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Blow up of solutions for 3D quasi-linear wave equations with positive initial energy

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ABSTRACT. In this paper we investigate blow up property of solutions for a system of nonlinear wave equations with nonlinear dissipations and positive initial energy in a bounded domain in $\mathbb{R}^3$. Our result improves and extends earlier results in the literature such as the ones in [Zhou, J. and Mu, C., *The lifespan for 3D quasilinear wave equations with nonlinear damping terms*, Nonlinear Anal., **74** (2011), 5455–5466] and [Pişkin, E., *Uniform decay and blow-up of solutions for coupled nonlinear Klein-Gordon equations with nonlinear damping terms*, Math. Meth. Appl. Scie., **37** (2014), No. 18, 3036–3047] in which the nonexistence results obtained only for negative initial energy or the one in [Ye, Y., *Global existence and nonexistence of solutions for coupled nonlinear wave equations with damping and source terms*, Bull. Korean Math. Soc., **51** (2014), No. 6, 1697–1710] where blow up results have been not addressed. Estimate for the lower bound of the blow up time is also given.

1. INTRODUCTION

This paper deals with the blow up of solutions for the following problem:

$$\begin{cases} \partial_t^2 u_i - \Delta u_i + |\partial_t u_i|^{p_i-1}\partial_t u_i + m_i^2 u_i = f_i(u_1, \ldots, u_n), \text{ in } \Omega_T, \ (i = 1, \ldots, n), \\ u_i = 0, \quad \text{on} \quad \Gamma_T, \ (i = 1, \ldots, n), \\ u_i(x, 0) = \varphi_i(x), \quad \text{in} \quad \Omega, \ (i = 1, \ldots, n), \\ \partial_t u_i(x, 0) = \psi_i(x), \quad \text{in} \quad \Omega, \ (i = 1, \ldots, n), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded open set with smooth boundary $\partial\Omega$ and Δ denotes the Laplacian operator in $\mathbb{R}^3$, T is a positive constant, $\Omega_T = \Omega \times (0, T)$, $\Gamma_T = \partial\Omega \times (0, T)$, $n \geq 2$ is an integer, $m_i \geq 0$, $p_i \geq 1$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$ are given functions such that $f_i = \partial_{u_i} F$, $(i = 1, \ldots, n)$ where

$$F(u_1, \ldots, u_n) = a\left|\sum_{i=1}^n u_i\right|^4 + 2b\left|\prod_{i=1}^n u_i\right|^2,$$

with $a, b > 0$.

When $n = 2$ and $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ the problem (1.1) has been investigated by many authors. In the case $m_i = 0$ $(i = 1, \ldots, n)$ and $N = 1, 2, 3$, Agre and Rammaha [1] proved the existence of global solutions if $\min\{p_1, p_2\} \geq 3$ and showed the blow up of solutions if $3 > \min\{p_1, p_2\}$ when the initial energy is negative. Later, these results improved by Alves et al. [2] where global existence result obtained by a method involving the Nehari manifold and a blow up result established when the initial energy is considered to be non-negative. Recently, this blow up result has been improved by Said-Houari [21] in which a certain class of initial data with positive initial energy is considered. In this regard and in connecting with global existence and nonexistence of solutions for coupled nonlinear wave equations with damping and source terms we refer to the studies by Messaoudi and

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Houari [14], Said-Houari et al. [23], Said-Houari [22], Wu [27], Kafini and Messaoudi [5], Li et al. [6], the studies by Liu [10, 11, 12] and a recent work by Ye [29] and the references in these works.

