalize the results of Piri et al. [16] by introducing Suzuki-Berinde type F-contraction in the setting of complete metric spaces. Then we give the notion of Suzuki type rational F-contraction and establish some new fixed point results regarding F-contraction and rational expressions. We also prove a fixed point theorem for a sequence of multi-valued mappings in ε-chainable metric spaces.

2. Fixed point results for Suzuki-Berinde type F-contraction

In the present section, we define Suzuki-Berinde type F-contractions to prove some fixed point theorems in the context of complete metric spaces. Our new results are proper generalization of Piri et al. [16].

Definition 2.3. Let (X, d) be a metric space and T be a self-mapping on X. We say that T is Suzuki-Berinde type F-contraction if there exist $F \in \digamma$, $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have

$$\frac{1}{2}d(x, Tx) < d(x, y) \tag{2.1}$$

implies

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}. \tag{2.2}$$

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying Suzuki-Berinde type F-contraction. Then T has a unique fixed point $z \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ is convergent to z.*

Proof. Let $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. If there exist $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is fixed point of T and we have nothing to prove. So we assume that $x_n \neq x_{n+1}$ or

$$0 < d(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Therefore

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, Tx_n) \tag{2.3}$$

for all $n \in \mathbb{N}$. It follows from assumption of theorem that

$$\tau + F\big(d(x_n, Tx_n)\big) \leq F\big(d(x_{n-1}, Tx_{n-1})\big) + L \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}$$

which implies that,

$$\begin{aligned}F\big(d(x_n,Tx_n)\big) &\leq F\big(d(x_{n-1},Tx_{n-1})\big)+L\min\{d(x_{n-1},x_n),d(x_{n-1},x_{n+1}),d(x_n,x_n)\}-\tau\\ &\leq F\big(d(x_{n-1},Tx_{n-1})\big)-\tau.\end{aligned}$$

Therefore

$$F\big(d(x_n,Tx_n)\big)\leq F\big(d(x_{n-1},Tx_{n-1})\big)-\tau\leq\ldots\leq F(d(x_0,Tx_0))-n\tau \tag{2.4}$$

for all $n\in\mathbb{N}$. Since $F\in\digamma$, so by taking limit as $n\to\infty$ in (2.4) we have,

$$\lim_{n\to\infty}F\big(d(x_n,Tx_n)\big)=-\infty\Longleftrightarrow\lim_{n\to\infty}d(x_n,Tx_n)=0. \tag{2.5}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence, then we assume that there exists $\varepsilon>0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n)>q(n)>n$, we have

$$d(x_{p(n)},x_{q(n)})\geq\varepsilon. \tag{2.6}$$

Then

$$d(x_{p(n)-1},x_{q(n)})<\varepsilon$$

for all $n\in\mathbb{N}$. So, by triangle inequality and (2.6), we have

$$\begin{aligned}\varepsilon &\leq d(x_{p(n)},x_{q(n)})\leq d(x_{p(n)},x_{p(n)-1})+d(x_{p(n)-1},x_{q(n)})\\ &\leq d(x_{p(n)-1},Tx_{p(n)-1})+\varepsilon.\end{aligned}$$

By taking the limit and using inequality (2.5), we get

$$\lim_{n\to\infty}d(x_{p(n)},x_{q(n)})=\varepsilon. \tag{2.7}$$

From (2.5) and (2.7), we can choose a natural number $n_0\in\mathbb{N}$ such that

$$\frac{1}{2}d(x_{p(n)},Tx_{p(n)})<\frac{\varepsilon}{2}<d(x_{p(n)},x_{q(n)}) \tag{2.8}$$

for all $n\geq n_0$. Then by the assumption, we get

$$\begin{aligned}\tau+F\big(d(Tx_{p(n)},Tx_{q(n)})\big) &\leq F\big(d(x_{p(n)},x_{q(n)})\big)\\ &\quad+L\min\{d(x_{p(n)},Tx_{p(n)}),d(x_{p(n)},Tx_{q(n)}),d(x_{q(n)},Tx_{p(n)})\}\\ &= F\big(d(x_{p(n)},x_{q(n)})\big)\\ &\quad+L\min\{d(x_{p(n)},x_{p(n)+1}),d(x_{p(n)},x_{q(n)+1}),d(x_{q(n)},x_{p(n)+1})\}\end{aligned}$$

