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An iterative process for a hybrid pair of a Bregman strongly nonexpansive single-valued mapping and a finite family of Bregman relative nonexpansive multi-valued mappings in Banach spaces

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ABSTRACT. In this paper, we construct an iterative process involving a hybrid pair of a Bregman strongly nonexpansive single-valued mapping and a finite family of Bregman relative nonexpansive multi-valued mappings and prove strong convergence theorems of the proposed iterative process in reflexive Banach spaces under appropriate conditions. Our main results can be viewed as an improvement and extension of the several results in the literature.

1. INTRODUCTION

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a reflexive Banach space, and let C be a nonempty, closed and convex subset of E and $T : C \to C$ be a mapping. Denote by $F(T) = \{x \in C : x = Tx\}$ is the set of fixed points of T. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1967, Bregman [4] has discovered an elegant and effective technique for the use of the Bregman distance function D_f in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregmans technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see, e.g.,[5], [15],[17], and the references therein).

Many researchers used the Bregman distances for approximating fixed points of nonlinear mappings in several iterative methods. In 2012, Suantai et al. [21] used the following Halpern's iterative scheme for a Bregman strongly nonexpansive self mapping T on E. For $u, x_1 \in E$, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Tx_n)), \ \ \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of T. Later, Li et al. [9] extended a Bregman strongly nonexpansive self mapping T on E for Halpern's iteration method to Bregman strongly nonexpansive multi-valued mapping $T : C \to N(C)$ as follows:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(z_n)), \ \ z_n \in Tx_n \ \ \forall n \geq 1, \tag{1.2}$$

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where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $N(C)$ is the family of nonempty subsets of C. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of T.

Very recently, Senakka and Cholamjiak [19] studied the strong convergence for a common fixed point of $T, S : C \to C$ which are Bregman strongly nonexpansive mappings in a reflexive Banach space E. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = P_c^f[\nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n))] \\ x_{n+1} = P_c^f[\nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Sy_n))], \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\} \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{\mathcal{F}}^f(u)$ where $\mathcal{F} = F(T) \cap F(S)$ and $P_{\mathcal{F}}^f(u)$ is a Bregman projection from E onto $\mathcal{F}$.

In this paper, we construct an iterative process involving a hybrid pair of a Bregman strongly nonexpansive single-valued mapping and a finite family of Bregman relative nonexpansive multi-valued mappings and prove strong convergence theorems of the proposed iterative process in reflexive Banach spaces under appropriate situations. Our main results can be viewed as an improvement and extension of the several results in the literature.

2. Preliminaries

Let E be a real reflexive Banach space with the dual space of E^*, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. Let $f : E \to (-\infty, +\infty]$ be a function. The effective domain of f is defined by

$$domf := \{x \in E : f(x) < +\infty\}.$$

We say that f is proper if $domf \neq \emptyset$. We denote by $int(domf)$ the interior of the effective domain of f. We denote by $ranf$ the range of f.

Let $x \in int(domf)$. The subdifferential of f at x is the convex set defined by:

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \ \forall y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}$. We know that $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x, x^* \rangle$ for all $x \in E$. A function f on E is said to be strongly coercive if $lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$, for any $x \in int(domf)$ (see [24]). Let $B_r := \{x \in E : \|z\| \leq r\}$. A function f on E is said to be locally bounded if $f(B_r)$ is bounded for all $r > 0$.

Let $f : E \to (-\infty, +\infty]$ be a convex function and $x \in int(dom)f$. The gradient $\nabla f(x)$ is defined to be the linear functional in E^* such that

$$\langle y, \nabla f(x) := \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}, \ \forall y \in E.$$

The function f is said to be Gâteaux differentiable at x if $\nabla f(x)$ is well defined, and f is Gâteaux differentiable if it is Gâteaux differentiable every where on E. The function f is said to be Frèchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Frèchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : domf \times int(domf) \to (-\infty, +\infty]$ defined as follows:

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

is called the Bregman distance with respect to f (see [4]).

Remark 2.1. The Bregman distance D_f does not satisfy the well-known properties of a metric because D_f is not symmetric and does not satisfy the triangle inequality.