In the case $m_i \geq 0$ $(i = 1, \ldots, n)$ and $n = 2$, the system in (1.1) converts to a Klein-Gordon type model. Reed in [20] considered the problem with

$$f_1(u_1, u_2) = -4\lambda(u_1 + \alpha u_2)^3 - 2\beta u_1 u_2^2,$$
$$f_2(u_1, u_2) = -4\alpha\lambda(u_1 + \alpha u_2)^3 - 2\beta u_1^2 u_2,$$

which defines the motion of charged mesons in an electro-magnetic field (see [24]). For more results concerning the existence of weak, global and nonglobal solutions we refer to Jörgens [4], Makhankov [13], Medeiros and Menzala [16] and the works by Miranda and Medeiros [15, 17]. In [7], Li and Tsai considered the system

$$\partial_t^2 u_i - \Delta u_i + a_i|\partial_t u_i|^{p_i-1}\partial_t u_i + m_i^2 u_i = f_i(u_1, u_2), \quad a_i \geq 0, \quad i = 1, 2, \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ when $a_i = 0$, $(i = 1, 2)$. Under Dirichlet boundary conditions, some considerations on initial data and the source terms, they obtained global existence, uniqueness and blow-up of solutions. Recently, when $a_i > 0, p_i = 1$ $(i = 1, 2)$ and without setting any restriction on upper bound of the initial energy, Wu in [26] extended their blow up result and investigated the local existence and established a sufficient condition of the initial data with arbitrarily high initial energy such that the corresponding local solution blows up in finite time. More recently, Ye [28] considered (1.2) for the case $a_i > 0, p_i > 1$ $(i = 1, 2)$ with the sources $f_1(u_1, u_2) = b|u_1|^\beta u_1 |u_2|^{\beta+2}$, $f_2(u_1, u_2) = b|u_2|^\beta u_2 |u_1|^{\beta+2}$, $b, \beta > 0$, and obtained the existence of global solutions and the asymptotic stability of solutions by using the potential well method. However, a blow up result has been not considered. In this regard, we may also mention to an other work by Pişkin [18] in which the author investigated (1.2) for some class of sources in a bounded domain $\Omega \subset \mathbb{R}^N$, $(N = 1, 2, 3)$ when $a_i > 0, p_i > 1$ $(i = 1, 2)$ and obtained a blow up of solutions with negative initial energy.

When $m_i \geq 0$ $(i = 1, \ldots, n)$ and $n \geq 2$, Zhou and Mu [31] recently investigated (1.1) in a bounded domain $\Omega \subset \mathbb{R}^3$. They extended the results in [1] and [7] and discussed the blow up of solutions in linear and nonlinear damping cases. They obtained the nonexistence of global solutions under some conditions on the parameters and showed the blow up of weak solutions with different range of initial energy. For this purpose, the authors considered the following additional assumption on the source term:

$$\sum_{i=1}^{n} m_i^2 \xi_i^2 - 2F(\xi_1, \ldots, \xi_n) \leq 0, \quad \text{for all} \quad (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{1.3}$$

In nonlinear damping case they also proved a nonexistence result with negative initial energy. Motivated to the above studies, in this paper we extend and improve the blow up results in the recent works [18, 31] by proving a blow up result for the problem (1.1) with not necessarily negative initial energy. In this way we don't need to consider the assumption (1.3) in the case of weak damping. An estimate for the lower bound of the blow up time is also given. We are inspired by some earlier studies such as the works by Philippin [19], Lili et al. [9] and Zhou [30] to obtain this estimate.

2. Preliminaries

In this section we present some notations, assumptions and lemmas needed for our work. In order to obtain our results, similar as in [31], we consider the following assumptions on the problem (1.1):

(A_1) $\varphi_i \in H_0^1(\Omega), \psi_i \in L^2(\Omega), (i = 1, \ldots, n)$.
(A_2) There exists constants $c_0, c_1 > 0$ such that

$$c_0 \sum_{i=1}^{n} |u_i|^4 \leq F(u_1, \ldots, u_n) \leq c_1 \sum_{i=1}^{n} |u_i|^4, \quad \text{for all} \quad (u_1, \ldots, u_n) \in \mathbb{R}^n.$$

Using the Faedo-Galerkin approximations and following the arguments in [1, 3] and [7] we can obtain the local existence of weak solutions:

Theorem 2.1. *Suppose that the assumptions (A_1) and (A_2) hold. Then there exists a unique local weak solution $(u_1, \ldots, u_n)$ of* (1.1) *in the class*

$$u_i \in C\big([0,T), H_0^1(\Omega)\big), \quad (i = 1, \ldots, n),$$
$$\partial_t u_i \in C\big([0,T), L^2(\Omega)\big) \cap L^{p_i+1}\big([0,T), L^{p_i+1}(\Omega)\big), \quad (i = 1, \ldots, n),$$

for some $T > 0$.