By taking limit as $n\to+\infty$ and using $(F3')$, (2.5) and (2.7), we get

$$\tau+F(\varepsilon)\leq F(\varepsilon)$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $z\in X$ such that, $x_n\to z$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty}d(x_n,z)=0. \tag{2.9}$$

Next, we claim that

$$\frac{1}{2}d(x_n,Tx_n)<d(x_n,z)\text{ or }\frac{1}{2}d(Tx_n,T^2x_n)<d(Tx_n,z) \tag{2.10}$$

for all $n\in\mathbb{N}$. We suppose on the contrary that there exists $m\in\mathbb{N}$ such that

$$\frac{1}{2}d(x_m,Tx_m)\geq d(x_m,z)\text{ or }\frac{1}{2}d(Tx_m,T^2x_m)\geq d(Tx_m,z). \tag{2.11}$$

Therefore

$$\begin{aligned} 2d(x_m, z) &\leq d(x_m, Tx_m) \\ &\leq d(x_m, z) + d(z, Tx_m) \end{aligned}$$

which implies that

$$d(x_m, z) \leq d(z, Tx_m). \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$d(x_m, z) \leq d(z, Tx_m) \leq \frac{1}{2} d(Tx_m, T^2 x_m). \tag{2.13}$$

Since

$$\frac{1}{2} d(x_m, Tx_m) < d(x_m, Tx_m).$$

So by assumption, we get

$$\tau + F\big(d(Tx_m, T^2x_m)\big) \leq F\big(d(x_m, Tx_m)\big) + L\min\{d(x_m, Tx_m), d(x_m, T^2x_m), d(Tx_m, Tx_m)\}$$

which implies that

$$\tau + F\big(d(Tx_m, T^2x_m)\big) \leq F\big(d(x_m, Tx_m)\big).$$

Since F is strictly increasing, so we have

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m). \tag{2.14}$$

It follows from (2.11), (2.13) and (2.14) that

$$\begin{aligned} d(Tx_m, T^2x_m) &< d(x_m, Tx_m) \\ &\leq d(x_m, z) + d(z, Tx_m) \\ &\leq \frac{1}{2} d(Tx_m, T^2x_m) + \frac{1}{2} d(Tx_m, T^2x_m) \\ &= d(Tx_m, T^2x_m). \end{aligned}$$

This is contradiction. Hence (2.10) holds. So from (2.10), for every $n \in \mathbb{N}$, we have

$$\tau + F\big(d(Tx_n, Tz)\big) \leq F\big(d(x_n, z)\big) + L\min\{d(x_n, Tx_n), d(x_n, Tz), d(z, Tx_n)\}$$

which implies that

$$\tau + F\big(d(Tx_n, Tz)\big) \leq F\big(d(x_n, z)\big) + L\min\{d(x_n, x_{n+1}), d(x_n, Tz), d(z, x_{n+1})\}. \tag{2.15}$$

Using (2.15), $(F2^{'})$ and Lemma 1.1, we get

$$\lim_{n\to\infty} F\big(d(Tx_n, Tz)\big) = -\infty.$$

It follows from $(F2^{'})$ and Lemma 1.1, we obtain

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0. \tag{2.16}$$

So

$$d(z, Tz) = \lim_{n\to\infty} d(x_{n+1}, Tz) = \lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

Hence, z is a fixed point of T. Now we show the uniqueness of fixed point. We suppose on the contrary that there exist an other fixed point u of T distinct from z that is

$$Tz = z \neq u = Tu.$$

Then

$$d(Tz, Tu) > 0.$$

So we get

$$0 = \frac{1}{2}d(z, Tz) < d(z, u).$$

Then from assumption of theorem, we obtain

$$\begin{aligned} F\big(d(z,u)\big) &= F\big(d(Tz,Tu)\big) < \tau F\big(d(Tz,Tu)\big) \\ &\leq F\big(d(z,u)\big) + L\min\{d(z,Tz), d(z,Tu), d(u,Tz)\} \end{aligned}$$

which further implies that

$$F\big(d(z,u)\big) < F\big(d(z,u)\big).$$

This is contradiction. Thus z is the unique fixed point of T. □

Corollary 2.1. [16]*Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $\tau > 0$ and $F \in \digamma$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have*

$$\tfrac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow \tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

Then T has a unique fixed point $z \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ is convergent to z.