A Bregman projection [4] of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\left(P_C^f(x), x\right) = \inf\{D_f(y, x) : y \in C\}.$$

Let C be a nonempty, closed and convex subset of $int(domf)$. A point $p \in C$ is called an asymptotic fixed point of T (see[15]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

Definition 2.1. ([3]) The function f is called to be

(i) essentially smooth if f is both locally bounded and single-valued on its domain.
(ii) essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $domf$.
(ii) Legendre if it is both essentially smooth and essentially strictly convex.

It is well known that in a reflexive Banach space E, if f is a Legendre function, then satisfies the following conditions:

(L1) f is essentially smooth if and only if f^* is essentially strictly convex.
(L2) f is Legendre if and only if f^* is Legendre.
(L3) $(\partial f)^{-1} = \partial f^*$.
(L4) If f is Legendre, then ∇f is a bijection satisfying:

$$\nabla f = (\nabla f^*)^{-1}, ran\nabla f = dom\nabla f^* = int(domf^*) \text{ and } \nabla f^* = dom\nabla f = int(domf).$$

If E is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. In this case, the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In this article, we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

Definition 2.2. Let C be a nonempty and convex subset of $int(domf)$. A mapping $T : C \to int(domf)$ with $F(T) \neq \emptyset$ is called to be

(i) Bregman quasi-nonexpansive, if

$$D_f(p, Tx) \leq D_f(p, x), \ \forall x \in C, p \in F(T).$$

(ii) Bregman relatively nonexpansive with respect to f, if $F(T) = \hat{F}(T)$,

$$D_f(p, Tx) \leq D_f(p, x), \ \forall x \in C, p \in F(T).$$

(iii) Bregman strongly nonexpansive with respect to f, if $F(T) = \hat{F}(T)$,

$$D_f(p, Tx) \leq D_f(p, x), \ \forall x \in C, p \in F(T).$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0.$$

It is obvious that any Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping, but the converse is not true in general. Pang et al. [13] showed that there exists a Bregman relatively nonexpansive mapping which is not a Bregman strongly nonexpansive mapping.

Let $N(C)$ and $CB(C)$ denote the families of nonempty subsets and nonempty closed bounded subsets of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} dist(x,B), \sup_{y\in B} dist(y,A)\} \text{ for } A,B \in CB(C),$$

for all $A,B \in CB(C)$, where $dist(x,B) = \inf\{\|x-y\| : y \in B\}$ is the distance from a point x to a subset B.

Definition 2.3. A multi-valued mapping $T : C \to CB(C)$ is said to be

(i) nonexpansive if $H(Tx,Ty) \le \|x-y\|$, for all $x,y \in C$.
(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx,Tp) \le \|x-p\|$, for all $x \in C$ and $p \in F(T)$.

Let $T : C \to CB(C)$. A point $p \in C$ is said to be a *fixed point* of T, if $p \in F(T)$, where $F(T) = \{p \in T : p \in Tp\}$. A point $p \in C$ is said to be an *asymptotic fixed point* [15] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 2.4. ([20]) A mapping $T : C \to CB(C)$ with $F(T) \neq \emptyset$ is called to be

(i) Bregman quasi-nonexpansive, if

$$D_f(p,z) \le D_f(p,x), \ \forall z \in Tx, x \in C \text{ and } p \in F(T).$$

(ii) Bregman relatively nonexpansive, if $F(T) = \hat{F}(T)$,

$$D_f(p,z) \le D_f(p,x), \ \forall z \in Tx, x \in C \text{ and } p \in F(T).$$

The following is an example of multi-valued Bregman relatively nonexpansive mapping given by (see [20]).

Example 2.1. Let $I = [0,1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \ge 0, \forall x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$\begin{cases} \{h \in C : f(x) - \frac{1}{2} \le h(x) \le f(x) - \frac{1}{4}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise.} \end{cases} \tag{2.4}$$

Then T defined by (2.4) is a multi-valued Bregman relatively nonexpansive mapping (see [20]).

