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Maia-Perov type fixed point results for Prešić type operators

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ABSTRACT. Starting from the results, established in [Albu, M., *A fixed point theorem of Maia-Perov type.* Studia Univ. Babeş-Bolyai Math., 23 (1978), No. 1, 76–79] and [Mureşan, V., *Basic problem for Maia-Perov's fixed point theorem*, Seminar on Fixed Point Theory, Babeş Bolyai Univ., Cluj-Napoca, (1988), Preprint Nr. 3, pp. 43–48] where fixed point theorems of Maia-Perov type are proved, the main aim of this paper is to extend this results to product metric spaces, using Prešić type operators. An existence, uniqueness and data dependence theorem related to the solution of the system of integral equations of Fredholm type in product metric spaces, is also presented.

1. INTRODUCTION

In 1966, Perov A. I. and Kibenco A. V. [10], proved a fixed point result related to Banach contraction mapping principle, where α is replaced with a convergent matrix towards 0, $A \in M_{nn}(\mathbb{R})$. Their fixed point theorem can be stated as follows:

Theorem 1.1. [1] *Let (X, d) be a generalized complete metric space with the metric $d : X \times X \to \mathbb{R}^n$ and $f : X \to X$ an mapping such that there exists a convergent matrix towards 0, $A \in M_{nn}(\mathbb{R})$, having the property of $d(f(x), f(y)) \le Ad(x, y),\ for\ any\ x, y \in X$.*

Under these conditions the mapping f has a unique fixed point x^* which can be obtained by the succesive approximations method starting from any element $x_0 \in X$. Moreover, the estimation $d(x_m, x^*) \le A^m(I - A)^{-1}d(x_0, x_1)$ takes place.

In 1968, M. G. Maia [5] generalized the Banach contraction mapping principle for sets endowed with two comparable metrics and is connected with Bielecki's method of changing the norm in the theory of differential equations. Maia type fixed point results for singlevalued or multivalued operators have been studied in [12], [14], [15], [16], [17], [18]. An enriched variant of Maia fixed point theorem can be stated as follows:

Theorem 1.2. [17]

Let X be a nonempty set, d and ρ two metrics on X and $f : X \to X$ an operator. We suppose that:

(i) $d(x, y) \le \rho(x, y)$ for all $x, y \in X$;

(ii) (X, d) is a complete metric space;

(iii) $f : (X, d) \to (X, d)$ is continuous;

(iv) $f : (X, \rho) \to (X, \rho)$ is an α-contraction.

Then:

(a) $F_f = \{x^*\}$;

(b) $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$, for all $x \in X$;

(c) $f^n(x) \xrightarrow{\rho} x^*$ as $n \to \infty$, for all $x \in X$;

(d) $\rho(x, x^*) \le \frac{1}{1-\alpha}\rho(x, f(x))$, for each $x \in X$.

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In 1978, Albu M.[1], established a fixed point theorem of Maia-Perov type by which she obtained a new existence and uniqueness theorem related to a system of integral equations of Fredholm type.

Theorem 1.3. [1] *Let X be a set endowed with the generalized metrics $d, \rho : X \times X \to \mathbb{R}^n$. If the conditions are fulfilled:*
(1) $d(x,y) \leq \rho(x,y),\ for\ any\ x,y \in X$;
(2) (X,d) *is a complete metric space;*
(3) $f : (X,d) \to (X,d)$ *is continuous;*
(4) *there exists a matrix convergent towards 0*

$$A \in M_{nn}(\mathbb{R})\ such\ that$$

$$\rho(f(x), f(y)) \leq A\rho(x,y)\ for\ any\ x,y \in X,$$

then the application f has a unique fixed point x^*, which can be obtained by the succesive approximations method starting from any element $x_0 \in X$.
Moreover, the following estimation takes place:

$$\rho(x_m, x^*) \leq A^m(I-A)^{-1}\rho(x_0, x_1).$$

Remark 1.1. From Theorem 1.3 we obtain the following two known results:
1. For $d = \rho$ we get the Perov fixed point theorem.
2. For $n = 1$ and $A = \alpha$ we get the Maia fixed point theorem.