Corollary 2.2. [16]*Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exist $F \in \digamma$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have*

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ is convergent to x^*.

3. Fixed point results for Suzuki type rational F-contractions

In this section, we first introduce Suzuki type rational F-contractions and then establish some fixed point results regarding these contractions.

Definition 3.4. Let (X, d) be a metric space and T be a self-mapping on X. We say T is Suzuki type rational $F-$contraction if for all $x, y \in X$ with $Tx \neq Ty$, we have

$$\tfrac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow \tau + F\big(d(Tx,Ty)\big) \leq F\big(R(x,y)\big) \tag{3.17}$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(Ty,y), \frac{d(Ty,y)(1 + d(x,Tx))}{1 + d(x,y)}\}$$

for some $\tau > 0$ and $F \in \digamma$.

Theorem 3.2. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a Suzuki type rational F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ is convergent to x^*.*

Proof. Let x_0 be an arbitrary but fixed element X. We define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. If there exists some $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is the required fixed point of T. So we assume that $x_n \neq x_{n+1}$ or

$$0 < d(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Therefore

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, Tx_n)$$

for all $n \in \mathbb{N}$. Then by the given assumption, we have

$$\begin{aligned}\tau + F\big(d(Tx_{n-1}, Tx_n)\big) &\leq F\Big(\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n),\\ &\qquad \frac{d(x_n, Tx_n)(1 + d(x_{n-1}, Tx_{n-1}))}{1 + d(x_{n-1}, x_n)}\Big)\\ &= F\Big(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}),\\ &\qquad \frac{d(x_n, x_{n+1})(1 + d(x_{n-1}, x_n))}{1 + d(x_{n-1}, x_n)}\Big)\\ &= F\Big(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\Big).\end{aligned}$$

If

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}).$$

Then

$$\tau + F\big(d(x_n, x_{n+1})\big) = \tau + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_n, x_{n+1})\big)$$

we get a contradiction to the fact that F is strictly increasing and $\tau > 0$. Thus

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Hence

$$\tau + F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big)$$

which implies that

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

Therefore

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau \leq \ldots \leq F(d(x_0, x_1)) - n\tau \tag{3.18}$$

for all $n \in \mathbb{N}$. Since $F \in \digamma$, so by taking limit as $n \to \infty$ in (3.18) we have,

$$\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty \iff \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.19}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not Cauchy then we assume there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d(x_{p(n)}, x_{q(n)}) \geq \varepsilon. \tag{3.20}$$

Then

$$d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (3.20), we have

$$\begin{aligned}\varepsilon &\leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})\\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.\end{aligned}$$

By taking the limit and using inequality (3.19), we get

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{3.21}$$

From (3.19) there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2} d(x_{p(n)}, Tx_{p(n)}) < \frac{\varepsilon}{2} < d(x_{p(n)}, x_{q(n)}) \tag{3.22}$$

for all $n \geq n_0$. Thus by the given assumptions, we get

$$\begin{aligned}\tau + F\big(d(Tx_{p(n)}, Tx_{q(n)})\big) &\leq F\Big(\max\{d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}),\\ &\qquad \frac{d(x_{q(n)}, Tx_{q(n)})[1 + d(x_{p(n)}, Tx_{p(n)})]}{1 + d(x_{p(n)}, x_{q(n)})}\Big).\end{aligned}$$

If

$$\max\{d(x_{p(n)},x_{q(n)}),d(x_{p(n)},Tx_{p(n)}),d(x_{q(n)},Tx_{q(n)}),\frac{d(x_{q(n)},Tx_{q(n)})[1+d(x_{p(n)},Tx_{p(n)})]}{1+d(x_{p(n)},x_{q(n)})}$$
$$= d(x_{p(n)},x_{q(n)}).$$