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Define a function $V_f : E \times E^* \to [0,+\infty)$ associated with f by

$$V_f(x,x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \ \forall x \in E, x^* \in E^*. \tag{2.5}$$

Then V_f is nonnegative and

$$V_f(x,x^*) = D_f(x, \nabla f(x^*) \ \forall x \in E, x^* \in E^*. \tag{2.6}$$

Moreover, by the subdifferential inequality,

$$V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \le V_f(x, x^* + y^*) \ \forall x \in E, x^*, y^* \in E^*, \tag{2.7}$$

(for more details see [1] and [8]).

Let $f : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. The function f is called totally convex if it is totally convex at any point $x \in int(domf)$ and is said to be totally convex on bounded if $v_f(B,t) > 0$ for any nonempty bounded subset B of E and

$t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : int(domf) \times [0, +\infty] \to [0, +\infty]$ defined by

$$v_f(B, t) := \inf\{V_f(x, t) : x \in B \cap domf\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [6]).

The now recall the following lemmas that will be used in the sequel.

Lemma 2.1. ([18]) Let C be a nonempty closed and convex subset of $int(domf)$ and $T : C \to C$ be a quasi-Bregman nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.

Lemma 2.2. ([20]) Let E be a real reflexive Banach space, and let $f : E \to \mathbb{R}$ be a uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T : C \to CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.

Lemma 2.3. ([6]) Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a G$\hat{a}$teaux differentiable and totally convex function and let $x \in E$. Then

$$z \in P_C^f(x) \text{ if and only if } \langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0, \ \forall y \in C.$$

Lemma 2.4. ([24]) Let $f : E \to \mathbb{R}$ be a strongly coercive and uniformly convex on bounded subsets of E, then f^* is bounded and uniformly Fr$\acute{e}$chet differentiable on bounded subsets of E^*.

Lemma 2.5. ([16]) Let $f : E \to (-\infty, +\infty]$ be a uniformly Fr$\acute{e}$chet differentiable and bounded on bounded sets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Lemma 2.6. ([17]) Let$f : E \to \mathbb{R}$ be a G$\hat{a}$teaux differentiable on $int(domf)$ such that ∇f^* is bounded on bounded subset of $domf^*$. Let x^* and $\{x_n\} \subset int(E)$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.

Lemma 2.7. ([14]) Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is proper, weak*lower semi-continuous and convex function. Thus, for all $z \in E$, we have:

$$D_f\Big(z, \nabla f^*\Big(\sum_{i=1}^{N} t_i \nabla f(x_i)\Big)\Big) \le \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.8. ([12]) Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then

$$f\Big(\sum_{k=0}^{n} \alpha_k x_k\Big) \le \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - y_i\|),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.9. ([12]) Let E be a Banach space and $f : E \to \mathbb{R}$ be a G$\hat{a}$teaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;
(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.10. ([23]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \;\; n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. ([10]) Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$.

$$a_{m_k} \leq a_{m_k+1} \;\; a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3. Main results

Theorem 3.1. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and subset of $int(domf)$ and $S : C \to C$ be a Bregman strongly nonexpansive mapping with respect to f. Let $\{T_i\}_{i=1}^N$ be a finite family of Bregman relative nonexpansive multi-valued mappings of C into $CB(C)$. Assume that $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\text{(3.8)} \qquad \begin{cases} y_n = P_c^f \nabla f^*[\beta_n^{(0)}\nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)}\nabla f(u_n^{(i)})], \;\; u_n^{(i)} \in T_i x_n, \\ x_{n+1} = P_c^f \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Sy_n)], \;\; n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{\beta_n^{(i)}\}_{i=1}^N \subset [a,b] \subset (0,1)$ and $\sum_{i=1}^N \beta_n^{(i)} = 1$. Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

Proof. From Lemma 2.1 and Lemma 2.2, we know that $F(S)$ and $F(T_i)$ for all $i = 1, 2, ..., N$ are closed and convex, hence $\mathcal{F}$ is closed and convex. Let $p = P_{\mathcal{F}}^f(u)$. Then

$$\text{(3.9)} \;\; D_f(p, y_n) = D_f(p, P_C^f \nabla f^*[\beta_n^{(0)}\nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)}\nabla f(u_n^{(i)})]) \leq D_f(p, \nabla f^*(\beta_n^{(0)}\nabla f(x_n)$$
$$+ \sum_{i=1}^N \beta_n^{(i)}\nabla f(u_n^{(i)}))) \leq \beta_n^{(0)}D_f(p, x_n) + \sum_{i=1}^N \beta_n^{(i)}D_f(p, u_n^{(i)}) \leq \beta_n^{(0)}D_f(p, x_n) + \sum_{i=1}^N \beta_n^{(i)}D_f(p, x_n)$$
$$= D_f(p, x_n).$$