Prešić S. B. [13] extended the famous Banach contraction principle [3] to the case of product spaces in 1965. Recently, in 2007, Ćirić and Prešić [4], generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Other important Prešić fixed point theorem generalizations and some related results can be found in Păcurar's papers [7], [8].

In an extended version [9], Prešić's result may be stated as follows:

Theorem 1.4. [9] *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a Prešić operator, it is, a mapping for which there exists $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ such that:*

$$d\big(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)\big) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i), \tag{1.1}$$

for all $x_0, \ldots, x_k \in X$.
Then:
1) *f has a unique fixed point x^*;*
2) *the sequence $\{y_n\}_{n\geq 0}$,*

$$y_{n+1} = f(y_n, y_n, \ldots, y_n), n \geq 0,$$

converges to x^*;
3) *the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \geq k,$$

also converges to x^*, with a rate estimated by

$$d(x_{n+1}, x^*) \leq \alpha d(x_n, x^*) + M \cdot \theta^n, n \geq 0,$$

where $M > 0$ is constant.

Remark 1.2. Particular cases:
1. From Maia fixed point theorem, when $d \equiv \rho$, we get Banach contraction mapping principle.
2. From Prešić fixed point theorem, when k=1, we get Banach contraction mapping principle.
3. From Perov fixed point theorem, when $A = \alpha$, $n = 1$, we get Banach contraction mapping principle.

Starting from the previous results, the aim of this paper is to unify Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 by proving a Maia-Perov fixed point theorem for Prešić type mappings.

To this end we need the following concepts:

Definition 1.1. Let (X, d) be a metric space, $d : X \times X \to \mathbb{R}^n$, k a positive integer, $A_1, A_2, \ldots, A_k \in M_{nn}(\mathbb{R}_+)$, $\sum_{i=1}^{k} A_i = A < I_n$, A is a matrix convergent towards 0 and $f : X^k \to X$ a mapping satisfying

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq A_1 \cdot d(x_0, x_1) + A_2 \cdot d(x_1, x_2) +$$

$$+ \cdots + A_k \cdot d(x_{k-1}, x_k), \tag{1.2}$$

for any $x_0, x_1, \ldots, x_k \in X$.

Then f is called a Prešić-Perov type operator.

The following lemma is an extension of Prešić [13] result using Perov fixed point theorem and we shall use it in the proof of our main result.

Lemma 1.1. *Let $k \in \mathbb{N}, k \neq 0$ and $A_1, A_2, \ldots, A_k$ matrices convergent towards 0 such that $\sum_{i=1}^{k} A_i = A < I_n$. If $\{\Delta_n\}_{n \geq 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \leq A_1 \Delta_n + A_2 \Delta_{n+1} + \cdots + A_k \Delta_{n+k-1}, n \geq 1, \tag{1.3}$$

then there exist $L > 0$ and $\theta \in (0, 1)$ such that

$$\Delta_n \leq L \cdot \theta^n, \; for \; all \; n \geq 1. \tag{1.4}$$

Remark 1.3. Let $f : X^k \to X$ be a mapping, k a positive integer. We consider

$$\bar{d}((x_1, x_2, \ldots, x_k), (a_1, a_2, \ldots, a_k)) = d(x_1, a_1) + d(x_2, a_2) + \cdots + d(x_k, a_k).$$

$f : (X^k, \bar{d}) \to (X, d)$ is said to be continuous in $a = (a_1, a_2, \ldots, a_k) \in X^k$ if

For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x = (x_1, x_2, \ldots, x_k) \in X^k$ with $\bar{d}((x_1, x_2, \ldots, x_k), (a_1, a_2, \ldots, a_k)) < \delta$, we have $d(f(x_1, x_2, \ldots, x_k), f(a_1, a_2, \ldots, a_k)) < \varepsilon$.

Remark 1.4. [14] For any operator $f : X^k \to X$, k a positive integer, we can define its associate operator $F : X \to X$ by

$$F(x) = f(x, \ldots, x), \; x \in X.$$

$x \in X$ is a fixed point of $f : X^k \to X$ if and only if x is a fixed point of its associate operator F.