By taking limit as $n\to+\infty$ and using $(F_3{}^{'})$,(3.17), (3.19) and (3.21), we get

$$\tau+F(\varepsilon)\le F(\varepsilon)$$

which is a contradiction. If

$$\max\{d(x_{p(n)},x_{q(n)}),d(x_{p(n)},Tx_{p(n)}),d(x_{q(n)},Tx_{q(n)}),\frac{d(x_{q(n)},Tx_{q(n)})[1+d(x_{p(n)},Tx_{p(n)})]}{1+d(x_{p(n)},x_{q(n)})}$$
$$= d(x_{p(n)},Tx_{p(n)}).$$

By taking limit as $n\to+\infty$ and using $(F_3{}^{'})$ (3.17), (3.19) and (3.21), we get

$$\tau+F(\varepsilon)\le F(\tfrac{\varepsilon}{2})$$

which is a contradiction. If

$$\max\{d(x_{p(n)},x_{q(n)}),d(x_{p(n)},Tx_{p(n)}),d(x_{q(n)},Tx_{q(n)}),\frac{d(x_{q(n)},Tx_{q(n)})[1+d(x_{p(n)},Tx_{p(n)})]}{1+d(x_{p(n)},x_{q(n)})}$$
$$= d(x_{q(n)},Tx_{q(n)}).$$

By taking limit as $n\to+\infty$ and using $(F_3{}^{'})$,(3.17), (3.19) and (3.21), we get

$$\tau+F(\varepsilon)\le F(\tfrac{\varepsilon}{2})$$

a contradicition. If

$$\max\{d(x_{p(n)},x_{q(n)}),d(x_{p(n)},Tx_{p(n)}),d(x_{q(n)},Tx_{q(n)}),\frac{d(x_{q(n)},Tx_{q(n)})[1+d(x_{p(n)},Tx_{p(n)})]}{1+d(x_{p(n)},x_{q(n)})}$$
$$= \frac{d(x_{q(n)},Tx_{q(n)})[1+d(x_{p(n)},Tx_{p(n)})]}{1+d(x_{p(n)},x_{q(n)})}.$$

By taking limit as $n\to+\infty$ and using $(F_3{}^{'})$,(3.17), (3.19) and (3.21), we get

$$\tau+F(\varepsilon)\le F(\varepsilon).$$

This is also a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^*\in X$ such that, $x_n\to x^*$ as $n\to\infty$ that is $\lim_{n\to\infty}d(x_n,x^*)=0$. Now we prove that $x^*=Tx^*$. For this we claim that

$$\frac{1}{2}d(x_n,Tx_n)<d(x_n,x^*) \text{ or } \frac{1}{2}d(Tx_n,T^2x_n)<d(Tx_n,x^*) \tag{3.23}$$

for all $n\in\mathbb{N}$. We suppose on the contrary that there exists $k\in\mathbb{N}$ such that

$$\frac{1}{2}d(x_k,Tx_k)\ge d(x_k,z) \text{ or } \frac{1}{2}d(Tx_k,T^2x_k)\ge d(Tx_k,x^*). \tag{3.24}$$

Therefore

$$\begin{aligned} 2d(x_k,x^*) &\le d(x_k,Tx_k)\\ &\le d(x_k,x^*)+d(x^*,Tx_k) \end{aligned}$$

which implies that

$$d(x_k,x^*)\le d(x^*,Tx_k). \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$d(x_k, x^*) \leq d(x^*, Tx_k) \leq \frac{1}{2}d(Tx_k, T^2x_k). \tag{3.26}$$

Since

$$\frac{1}{2}d(x_k, Tx_k) < d(x_k, Tx_k).$$

So by assumption, we get

$$\begin{aligned}\tau + F\big(d(Tx_k, T^2x_k)\big) \quad &\leq \quad F\big(\max\{d(x_k, Tx_k), d(x_k, Tx_k), d(Tx_k, T^2x_k),\\ &\qquad \frac{d(Tx_k, T^2x_k)(1 + d(x_k, Tx_k))}{1 + d(x_k, Tx_k)}\big)\end{aligned}$$

which implies that

$$\tau + F\big(d(Tx_k, T^2x_k)\big) \leq F\big(\max\{d(x_k, Tx_k), d(Tx_k, T^2x_k)\}\big).$$