Next, we define the following functionals on $H_0^1(\Omega)$:

$$K(t) = K(u_1, \ldots, u_n) = \sum_{i=1}^{n} \Big(\|\nabla u_i\|_2^2 + \|m_i u_i\|_2^2 \Big) - 4 \int_\Omega F(u_1, \ldots, u_n) dx,$$
$$J(t) = J(u_1, \ldots, u_n) = \frac{1}{2} \sum_{i=1}^{n} \Big(\|\nabla u_i\|_2^2 + \|m_i u_i\|_2^2 \Big) - \int_\Omega F(u_1, \ldots, u_n) dx,$$
$$E(t) = E(u_1, \ldots, u_n) = \frac{1}{2} \sum_{i=1}^{n} \|\partial_t u_i\|_2^2 + J(u_1, \ldots, u_n).$$

Lemma 2.1. *E(t) is a non-increasing function for $t \geq 0$ and*

$$E(t) = E(0) - \sum_{i=1}^{n} \int_0^t \int_\Omega |\partial_t u_i(s)|^{p_i+1} dx ds. \tag{2.4}$$

Proof. Multiplying the equations in (1.1) by $\partial_t u_i$ $(i = 1, \ldots, n)$, integrating over Ω, and using the initial and boundary conditions we obtain (2.4). □

Consider the space

$$\mathcal{W}_T = \Big\{ (u_1, \ldots, u_n) \ : u_i \in C\big([0,T), H_0^1(\Omega)\big), \\ \partial_t u_i \in C\big([0,T), L^2(\Omega)\big) \cap L^{p_i+1}\big([0,T), L^{p_i+1}(\Omega)\big), \ i = 1, \ldots, n \Big\},$$

with the norm

$$\|(u_1, \ldots, u_n)\|_{\mathcal{W}_T}^2 = \max_{0 \leq t \leq T} \Big\{ \sum_{i=1}^{n} \Big(\|\partial_t u_i(t)\|_2^2 + \|\nabla u_i(t)\|_2^2 \Big) \Big\} + \sum_{i=1}^{n} \bigg(\int_0^T \|u_i(s)\|_{p_i+1}^{p_i+1} ds \bigg)^{\frac{1}{p_i+1}}.$$

Definition 2.1. Let the assumptions (A_1) and (A_2) hold, $(u_1, \ldots, u_n)$ be a solution of (1.1) and

$$T^\star = \sup \Big\{ T > 0 \ : \ (u_1, \ldots, u_n) \in \mathcal{W}_T \text{ exists on } [0,T) \Big\}.$$

If $T^\star = +\infty$ then we say that the solution of (1.1) exists globally and if $T^\star < +\infty$ we say that the solutions blow up at the finite time $T^\star$ in the sense

$$\sum_{i=1}^{n} \Big(\|\partial_t u_i(t)\|_2^2 + \|\nabla u_i(t)\|_2^2 \Big) \to +\infty \quad as \quad t \to {T^\star}^-. \tag{2.5}$$

Remark 2.1. In the case $T^\star = +\infty$ we refer to the arguments in [1, 3] and [7] to obtain global existence of weak solutions when $p_i \geq 3$. In the case $T^\star < +\infty$ Zhou and Mu in [31] considered (1.1) and proved the blow up results in linear damping case ($p_i = 1$) with different range of initial energy and nonlinear damping case ($1 < p_i < 3$) with negative initial energy.