2. Main results

The purpose of our first result is to solve the basic problem of the metrical fixed point theory for the Maia-Perov's fixed point theorem in product metric spaces. For this to be done, we start from the results obtained by Mureşan V. in [6], Albu M. in [1] and Rus I. A. in [15].

Theorem 2.5. *Let X be a nonempty set endowed with the generalized metrics $d, \rho : X \times X \to \mathbb{R}^n$ and $f : X^k \to X$ a Prešić-Perov operator w.r.t. ρ.*

We suppose that:

(i) *there exists a matrix $C \in M_{n,n}(\mathbb{R}_+)$ such that*

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq C \cdot \rho(x_0, x_k),$$

for all $x_0, x_1, \ldots, x_k \in X$;

(ii) *there exists $c_1 > 0$ such that*

$$\rho(x_0, x_1) \leq c_1 \cdot d(x_0, x_1),$$

for all $x_0, x_1 \in X$;

(iii) *(X, d) is a complete metric space;*

(iv) *$f : (X^k, \bar{d}) \to (X, d)$ is continuous.*

Then:

(a) *f has a unique fixed point $x^* = f(x^*, \ldots, x^*)$;*

(b) *the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1})$, $n \geq k-1$, converges to x^* w.r.t. d, with a rate estimated by*

$$d(x_{n+1}, x^*) \leq c_1 \cdot C \cdot A \cdot d(x_{n-k}, x^*) + c_1 \cdot C^2 \cdot M \cdot \theta^{n-k-1}, \tag{2.5}$$

where $M = L[A_1 + (A_1 + A_2)\theta^{-1} + \cdots + (A_1 + \cdots + A_{k-1})\theta^{-k}] > 0$.

(c) *let $g : X^k \to X$ be a mapping which approximates the mapping f. More precisely we assume that there exists $\eta \in \mathbb{R}^n$ such that for all $x \in X$*

$$d(f(x, x, \ldots, x), g(x, x, \ldots, x)) \leq \eta,$$

If $x_g^* \in F_g$, $x_g^* = g(x_g^*, x_g^*, \ldots, x_g^*)$, then

$$d(f(x_f^*, x_f^*, \ldots, x_f^*), g(x_g^*, x_g^*, \ldots, x_g^*)) \leq \frac{\eta}{I_n - c_1 \cdot C \cdot A}. \tag{2.6}$$

(d) *If $(X^k, \|\cdot\|_1, \|\cdot\|_2)$ is a linear space with two norms and*

$$d(x, y) = \|x - y\|_1 = \max_{t\in X}|x(t) - y(t)|,$$

$$\rho(x, y) = \|x - y\|_2 = (\int_X |x(t) - y(t)|^2 dt)^{\frac{1}{2}},$$

where $x, y \in X^k$, and $(i) - (iv)$ holds, then $1_{X^k} - f : X^k \to X$ is a bijective mapping.

Moreover, if $det(I_n - c_1 C) \neq 0$, then $1_{X^k} - f : (X^k, \|\cdot\|_1) \to (X, \|\cdot\|_1)$ is a homeomorphism.

Proof. (a) and (b):

Let $\{x_n\}_{n\geq 0}$, $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}), n \geq k-1$,

$$\rho(x_n, x_{n+1}) = \rho\big(f(x_{n-1}, x_{n-2}, \ldots, x_{n-k}), f(x_n, x_{n-1}, \ldots, x_{n-k+1})\big) \leq$$

$$\leq A_1\rho(x_{n-1}, x_n) + A_2\rho(x_{n-2}, x_{n-1}) + \cdots + A_k\rho(x_{n-k}, x_{n-k+1})$$

By Lemma 1.1, we have

$$\rho(x_{n-1}, x_n) \leq L \cdot \theta^n, n \geq 1, \tag{2.7}$$

For $n \geq 1, m \geq 1$, by (2.7) we obtain:

$$\rho(x_n, x_{n+m}) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+m-1}, x_{n+m}) \leq$$

$$\leq L \cdot \theta^{n+1} + L \cdot \theta^{n+2} + \cdots + L \cdot \theta^{n+m} =$$

$$= L \cdot \theta^{n+1}(1 + \theta + \theta^2 + \cdots + \theta^{m-1})$$

so

$$\rho(x_n, x_{n+m}) \leq L \cdot \theta^{n+1} \cdot \frac{1-\theta^m}{1-\theta}, n \geq 1, m \geq 1.$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X, ρ).