If

$$\max\{d(x_k, Tx_k), d(Tx_k, T^2x_k)\} = d(Tx_k, T^2x_k)$$

then

$$\tau + F\big(d(Tx_k, T^2x_k)\big) \leq F\big(d(Tx_k, T^2x_k)\big)$$

and we get a contradiction to the fact that $\tau > 0$. If

$$\max\{d(x_k, Tx_k), d(Tx_k, T^2x_k)\} = d(x_k, Tx_k)$$

then

$$\tau + F\big(d(Tx_k, T^2x_k)\big) \leq F\big(d(x_k, Tx_k)\big).$$

Since F is strictly increasing, so we have

$$d(Tx_k, T^2x_k) < d(x_k, Tx_k). \tag{3.27}$$

It follows from (3.24), (3.26) and (3.27) that

$$\begin{aligned}d(Tx_k, T^2x_k) \quad &< \quad d(x_k, Tx_k)\\ &\leq \quad d(x_k, x^*) + d(x^*, Tx_k)\\ &\leq \quad \frac{1}{2}d(Tx_k, T^2x_k) + \frac{1}{2}d(Tx_k, T^2x_k)\\ &= \quad d(Tx_k, T^2x_k)\end{aligned}$$

which is a contradiction. Hence the inequality (3.23) holds and by the given assumption, we get

$$\begin{aligned}&\tau + F\big(d(x_{n+1}, Tx^*)\big) = \tau + F\big(d(Tx_n, Tx^*)\big)\\ &\leq F\big(\max\{d(x_n, x^*), d(x_n, Tx_n), d(Tx^*, x^*), \frac{d(Tx^*, x^*)[1+d(x_n, Tx_n)]}{1+d(x_n, x^*)}\}\big)\\ &\leq F\big(\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(Tx^*, x^*), \frac{d(Tx^*, x^*)[1 + d(x_n, x_{n+1})]}{1+d(x_n, x^*)}\}\big).\end{aligned}$$

If

$$\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(Tx^*, x^*), \frac{d(Tx^*, x^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)}\} = d(x_n, x^*)$$

then

$$\tau + F\big(d(x_{n+1}, Tx^*)\big) \leq F\big(d(x_n, x^*)\big).$$

which further implies that

$$F\big(d(x_{n+1}, Tx^*)\big) \leq F\big(d(x_n, x^*)\big) - \tau.$$

Since F is strictly increasing, so we get

$$d(x_{n+1}, Tx^*) < d(x_n, x^*).$$

Taking limit as $n \to +\infty$, we get

$$d(x^*, Tx^*) \leq 0.$$

Thus x^* is a fixed point of T. If

$$\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x^*, Tx^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)}\} = d(x_n, x_{n+1}).$$

Then by the same procedure as above, one can easily get x^* as a fixed point of T. If

$$\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x^*, Tx^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)}\} = d(x^*, Tx^*).$$

Then

$$\tau + F\big(d(x_{n+1}, Tx^*)\big) \leq F\big(d(x^*, Tx^*)\big)$$

which further implies that

$$F\big(d(x_{n+1}, Tx^*)\big) \leq F\big(d(x^*, Tx^*)\big) - \tau.$$

Since F is strictly increasing, so we get

$$d(x_{n+1}, Tx^*) < d(x^*, Tx^*).$$

Taking limit as $n \to +\infty$, we get

$$d(x^*, Tx^*) \leq 0.$$

Thus by (3.26), we get

$$\tau + F\big(d(x_{n+1}, Tx^*)\big) \leq F\big(d(x^*, Tx^*)\big)$$

for all $n \geq \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in above inequality, we get

$$\tau + F\big(d(x^*, Tx^*)\big) \leq F\big(d(x^*, Tx^*)\big)$$

which is a contradiction. Therefore $d(x^*, Tx^*) = 0$, that is x^* is a fixed point of T. The uniqueness is similar to the above main result. □

4. Fixed point results for multivalued mappings

Now we state the main result of this section as follows.

Theorem 4.3. *Let (X, d) be a complete ε-chainable metric space and $\{T_n\}_{n=1}^{\infty} : X \to CB(X)$ be the sequence of mappings. Assume that there exist a function $F \in \digamma$ which is continuous from right and $\tau > 0$ such that*

$$\forall x, y \in X \text{ and } 0 < d(x, y) < \varepsilon \Rightarrow 2\tau + F\big(H(T_n x, T_m y)\big) \leq F\big(d(x, y)\big) \tag{4.28}$$

for all $n, m = 1, 2, ...$Then there exists a point $u^* \in X$ such that $u^* \in \cap_{n=1}^{\infty} T_n u^*$.