3. Blow up

In this section, we study the blow up of the solutions to the system (1.1). First, we introduce the following:

$$\hat{c} = 4\pi^{-2}3^{-3/2}, \quad \frac{1}{\alpha} = \max\left\{\frac{1}{m_1^2}, \dots, \frac{1}{m_n^2}\right\}, \quad \gamma_1 = \frac{1}{2\sqrt{3}c_1\hat{c}}\sqrt{\frac{\alpha}{n}}, \quad E_1 = \frac{1}{6}\gamma_1.$$

Our main result reads in the following theorem:

Theorem 3.2. *Suppose that the assumptions (A_1) and (A_2) hold, $1 \leq p_i < 3$, $(i = 1, \dots, n)$, and $(u_1, \dots, u_n)$ is a solution of (1.1). Moreover, assume that*

$$E(0) < E_1, \qquad \sum_{i=1}^{n}\Big(\|\nabla\varphi_i\|_2^2 + \|m_i\varphi_i\|_2^2\Big) > \gamma_1. \tag{3.6}$$

Then, the solution of (1.1) blows up at a finite time $T^\star$. Furthermore, the finite blow-up time $T^\star$ satisfies in the following estimate

$$T^\star > \int_{\Phi(0)}^{+\infty} \frac{27\pi^4 d\zeta}{2^{10}n\big[\big(E(0)\big)^3 + c_1^3\zeta^3\big] + 108\pi^4\big(E(0) + c_1\zeta\big)}. \tag{3.7}$$

where $\Phi(0) = \sum_{i=1}^{n}\|\varphi_i\|_4^4$.

To prove the above theorem we need the following lemma:

Lemma 3.2. *Suppose that (3.6) and the assumptions (A_1) and (A_2) hold. Then, there exists a constant $\gamma_2 > \gamma_1$ such that*

$$\sum_{i=1}^{n}\Big(\|\nabla u_i(t)\|_2^2 + \|m_i u_i(t)\|_2^2\Big) > \gamma_2, \quad \forall t \geq 0, \tag{3.8}$$

and

$$(c_1\hat{c})^{-2/3}\sqrt[3]{\frac{\alpha}{n}}\left(a\sum_{i=1}^{n}\|u_i(t)\|_4^4 + 2b\Big\|\prod_{i=1}^{n}u_i(t)\Big\|_2^2\right)^{1/3} \geq \gamma_2, \quad \forall t \geq 0. \tag{3.9}$$

Proof. By the assumption (A_2) we have

$$E(t) \geq \frac{1}{2}\sum_{i=1}^{n}\Big(\|\nabla u_i(t)\|_2^2 + \|m_i u_i(t)\|_2^2\Big) - c_1\sum_{i=1}^{n}\|u_i\|_4^4. \tag{3.10}$$

Using the Young's inequality, for any $\varepsilon > 0$, we get

$$\int_\Omega |u_i|^4 dx \leq \frac{\varepsilon}{2}\int_\Omega |u_i|^2 dx + \frac{1}{2\varepsilon}\int_\Omega |u_i|^6 dx, \quad i = 1, \dots, n. \tag{3.11}$$

From the Talenti-Sobolev Theorem (see [25] with $m = 3$ and $p = 2$) we have

$$\int_\Omega |u_i|^6 dx \leq (\hat{c})^2\left(\int_\Omega |\nabla u_i|^2 dx\right)^3, \quad i = 1, \dots, n. \tag{3.12}$$

Therefore, from (3.11) and (3.12) we obtain

$$\sum_{i=1}^{n}\|u_i\|_4^4 \leq \frac{\varepsilon}{2\alpha}\sum_{i=1}^{n}\xi_i(t)+\frac{n(\hat{c})^2}{2\varepsilon}\left(\sum_{i=1}^{n}\xi_i(t)\right)^3, \tag{3.13}$$

where $\xi_i(t)=\|\nabla u_i(t)\|_2^2+\|m_iu_i(t)\|_2^2,\ i=1,\dots,n$. Then, using (3.13) for $\varepsilon=\frac{\alpha}{2c_1}$, the inequality (3.10) takes the form

$$E(t)\geq \frac{1}{4}\gamma(t)-\frac{n(c_1)^2(\hat{c})^2}{\alpha}\big(\gamma(t)\big)^3 =: G\big(\gamma(t)\big), \tag{3.14}$$