For $n \leq m$,

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})$$

From relation (i) we obtain:

$$d(x_n, x_{n+1}) = d(x_{n+1}, x_n) = d(f(x_n, x_{n-1}, \ldots, x_{n-k+1}), f(x_{n-1}, x_{n-2} \ldots, x_{n-k+1}, x_{n-k})) \leq$$
$$\leq C \cdot \rho(x_n, x_{n-k})$$

and

$$\rho(x_n, x_{n-k}) \leq L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^k}{1-\theta}.$$

Since $\frac{1-\theta^k}{1-\theta} < 1$, we have

(2.8) $$d(x_{n+1}, x_n) \leq C \cdot L \cdot \theta^{n-k+1}.$$

Similarly

$$d(x_{n+2}, x_{n+1}) \leq C \cdot L \cdot \theta^{n-k+2}, \ldots, d(x_{n+m}, x_{n+m-1}) \leq C \cdot L \cdot \theta^{n-k+m}.$$

So,

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \leq$$

$$\leq C \cdot L \cdot \theta^{n-k+1} + C \cdot L \cdot \theta^{n-k+2} + \cdots + C \cdot L \cdot \theta^{n-k+m} =$$

$$= C \cdot L \cdot \theta^{n-k} \cdot (\theta + \theta^2 + \cdots + \theta^m) =$$

$$= C \cdot L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^m}{1-\theta}.$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, d) so $\{x_n\}_{n\geq 0}$ is also convergent: there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

By continuity of f and considering the associate operator $F : X \to X, F(x) = f(x, x, \ldots, x)$, for any $x \in X$ we have:

$$d(F(x^*), x^*) = d(f(x^*, x^*, \ldots, x^*), x^*) =$$
$$= d(f(\lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_{n-k+1}), x^*) =$$
$$= \lim_{n\to\infty} d(f(x_n, \ldots, x_{n-k+1}), x^*) =$$

$$= \lim_{n\to\infty} d(x_{n+1}, x^*) =$$
$$= 0.$$

Therefore $x^* = f(x^*, x^*, \ldots, x^*) = F(x^*)$ is a fixed point of f.

We suppose that f has another fixed point $y^* = f(y^*, y^*, \ldots, y^*)$.

$$\rho(x^*, y^*) = \rho(f(x^*, x^*, \ldots, x^*), f(y^*, y^*, \ldots, y^*)) \leq$$

$$\leq \rho(f(x^*, x^*, \ldots, x^*), f(x^*, x^*, \ldots, y^*)) +$$

$$+\rho(f(x^*, x^*, \ldots, y^*), f(x^*, \ldots, x^*, y^*, y^*)) +$$

$$+\cdots + \rho(f(x^*, y^*, \ldots, y^*), f(y^*, y^*, \ldots, y^*)) \leq$$

$$\leq A_k \cdot \rho(x^*, y^*) + A_{k-1} \cdot \rho(x^*, y^*) + \cdots + A_1 \cdot \rho(x^*, y^*) =$$

$$= \rho(x^*, y^*) \cdot \sum_{i=1}^{k} A_i.$$

As $\sum_{i=1}^{k} A_i < I_n$, we obtain $\rho(x^*, y^*) = 0$, so $x^* = y^*$. The uniqueness of fixed point is proved.

To obtain the estimation (2.5) we use (i), the Prešić-Perov type contraction condition and (ii):