Proof. Let $y_0 \in X$ be an arbitrary but fixed element. We define a sequence $\{y_n\}$ of points of X in the following way. Let $y_1 \in X$ be such that $y_1 \in Ty_0$ and

$$y_0 = x_{(1,0)}, x_{(1,1)}, x_{(1,2)}, ..., x_{(1,m)} = y_1 \in Ty_0 \tag{4.29}$$

be an arbitrary ε-chain from y_0 to y_1. Rename the following y_1 as $x_{(2,0)}$. Since $x_{(2,0)} \in Tx_{(1,0)}$, so from (4.28), we have

$$2\tau + F\big(H(T_1 x_{(1,0)}, T_2 x_{(1,1)})\big) \leq F\big(d(x_{(1,0)}, x_{(1,1)})\big).$$

By F_1, we get

$$\begin{aligned} H(T_1x_{(1,0)}, T_2x_{(1,1)}) &< d(x_{(1,0)}, x_{(1,1)}) \\ &< \epsilon. \end{aligned}$$

Since F is continuous from the right, so there exists a real number $r > 1$ such that

$$F\big(rH(T_1x_{(1,0)}, T_2x_{(1,1)})\big) < F\big(H(T_1x_{(1,0)}, T_2x_{(1,1)})\big) + \tau.$$

We can choose $x_{(2,1)} \in T_2x_{(1,1)}$ such that

$$d(x_{(2,0)}, x_{(2,1)}) \le rH(T_1x_{(1,0)}, T_2x_{(1,1)}).$$

Consequently, we have

$$\begin{aligned} F(d(x_{(2,0)}, x_{(2,1)})) &\le F(rH(T_1x_{(1,0)}, T_2x_{(1,1)})) \\ &< F\big(H(T_1x_{(1,0)}, T_2x_{(1,1)})\big) + \tau \end{aligned}$$

which implies

$$\begin{aligned} 2\tau + F(d(x_{(2,0)}, x_{(2,1)})) &< 2\tau + F\big(H(T_1x_{(1,0)}, T_2x_{(1,1)})\big) + \tau \\ &\le F\big(d(x_{(1,0)}, x_{(1,1)})\big) + \tau. \end{aligned} \tag{4.30}$$

Since F is strictly increasing, we deduce

$$d(x_{(2,0)}, x_{(2,1)})) < d(x_{(1,0)}, x_{(1,1)}) < \varepsilon.$$

Since $x_{(2,1)} \in T_2x_{(1,1)}$, so from (4.28), we get

$$2\tau + F\big(H(T_2x_{(1,1)}, T_2x_{(1,2)})\big) \le F\big(d(x_{(1,1)}, x_{(1,2)})\big).$$

Since F is strictly increasing, we have

$$\begin{aligned} H(T_2x_{(1,1)}, T_2x_{(1,2)}) &< d(x_{(1,1)}, x_{(1,2)}) \\ &< \epsilon. \end{aligned}$$

Since F is continuous from the right, there exists a real number $r > 1$ such that

$$F\big(rH(T_2x_{(1,1)}, T_2x_{(1,2)})\big) < F\big(H(T_2x_{(1,1)}, T_2x_{(1,2)})\big) + \tau$$

We can choose $x_{(2,2)} \in T_2x_{(1,2)}$ such that $d(x_{(2,1)}, x_{(2,2)}) \le rH(T_2x_{(1,1)}, T_2x_{(1,2)})$. Consequently, we get

$$\begin{aligned} F(d(x_{(2,1)}, x_{(2,2)})) &\le F(rH(T_2x_{(1,1)}, T_2x_{(1,2)})) \\ &< F\big(H(T_2x_{(1,1)}, T_2x_{(1,2)})\big) + \tau \end{aligned}$$

which implies

$$\begin{aligned} 2\tau + F(d(x_{(2,1)}, x_{(2,2)})) &< 2\tau + F\big(H(T_2x_{(1,1)}, T_2x_{(1,2)})\big) + \tau \\ &\le F\big(d(x_{(1,1)}, x_{(1,2)})\big) + \tau. \end{aligned} \tag{4.31}$$