where $\gamma(t)=\sum_{i=1}^{n}\xi_i(t)$ and $G(\gamma)=\frac{1}{4}\gamma-\frac{n(c_1)^2(\hat{c})^2}{\alpha}\gamma^3$. It is not difficult to see that G is strictly increasing in $(0,\gamma_1)$, strictly decreasing in $(\gamma_1,+\infty)$, and $G(\gamma)\to-\infty$ as $\gamma\to+\infty$. Since $E(0)<E_1$, there exists $\gamma_2>\gamma_1$ such that $G(\gamma_2)=E(0)$. Therefore, by (3.14) we have

$$G(\gamma(0))\leq E(0)=G(\gamma_2).$$

Thus, $\gamma(0)\geq\gamma_2$. To show (3.8) we suppose that there exists $t_0>0$ such that $\gamma(t_0)\leq\gamma_2$ and by continuity of $\gamma(.)$ we can choose t_0 such that $\gamma_1<\gamma(t_0)$. Since G is decreasing on $(\gamma_1,+\infty)$ we have $G\big(\gamma(t_0)\big)\geq G(\gamma_2)=E(0)$ and by (3.14) we know that $G\big(\gamma(t_0)\big)\leq E(t_0)$ which yields $E(t_0)\geq E(0)$ and this contradicts (2.4). Hence (3.8) holds. To show (3.9), we use definition of the energy functional and (2.4) to obtain

$$E(0)+a\sum_{i=1}^{n}\|u_i(t)\|_4^4+2b\Big\|\prod_{i=1}^{n}u_i(t)\Big\|_2^2\geq\frac{1}{2}\gamma(t).$$

Then, by (3.8) we have

$$\begin{aligned}a\sum_{i=1}^{n}\|u_i(t)\|_4^4+2b\Big\|\prod_{i=1}^{n}u_i(t)\Big\|_2^2&\geq\frac{1}{2}\gamma_2-E(0)\\&\geq\frac{1}{4}\gamma_2-G(\gamma_2)=\frac{n(c_1)^2(\hat{c})^2}{\alpha}\gamma_2^3.\end{aligned}$$

Therefore, (3.9) is established. This completes the proof of lemma 3.2. □

Proof of Theorem 3.2. We define

$$\mathcal{L}(t)=\sum_{i=1}^{n}\int_\Omega u_i^2(t)dx,$$

then

$$\mathcal{L}'(t)=2\sum_{i=1}^{n}\int_\Omega u_i(t)\partial_t u_i(t)dx,$$

$$\begin{aligned}\mathcal{L}''(t)=&-2\sum_{i=1}^{n}\Big(\|\nabla u_i\|_2^2+\|m_iu_i\|_2^2\Big)-2\sum_{i=1}^{n}\int_\Omega u_i\partial_t u_i|\partial_t u_i|^{p_i-1}dx\\&+8\int_\Omega F(u_1,\dots,u_n)dx+8(n-2)b\Big\|\prod_{i=1}^{n}u_i\Big\|_2^2+2\sum_{i=1}^{n}\|\partial_t u_i(t)\|_2^2.\end{aligned}\tag{3.15}$$

Using Hölder's inequality, the left inequality in (A_2) and (3.9), for $i = 1, \ldots, n$, we get

$$\begin{aligned}\left|\int_\Omega u_i\partial_t u_i|\partial_t u_i|^{p_i-1}dx\right| &\le \|u_i\|_{p_i+1}\|\partial_t u_i\|_{p_i+1}^{p_i} \le |\Omega|^{\frac{3-p_i}{4(p_i+1)}}\|u_i\|_4\|\partial_t u_i\|_{p_i+1}^{p_i}\\ &\le |\Omega|^{\frac{3-p_i}{4(p_i+1)}}\frac{1}{\sqrt[4]{c_0}}\left(\int_\Omega F(u_1,\ldots,u_n)dx\right)^{1/4}\|\partial_t u_i\|_{p_i+1}^{p_i}\\ &\le k_i\left(\int_\Omega F(u_1,\ldots,u_n)dx\right)^{1/(p_i+1)}\|\partial_t u_i\|_{p_i+1}^{p_i},\end{aligned} \tag{3.16}$$

where

$$k_i = \frac{1}{\sqrt[4]{c_0}}\left(\frac{\alpha|\Omega|}{n(c_1)^2(\hat{c})^2\gamma_2^3}\right)^{\frac{3-p_i}{4(p_i+1)}}, \quad i = 1, \ldots, n.$$