$$d(x_{n+1}, x^*) = d(f(x_n, x_{n-1}, \ldots, x_{n-k+1}), f(x^*, x^*, \ldots, x^*)) \leq$$
$$\leq C \cdot \rho(x_n, x^*) = C \cdot \rho(f(x_{n-1}, x_{n-2}, \ldots, x_{n-k}), f(x^*, x^*, \ldots, x^*)) \leq$$
$$\leq C \cdot [\rho(f(x_{n-1}, x_{n-2}, \ldots, x_{n-k}), f(x_{n-2}, \ldots, x_{n-k}, x^*)) +$$
$$+\rho(f(x_{n-2}, \ldots, x_{n-k}, x^*), f(x_{n-3}, \ldots, x_{n-k}, x^*, x^*)) + \cdots +$$
$$+\rho(f(x_{n-k}, x^*, \ldots, x^*), f(x^*, x^*, \ldots, x^*))] \leq$$
$$\leq C \cdot [A_1\rho(x_{n-1}, x_{n-2}) + A_2\rho(x_{n-2}, x_{n-3}) + \cdots + A_k\rho(x_{n-k}, x^*) +$$
$$+A_1\rho(x_{n-2}, x_{n-3}) + A_2\rho(x_{n-3}, x_{n-4}) + \cdots + A_{k-1}\rho(x_{n-k}, x^*) + A_k\rho(x^*, x^*) +$$
$$+\cdots + A_1\rho(x_{n-k}, x^*) + A_2\rho(x^*, x^*) + \cdots + A_k\rho(x^*, x^*)] \leq$$
$$\leq c_1 \cdot C \cdot [A_1 d(x_{n-1}, x_{n-2}) + A_2 d(x_{n-2}, x_{n-3}) + \cdots + A_k d(x_{n-k}, x^*) +$$
$$+A_1 d(x_{n-2}, x_{n-3}) + A_2 d(x_{n-3}, x_{n-4}) + \cdots + A_{k-1} d(x_{n-k}, x^*) + A_k d(x^*, x^*) +$$
$$+\cdots + A_1 d(x_{n-k}, x^*) + A_2 d(x^*, x^*) + \cdots + A_k d(x^*, x^*)]$$

$$d(x_{n+1}, x^*) \leq c_1 \cdot C \cdot [A_1 d(x_{n-1}, x_{n-2}) + (A_1 + A_2) d(x_{n-2}, x_{n-3}) +$$
$$+(A_1+A_2+A_3) d(x_{n-3}, x_{n-4}) + \cdots + (A_1+A_2+\cdots+A_{k-1}) d(x_{n-k-1}, x_{n-k}) + A d(x_{n-k}, x^*)]$$

Now using (2.8) it follows that

$$d(x_{n+1}, x^*) \leq c_1 \cdot C \cdot [A_1 \cdot C \cdot L \cdot \theta^{n-k-1} + (A_1 + A_2) \cdot C \cdot L \cdot \theta^{n-k-2} +$$
$$+(A_1+A_2+A_3) \cdot C \cdot L \cdot \theta^{n-k-3} + \cdots + (A_1+A_2+\cdots+A_{k-1}) \cdot C \cdot L \cdot \theta^{n-2k-1} + A d(x_{n-k}, x^*)] =$$
$$= c_1 \cdot C \cdot A d(x_{n-k}, x^*) + c_1 \cdot C^2 \cdot L \cdot \theta^{n-k-1} \cdot [A_1 + (A_1 + A_2)\theta^{-1} + \cdots + (A_1 + \cdots + A_{k-1})\theta^{-k}]$$

Denoting $M = L[A_1 + (A_1 + A_2)\theta^{-1} + \cdots + (A_1 + \cdots + A_{k-1})\theta^{-k}]$, we obtain the estimation (2.5).

(c) :

$$d(x_f^*, x_g^*) = d(f(x_f^*, x_f^*, \ldots, x_f^*), g(x_g^*, x_g^*, \ldots, x_g^*)) \leq$$
$$\leq d(f(x_f^*, x_f^*, \ldots, x_f^*), f(x_g^*, x_g^*, \ldots, x_g^*)) + d(f(x_g^*, x_g^*, \ldots, x_g^*), g(x_g^*, x_g^*, \ldots, x_g^*)) \leq$$
$$\leq d(f(x_f^*, x_f^*, \ldots, x_f^*), f(x_g^*, x_g^*, \ldots, x_g^*)) + \eta \leq c \cdot \rho(f(x_f^*, x_f^*, \ldots, x_f^*), f(x_g^*, x_g^*, \ldots, x_g^*)) + \eta \leq$$
$$\leq C \cdot [\rho(f(x_f^*, x_f^*, \ldots, x_f^*), f(x_f^*, \ldots, x_f^*, x_g^*)) + \rho(f(x_f^*, \ldots, x_f^*, x_g^*), f(x_f^*, \ldots, x_f^*, x_g^*, x_g^*)) +$$
$$+ \cdots + \rho(f(x_f^*, x_g^*, \ldots, x_g^*), f(x_g^*, x_g^*, \ldots, x_g^*))] + \eta$$