Since F is strictly increasing, we deduce

$$d(x_{(2,1)}, x_{(2,2)}) < d(x_{(1,1)}, x_{(1,2)}) < \varepsilon.$$

Thus we can get a finite set of points $x_{(2,1)}, x_{(2,2)}, ..., x_{(2,m)}$ such that $x_{(2,0)} \in T_1x_{(1,0)}$ and $x_{(2,j)} \in T_2x_{(1,j)}$, for $j = 1, 2, ..., m$, with

$$d(x_{(2,j)}, x_{(2,j+1)}) < d(x_{(1,j)}, x_{(1,j+1)}) < \varepsilon$$

for $j = 0, 1, 2, ...m - 1$. Let $x_{(2,m)} = y_2$, then the set of points

$$y_1 = x_{(2,0)}, x_{(2,1)}, x_{(2,2)}, ..., x_{(2,m)} = y_2 \in T_2y_1$$

is an ε-chain from y_1 to y_2. Rename the following y_2 as $x_{(3,0)}$, then by the same procedure we obtain an ε-chain

$$y_2 = x_{(3,0)}, x_{(3,1)}, x_{(3,2)}, ..., x_{(3,m)} = y_3 \in T_3 y_2$$

from y_2 to y_3. Inductively, we obtain

$$y_n = x_{(n+1,0)}, x_{(n+1,1)}, x_{(n+1,2)}, ..., x_{(n+1,m)} = y_{n+1} \in T_{n+1} y_n$$

with

$$d(x_{(n+1,j)}, x_{(n+1,j+1)}) < d(x_{(n,j)}, x_{(n,j+1)}) < \varepsilon$$

for $j = 0, 1, 2, ..., m-1$ and $n = 0, 1, 2, ...$ Consequently, we generate a sequence $\{y_n\}$ of points of X with

$$\begin{array}{rcl} y_1 & = & x_{(1,m)} = x_{(2,0)} \in T_1 y_0 \\ y_2 & = & x_{(2,m)} = x_{(3,0)} \in T_2 y_1 \\ y_3 & = & x_{(3,m)} = x_{(4,0)} \in T_3 y_2 \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

that is

$$y_{n+1} = x_{(n+1,m)} = x_{(n+2,0)} \in T_{n+1} y_n$$

for $n = 0, 1, 2, ...$ and

$$\begin{array}{rcl} 2\tau + F(d(y_n, y_{n+1}) & < & 2\tau + F\big(H(T_n y_{n-1}, T_{n+1} y_n)\big) + \tau \\ & \leq & F\big(d(y_{n-1}, y_n)\big) + \tau \end{array}$$

that is

$$\begin{array}{rcl} F(d(y_n, y_{n+1})) & < & F\big(d(y_{n-1}, y_n)\big) - \tau \\ & & . \\ & & . \\ & & . \\ & < & F\big(d(y_0, y_1)\big) - n\tau \end{array}$$

for all $n = 1, 2, ...$ It follows by similar to the above Theorem 2.1 and Theorem 3.2 that $\{y_n\}$ is a Cauchy sequence. Since X is complete, so $y_n \to u^*$. Hence there exists an integer $M > 0$ such that $n > M$ implies $d(y_n, u^*) < \varepsilon$. Thus by the inequality (4.28), we have

$$\begin{array}{rcl} 2\tau + F\big(d(y_{n+1}, T_j u^*)\big) & \leq & 2\tau + F\big(H(T_{n+1} y_n, T_j u^*)\big) \\ & \leq & F\big(d(y_n, u^*)\big) + \tau. \end{array}$$

Since F is strictly increasing, we have

$$d(y_{n+1}, T_j u^*) < d(y_n, u^*).$$

Letting $n \to +\infty$ in the previous inequality, we get $d(u^*, T_j u^*) = 0$ which implies that $u^* \in T_j u^*$. Therefore $u^* \in \cap_{n=1}^{\infty} T_n u^*$. $\square$

Example 4.1. *Consider the sequence $\{S_n\}$ as follows:*

$S_n = 1.1! + 2.2! + 3.3! + ... + n.n! = (n+1)! - 1$.