Applying Young's inequality to (3.16), for $i = 1, \ldots, n$, we obtain

$$\begin{aligned}\left|\int_\Omega u_i\partial_t u_i|\partial_t u_i|^{p_i-1}dx\right| \le k_i\bigg\{&\frac{\varepsilon_i^{p_i+1}}{p_i+1}\int_\Omega F(u_1,\ldots,u_n)dx\\ &+\frac{p_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\int_\Omega|\partial_t u_i|^{p_i+1}dx\bigg\},\end{aligned} \tag{3.17}$$

where $\varepsilon_i > 0$ will be specified later. Therefore, from (3.17) the equality (3.15) turns into the following inequality

$$\begin{aligned}\mathcal{L}''(t) \ge\ & 2\sum_{i=1}^n\|\partial_t u_i\|_2^2 - 2\left(\sum_{i=1}^n\frac{k_i\varepsilon_i^{p_i+1}}{p_i+1}\right)\int_\Omega F(u_1,\ldots,u_n)dx - 2K(t)\\ &-2\sum_{i=1}^n\left(\frac{k_ip_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\int_\Omega|\partial_t u_i|^{p_i+1}dx\right) + 8(n-2)b\Big\|\prod_{i=1}^n u_i\Big\|_2^2.\end{aligned} \tag{3.18}$$

By the definition of $E(t)$ we have

$$\begin{aligned}-2K(t) &\ge -2K(t) + 2\sigma(E(t)-E(0))\\ &= \sigma\sum_{i=1}^n\|\partial_t u_i(t)\|_2^2 + (\sigma-2)\sum_{i=1}^n\left(\|\nabla u_i\|_2^2 + \|m_iu_i\|_2^2\right)\\ &\quad + 2(4-\sigma)\int_\Omega F(u_1,\ldots,u_n)dx - 2\sigma E(0).\end{aligned} \tag{3.19}$$

where $\sigma > 2$ is a constant to be specified later. Hence, by (3.18) and (3.19) we get

$$\begin{aligned}\mathcal{L}''(t) \ge\ & (\sigma+2)\sum_{i=1}^n\|\partial_t u_i\|_2^2 + (\sigma-2)\sum_{i=1}^n\left(\|\nabla u_i\|_2^2+\|m_iu_i\|_2^2\right)\\ &+2\left[4-\sigma-\left(\sum_{i=1}^n\frac{k_i\varepsilon_i^{p_i+1}}{p_i+1}\right)\right]\int_\Omega F(u_1,\ldots,u_n)dx - 2\sigma E(0)\\ &-2\sum_{i=1}^n\left(\frac{k_ip_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\int_\Omega|\partial_t u_i|^{p_i+1}dx\right) + 8(n-2)b\Big\|\prod_{i=1}^n u_i\Big\|_2^2.\end{aligned} \tag{3.20}$$

Since $E(0) < E_1$ we can choose σ such that

$$\frac{6E_1}{3E_1 - E(0)} < \sigma < 4. \tag{3.21}$$

Then, by lemma 3.2, (3.8) and (3.21) we have

$$(\sigma-2)\sum_{i=1}^{n}\left(\|\nabla u_i\|_2^2+\|m_iu_i\|_2^2\right)-2\sigma E(0)>6E_1(\sigma-2)-2\sigma E(0)>0.$$

We now fix $\varepsilon_i,\ i=1,\dots,n$ small enough such that

$$\mu:=4-\sigma-\left(\sum_{i=1}^{n}\frac{k_i\varepsilon_i^{p_i+1}}{p_i+1}\right)>0.$$