and further on

$$\leq C \cdot [A_1 \rho(x_f^*, x_f^*) + A_2 \rho(x_f^*, x_f^*) + \cdots + A_k \rho(x_f^*, x_g^*) +$$
$$+ A_1 \rho(x_f^*, x_f^*) + \cdots + A_{k-1} \rho(x_f^*, x_g^*) + A_k \rho(x_f^*, x_g^*) +$$
$$+ \cdots + A_1 \rho(x_f^*, x_f^*) + A_2 \rho(x_g^*, x_g^*) + \cdots + A_k \rho(x_g^*, x_g^*)] + \eta =$$
$$= C \cdot A \cdot \rho(x_f^*, x_g^*) + \eta$$
$$d(x_f^*, x_g^*) \leq c_1 \cdot C \cdot A \cdot d(x_f^*, x_g^*) + \eta$$
$$d(x_f^*, x_g^*) \leq \frac{\eta}{I_n - c_1 \cdot C \cdot A}.$$

(d) :

Let $x, y \in X^k$, $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k)$. $1_{X^k} - f : X^k \to X$ is injective if from $1_{X^k} - f(x) = 1_{X^k} - f(y)$, we have $x = y$.

In $1_{X^k} - f(x) = 1_{X^k} - f(y)$, $1_{X^k}(x, x, \ldots, x) = (x, x, \ldots, x)$, so $f(x) = f(y)$.

$$d(f(x), f(y)) = d((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_k - y_k|$$

and

$$d(f(x), f(y)) = \|f(x) - f(y)\| = 0$$

so

$$x_1 = y_1,\ x_2 = y_2,\ \ldots,\ x_k = y_k,\ that\ is\ x = y.$$

$1_{X^k} - f : X^k \to X$ is surjective if for any $y \in X$, there exists $x = (x_1, x_2, \ldots, x_k) \in X^k$ such that $1_{X^k} - f(x) = y$.

We define the mapping $g : X^k \to X$ by

$$(x_1, x_2, \ldots, x_k) \mapsto f(x_1, x_2, \ldots, x_k) + y.$$

We have

$$\|g(x_1, x_2, \ldots, x_k) - g(x_2, x_3, \ldots, x_{k+1})\|_2 = \|f(x_1, x_2, \ldots, x_k) - f(x_2, x_3, \ldots, x_{k+1})\|_2 \leq$$
$$\leq A_1 \|x_1 - x_2\|_2 + A_2 \|x_2 - x_3\|_2 + \cdots + A_k \|x_k - x_{k+1}\|_2$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$.

By Prešić-Perov type fixed point theorem, Corollary 2.2, the mapping g has a unique fixed point, $z^* = f(z^*, z^*, \ldots, z^*)$, and $z^* - f(z^*) = y$.

We obtain $1_{X^k} - f : X^k \to X$ is a bijective mapping, from (i), (iii), (iv) and f a Prešić-Perov operator.

From (iv), $1_{X^k} - f : (X^k, \bar{d}) \to (X, d)$ is continuous.

Let be $y_{n+1} \xrightarrow{\|\cdot\|_1} y$ for $n \to \infty$, where $y_{n+1} = f(y_n, y_n, \ldots, y_n)$, $y = f(y, y, \ldots, y)$.

There exists $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1})$ and $x = f(x, x, \ldots, x) \in X^k$ such that

$$x_{n+1} - f(x_{n+1}) = y_{n+1}\ and\ x - f(x) = y$$

Then

$$\|x - x_{n+1}\|_1 = \|y + f(x) - y_{n+1} - f(x_{n+1})\|_1 \leq \|y - y_{n+1}\|_1 + \|f(x) - f(x_{n+1})\|_1 \leq$$
$$\leq \|y - y_{n+1}\|_1 + C \cdot \|x - x_{n+1}\|_2 \leq \|y - y_{n+1}\|_1 + c_1 \cdot C \cdot \|x - x_{n+1}\|_1$$
$$\|x - x_{n+1}\|_1 \leq (I_n - c_1 \cdot C)^{-1} \|y - y_{n+1}\|_1$$

It follows that $x_{n+1} \xrightarrow{\|\cdot\|_1} x$, therefore $(1_{X^k} - f)^{-1} : (X^k, \bar{d}) \to (X, d)$ is continuous.