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mapping $T : X \to X$ by

$$T(S_1) = S_1, \quad T(S_n) = S_{n-1}, \quad \text{for all } n > 1.$$

First, let us consider the mapping $F(t) = \ln(t)$. The mapping T is not the F-contraction in this case (which actually means that T is not Suzuki- type F-contraction with $\tau = 8$).

Indeed, for $n = 1$ and $m = 4$, we get

$$\frac{1}{2}d(S_n, T(S_n)) \leq d(S_n, S_m)$$

implies

$$\tau + \ln(d(T(S_n), T(S_m))) \geq \ln(d(S_n, S_m))$$

because

$$8 + \ln(2.2! + 3.3!) \geq \ln(2.2! + 3.3! + 4.4!)$$

that is

$$8 + \ln(d(S_1, S_3)) \geq \ln(d(S_1, S_4)).$$

Let us consider the mapping $F(t) = \frac{-1}{t} + t$, we obtain that T is Suzuki-Berinde type F-contraction with $\tau = 8$.

To see this, let us consider the following calculations. We discuss our main result for $[(1 = n < m) \vee (1 = m < n) \vee (1 < n < m)]$.

For $1 = n < m$, we have

$$|T(S_m) - T(S_1)| = |S_{m-1} - S_1| = 2.2! + 3.3! + ... + (m-1).(m-1)! \tag{4.32}$$

$$d(S_m, S_1) = |S_m - S_1| = 2.2! + 3.3! + ... + m.(m)! \tag{4.33}$$

Since $m > 1$, so we have

$$\begin{aligned} & 8 - \frac{-1}{2.2! + 3.3! + ... + (m-1).(m-1)!} + 2.2! + 3.3! + ... + (m-1).(m-1)! \\ < \quad & -\frac{-1}{2.2! + 3.3! + ... + m.(m)!} + 2.2! + 3.3! + ... + (m-1).(m-1)! + m.(m)! \\ & + \min\{m.(m)!, 2.2! + 3.3! + ... + m.(m)!, 2.2! + 3.3! + ... + (m-1).(m-1)!\}. \end{aligned}$$

Thus from the equalities (4.32) and (4.33), we have

$$\begin{aligned} 8 - \frac{1}{|T(S_m), T(S_1)|} + |T(S_m), T(S_1)| < -\frac{1}{|S_m - S_1|} + |S_m - S_1| \\ + L\min\{|S_m, T(S_m)|, |S_m, T(S_1)|, |S_1, T(S_m)|\}. \end{aligned}$$

For every $m, n \in N$ with $m > n > 1$, we have

$$|T(S_m) - T(S_n)| = n.(n)! + (n+1).(n+1)! + ... + (m-1).(m-1)! \tag{4.34}$$

$$|S_m - S_n| = (n+1).(n+1)! + ... + (m).(m)!. \tag{4.35}$$

Since $m > n > 1$, we have

$$(m).(m)! \geq (n+1).(n+1)! = (n).(n+1)! + (n+1)! > (n).(n)! + (n)! + (n+1)! \geq (n).(n)! + 8.$$

From above, we get

$$\begin{aligned} & 8 - \frac{-1}{n.(n)! + (n+1).(n+1)! + ... + (m-1).(m-1)!} \\ & + n.(n)! + (n+1).(n+1)! + ... + (m-1).(m-1)! \\ < \quad & -\frac{-1}{(n+1).(n+1)! + ... + (m).(m)!} \\ & (n+1).(n+1)! + ... + (m).(m)! \\ & + \min\{m.(m)!, 2.2! + 3.3! + ... + m.(m)!, 2.2! + 3.3! + ... + (m-1).(m-1)!\}. \end{aligned}$$

So from the equalities (4.34) and (4.35), we have

$$8-\frac{1}{|T(S_m),T(S_1)|}+|T(S_m),T(S_1)|<-\frac{1}{|S_m-S_1|}+|S_m-S_1| \\ +L\min\{|S_m,T(S_m)|,|S_m,T(S_1)|,|S_1,T(S_m)|\}.$$

Hence all the conditions of Theorem 2.1 are satisfied and S_1 is a unique fixed point of mapping T. Notice that the above mentioned results can not be applied on this example as $L \neq 0$.

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