Integrating (3.20) over $(0,t)$ we obtain

$$\begin{aligned}\mathcal{L}'(t)>&2\mu\int_0^t\int_\Omega F\big(u_1(s),\dots,u_n(s)\big)dxds\\&-2\sum_{i=1}^{n}\left(\frac{k_ip_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\int_0^t\int_\Omega|\partial_tu_i(s)|^{p_i+1}dxds\right)+\mathcal{L}'(0).\end{aligned}\tag{3.22}$$

Taking (3.9) and (2.4) into account and using the fact that $E(0)-E(t)<E_1$, the inequality (3.22) takes the form

$$\mathcal{L}'(t)>\left(\frac{2\mu n\gamma_2^3(c_1)^2(\hat{c})^2}{\alpha}\right)t-2E_1\sum_{i=1}^{n}\left(\frac{k_ip_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\right)+\mathcal{L}'(0).\tag{3.23}$$

Finally, by integrating (3.23) from 0 to t we find

$$\mathcal{L}(t)>\left(\frac{\mu n\gamma_2^3(c_1)^2(\hat{c})^2}{\alpha}\right)t^2+\left\{\mathcal{L}'(0)-2E_1\sum_{i=1}^{n}\left(\frac{k_ip_i}{p_i+1}\varepsilon_i^{-\frac{p_i+1}{p_i}}\right)\right\}t+\mathcal{L}(0).\tag{3.24}$$

On the other hand by using Hölder's inequality and (2.4), for $i=1,\dots,n$, we have

$$\begin{aligned}\|u_i(t)\|_2\le\|\varphi_i\|_2+\int_0^t\|\partial_tu_i(s)\|_2ds&\le\|\varphi_i\|_2+C_i\int_0^t\|\partial_tu_i(s)\|_{p_i+1}ds\\&\le\|\varphi_i\|_2+C_iE_1^{\frac{1}{p_i+1}}t^{\frac{p_i}{p_i+1}},\end{aligned}$$

where $C_i,\ i=1,\dots,n$ are some positive constants. Consequently,

$$\|u_i(t)\|_2^2\le\|\varphi_i\|_2^2+C_i^2E_1^{\frac{2}{p_i+1}}t^{\frac{2p_i}{p_i+1}},\quad ,i=1,\dots,n,$$

which contradicts (3.24).

To show (3.7) we define

$$\Phi(t)=\sum_{i=1}^{n}\int_\Omega|u_i(t)|^4dx.$$

Then,

$$\Phi'(t)=4\sum_{i=1}^{n}\int_\Omega|u_i(t)|^2u_i\partial_tu_idx.$$

By using the Young's inequality we obtain

$$\Phi'(t)\le2\sum_{i=1}^{n}\int_\Omega|u_i(t)|^6dx+2\sum_{i=1}^{n}\int_\Omega|\partial_tu_i(t)|^2dx.\tag{3.25}$$

From (2.4) and (A_2) we have

$$
\begin{aligned}
\sum_{i=1}^{n}\|\partial_t u_i(t)\|_2^2+\sum_{i=1}^{n}\Big(\|\nabla u_i(t)\|_2^2+\|m_i u_i\|_2^2\Big) &= 2E(t)+2\int_\Omega F(u_1,\dots,u_n)dx \\
&\le 2E(0)+2\int_\Omega F(u_1,\dots,u_n)dx \\
&\le 2E(0)+2c_1\Phi(t).
\end{aligned}
\tag{3.26}
$$

It follows from (3.25), (3.26) and (3.12) that

$$
\begin{aligned}
\Phi'(t) &\le 2(\hat{c})^2\sum_{i=1}^{n}\Big(\int_\Omega|\nabla u_i|^2\Big)^3+4E(0)+4c_1\Phi(t) \\
&\le 2n(\hat{c})^2\Big(\sum_{i=1}^{n}\|\nabla u_i(t)\|_2^2\Big)^3+4E(0)+4c_1\Phi(t) \\
&\le 64n(\hat{c})^2\Big(E^3(0)+c_1^3\Phi^3(t)\Big)+4\Big(E(0)+c_1\Phi(t)\Big).
\end{aligned}
\tag{3.27}
$$