So $1_{X^k} - f : (X^k, \|\cdot\|_1) \to (X, \|\cdot\|_1)$ is a homeomorphism for $det(I_n - c_1 C) \neq 0$. □

Remark 2.5. We have the following important particular cases of Theorem 2.5:
1. If $k = 1$, by Theorem 2.5 we get a Maia-Perov type fixed point theorem, given by Mureşan V. in [6].
2. If $d = \rho$ and $k = 1$ by Theorem 2.5 we get Perov's [10] fixed point theorem 1.1.

Remark 2.6. If the conditions (i) and (ii) from Theorem 2.5 are replaced with (1) $d(x, y) \leq \rho(x, y)$, for all $x, y \in X^k$, then we obtain the following result, presented as a Corollary.

Corollary 2.1. *Let X be a nonempty set endowed with the generalized metrics $d, \rho : X \times X \to \mathbb{R}^n$ and $f : X^k \to X$ a Prešić-Perov operator w.r.t. ρ.*

If the conditions are fulfilled:
(1) $d(x, y) \leq \rho(x, y)$, for all $x, y \in X^k$;
(2) (X, d) is a complete metric space;
(3) $f : (X^k, \bar{d}) \to (X, d)$ is continuous.
Then
(a) f has a unique fixed point x^*, $x^* = f(x^*, x^*, \ldots, x^*)$;
(b) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}), n \geq k-1$, converges to x^* w.r.t. d;
(c) the sequence $\{y_n\}_{n \geq 0}$, $y_{n+1} = f(y_n, y_n, \ldots, y_n), n \geq 0$, converges to x^* w.r.t. ρ;
(d) the following estimation takes place: $\rho(x_{n+1}, x^*) \leq A \cdot \rho(x_{n-k+1}, x^*) + N \cdot \theta^{n-1}$, where $N = L \cdot [A_1 + (A_1 + A_2) \cdot \theta + \cdots + (A_1 + A_2 + \cdots + A_{k-1}) \cdot \theta^{-k+2}] > O_n$.

Remark 2.7. If $d = \rho$, from Corollary 2.1 we obtain a Prešić-Perov type fixed point theorem:

Corollary 2.2. *Let (X, d) be a generalized complete metric space, with the metric $d : X \times X \to \mathbb{R}^n$, k a positive integer and $f : X^k \to X$ a Prešić-Perov operator, it is, a mapping for which there exists $A_1, A_2, \ldots, A_k \in M_{nn}(\mathbb{R})$ matrices convergent towards 0, $\sum_{i=1}^{k} A_i = A < I_n$ such that:*

$$d\big(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)\big) \leq \sum_{i=1}^{k} A_i d(x_{i-1}, x_i), \tag{2.9}$$

for all $x_0, \ldots, x_k \in X$.
Then:
1) f has a unique fixed point x^*;
2) the sequence $\{y_n\}_{n \geq 0}$,

$$y_{n+1} = f(y_n, y_n, \ldots, y_n), n \geq 0,$$

converges to x^*;
3) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}), n \geq k-1$, converges to x^* with a rate estimated by

$$d(x_{n+1}, x^*) \leq A \cdot d(x_{n-k+1}, x^*) + N \cdot \theta^{n-1},$$

where $N = L \cdot [A_1 + (A_1 + A_2) \cdot \theta + \cdots + (A_1 + A_2 + \cdots + A_{k-1}) \cdot \theta^{-k+2}] > O_n$.

Remark 2.8. We have the following important particular cases of Corollary 2.1:
1. If $k = 1$, by Corollary 2.1 we get a Maia-Perov fixed point theorem from [1];
2. If $d = \rho$, by Corollary 2.1 we get a Prešić-Perov type fixed point theorem, Corollary 2.2.

Following the result in [6] and using Theorem 2.5, we can study the existence, uniqueness and dependence of data for the solution of the system of integral equations of Fredholm type:

$$\varphi(x) = \int_\Omega K(x, y, \varphi(y))dy + f(x) \tag{2.10}$$

where $K \in C(\bar{\Omega} \times \bar{\Omega} \times X^k, \mathbb{R}^m)$, $f \in X$ and $\Omega \in \mathbb{R}^m$ bounded domaine.

We denote $X = C(\bar{\Omega}, \mathbb{R}^m)$ and let $A : X^k \to X$ be the operator defined by

$$(A\varphi)(x) = \int_\Omega K(x, y, \varphi(y))dy + f(x)$$

We consider the generalized metric space X, endowed with the following two metrics

$$d(\varphi, \psi) = \|\varphi - \psi\|_{C(\bar{\Omega},\mathbb{R}^m)} = (\|\varphi_1 - \psi_1\|_{C(\bar{\Omega})}, \dots, \|\varphi_m - \psi_m\|_{C(\bar{\Omega})}),$$

$$\rho(\varphi, \psi) = \|\varphi - \psi\|_{L^2(\Omega,\mathbb{R}^m)} = (\|\varphi_1 - \psi_1\|_{L^2(\Omega)}, \dots, \|\varphi_m - \psi_m\|_{L^2(\Omega)}),$$

where

$$\|u_i\|_{C(\bar{\Omega})} = \max\{|u_i(t)| : t \in \bar{\Omega}\}, \|u_i\|_{L^2(\Omega)} = (\int_\Omega |u_i(t)|^2 dt)^{\frac{1}{2}}$$

We get the following existence and uniqueness theorem of the solution

Theorem 2.6. *If the following conditions are fulfilled*

(i) $K \in C(\bar{\Omega} \times \bar{\Omega} \times X^k, \mathbb{R}^m)$, $f \in X$

(ii) *there exists* $L : \Omega \times \Omega \to M_{n,n}(\mathbb{R}_+)$ *nonnegative, with*

$$\sup_{x\in\bar{\Omega}} (\int_\Omega |L_{ij}(x, y)|^2 dy)^{\frac{1}{2}} < \infty, \ for\ any\ i, j = \overline{1, m},$$

so that $|K(x, y, u) - K(x, y, v)| \le L(x, y)|u - v|$, *for any* $x, y \in \Omega, u, v \in X^k$ *and* $|K(x, y, 0)| \le r(x, y)$ *where* $r \in C(\bar{\Omega} \times \bar{\Omega}, \mathbb{R}^m)$ *nonnegative.*

(iii) *there exists a convergent matrix toward 0,* $S \in M_{n,n}(\mathbb{R})$ *such that*

$$(\{\int_{\Omega\times\Omega} |L_{ij}(x, y)|^2 dxdy\}^{\frac{1}{2}})_m^m \le S$$

then

(a) *the system of equations* (2.10) *has in* X^k *one and only one solution* φ^* *which can be obtained by the succesive approximations method starting from any element from* X^k.

(b) *If* $\psi \in X^k$ *is so that*

$$\psi(x) = \int_\Omega H(x, y, \psi(y))dy + f(x),$$

where $H \in C(\bar{\Omega} \times \bar{\Omega} \times X^k, \mathbb{R}^m)$ *and there exists* $\eta_1 \in \mathbb{R}^m$ *so that*

$$|K(x, y, u) - H(x, y, u)| \le \eta_1,$$

for any $x, y \in \Omega$ *and* $u \in X^k$

then

$$\|\varphi^* - \psi\|_{C(\bar{\Omega}\times\mathbb{R}^m)} \le \frac{m(\Omega)\eta_1}{I_n - c_1 \cdot C \cdot A}$$

where

$$A = (\{\int_{\Omega\times\Omega} |L_{ij}(x, y)|^2 dxdy\}^{\frac{1}{2}})_m^m$$

$$c_1 = (m(\Omega))^{\frac{1}{2}},$$
$$C = (\sup_{t\in\bar{\Omega}} \|L_{ij}(t,.)\|_{L^2(\Omega)})^m_m.$$

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