Therefore, integrating (3.27) over $(0,t)$ we arrive at

$$
t\ge\int_{\Phi(0)}^{\Phi(t)}\frac{d\zeta}{64n(\hat{c})^2\Big(\big(E(0)\big)^3+c_1^3\zeta^3\Big)+4\Big(E(0)+c_1\Phi(t)\Big)}.
\tag{3.28}
$$

Since, $\sum_{i=1}^{n}\|u_i(t)\|_2^2\to+\infty$ as $t\to T^{\star^-}$ and $\|u_i\|_2\le C\|u_i\|_4$ for $i=1,\dots,n$ and some $C>0$, then we can deduce that $\sum_{i=1}^{n}\|u_i(t)\|_4^4\to+\infty$ as $t\to T^{\star^-}$. Hence, (3.7) follows by letting $t\to T^{\star^-}$ in (3.28). This completes the proof of Theorem 3.2.

Example 3.1. (Coupled Nonlinear Klein-Gordon Equations with Nonlinear Damping Terms) When $n=2$ the equations in (1.1) converts to

$$
\begin{cases}
\partial_{tt}^2u_1-\Delta u_1+m_1^2u_1+|\partial_t u_1|^{p_1-1}\partial_t u_1=f_1(u_1,u_2), & \text{in} \quad \Omega_T,\\
\partial_{tt}^2u_2-\Delta u_2+m_2^2u_2+|\partial_t u_2|^{p_2-1}\partial_t u_2=f_2(u_1,u_2), & \text{in} \quad \Omega_T,
\end{cases}
\tag{3.29}
$$

where

$$
f_1(u_1,u_2)=4\big[a(u_1+u_2)^3+bu_1u_2^2\big],\quad f_2(u_1,u_2)=4\big[a(u_1+u_2)^3+bu_1^2u_2\big].
\tag{3.30}
$$

Pişkin in [18] considered (3.29) with the same initial and boundary conditions as in (1.1). The author proved that if $1\le p_1,p_2<3$ then the solution of (3.29) blows up at a finite time $T^\star$ and

$$
T^\star\le\frac{1-\sigma}{\xi\sigma\zeta^{\sigma/(1-\sigma)}(0)},\qquad \text{for some}\quad \xi>0,
$$

where $0<\sigma\le\min\left\{\frac{3-p_1}{4p_1},\frac{3-p_2}{4p_2},\frac{1}{4}\right\}$ and

$$
\zeta(0)=\Big(-E(0)\Big)^{1-\sigma}+\varepsilon\int_\Omega\Big(\phi_1(x)\psi_1(x)+\phi_2(x)\psi_2(x)\Big)dx,\qquad \varepsilon>0.
$$

This blow up result is obtained under the restriction $E(0)<0$ while as Theorem 3.2 shows, the initial energy dose not need to be necessarily negative.

Example 3.2. (Coupled Nonlinear Klein-Gordon Equations with Linear Damping Terms) Consider again (3.29) under the initial and boundary conditions in (1.1) when $p_1=p_2=1$. Using a concavity method, a nonexistence result of this problem has been investigated by

Wu [26] where he proved that the solution with arbitrary positive initial energy blows up at a finite $T^\star$ time in the sense

$$\lim_{t\nearrow T^\star}\left(\|u_1(t)\|_2^2+\|u_2(t)\|_2\right)=+\infty.$$

To obtain this result the author considered the following crucial assumption on the source terms:

$$u_1f_1(u_1,u_2)+u_2f_1(u_1,u_2)\geq(2+4\delta)F(u_1,u_2),\quad\forall(u_1,u_2)\in\mathbb{R}\times\mathbb{R}, \tag{3.31}$$

for some positive constant δ. As an example (see Example 3.6 in [26]) the author showed that for $0<\delta\leq 1/2$ the solutions of (3.29) with the source terms (3.30) blow up. Restrictions similar to (3.31) have been imposed on source terms in different problems of type (1.1) to obtain blow up results such as (1.3) in [31] by Zhou and Mu or the assumption $(A4)$ in [8] or the condition (3.16) in [7] by Li and Tsai. However, as the Theorem 3.6 indicates, we can neglect these kind of restrictions.

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