Now, using (3.9) and Bregman strongly nonexpansiveness of S, we have

$$D_f(p, x_{n+1}) \leq D_f(p, \nabla f^*[\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n)]) \leq \alpha_n D_f(p, u) + (1-\alpha_n)D_f(p, Sy_n)$$
$$\leq \alpha_n D_f(p, u) + (1-\alpha_n)D_f(p, y_n) \leq \alpha_n D_f(p, u) + (1-\alpha_n)D_f(p, x_n)$$
$$\leq \max\{D_f(p, u), D_f(p, x_n)\}.$$

By induction, we obtain that $\{D_f(p, x_n)\}$ is bounded. Using Lemma 2.6, we have the sequence $\{x_n\}$ is bounded. Let $z_n = \nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n))$, $n \geq 1$. We obtain that

$$D_f(p, x_{n+1}) = D_f(p, P_C^f[\nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n))])$$
$$\leq D_f(p, \nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n))) = V_f(p, \alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n))$$
$$\leq V_f(p, \alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Sy_n) - \alpha_n(\nabla f(u) - \nabla f(p))) + \alpha_n\langle\nabla f(u) - \nabla f(p), z_n - p\rangle$$
$$= V_f(p, \alpha_n\nabla f(p) + (1-\alpha_n)\nabla f(Sy_n)) + \alpha_n\langle\nabla f(u) - \nabla f(p), z_n - p\rangle \leq \alpha_n V_f(p, \nabla f(p))$$

$+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle=\alpha_n D_f(p,p)+(1-\alpha_n)D_f(p,Sy_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle$

$$= (1-\alpha_n)D_f(p,Sy_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle \tag{3.10}$$

$$\leq (1-\alpha_n)D_f(p,y_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle \tag{3.11}$$

$$\leq (1-\alpha_n)D_f(p,x_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle. \tag{3.12}$$

Moreover, we have

$$D_f(p,y_n)\leq D_f(p,\nabla f^*(\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)})))=V_f(p,\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)}))$$

$$=f(p)-\langle p,\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)})\rangle+f^*(\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)})). \tag{3.13}$$

Since f is a uniformly Fréchet differentiable function, we obtain that f is uniformly smooth. Hence by Theorem 3.5.5 of [24], we get that f^* is uniformly convex. This, with Lemma 2.8, and (3.13) yields

$$D_f(p,y_n)\leq f(p)-\langle p,\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)})+f^*(\beta_n^{(0)}\nabla f(x_n)+\sum_{i=1}^N\beta_n^{(i)}\nabla f(u_n^{(i)}))$$

$$\leq f(p)-\beta_n^{(0)}\langle p,\nabla f(x_n)\rangle+\sum_{i=1}^N\beta_n^{(i)}\langle p,\nabla f(u_n^{(i)})+\beta_n^{(0)}f^*(\nabla f(x_n))+\sum_{i=1}^N\beta_n^{(i)}f^*(\nabla f(u_n^{(i)}))$$

$$-\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)=\beta_n^{(0)}V_f(p,\nabla f(x_n))+\sum_{i=1}^N\beta_n^{(i)}V_f(p,\nabla f(u_n^{(i)}))$$

$$-\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)=\beta_n^{(0)}D_f(p,x_n)+\sum_{i=1}^N\beta_n^{(i)}D_f(p,u_n^{(i)})$$

$$-\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)=\beta_n^{(0)}D_f(p,x_n)+\sum_{i=1}^N\beta_n^{(i)}D_f(p,x_n)$$

$$-\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)=D_f(p,x_n)-\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|),$$

which implies that

$$\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)\leq D_f(p,x_n)-D_f(p,y_n).$$

By (3.11), we obtain that

$$\begin{aligned}\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)&\leq D_f(p,x_n)-D_f(p,x_{n+1})\\&\quad-\alpha_n D_f(p,y_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),z_n-p\rangle.\end{aligned} \tag{3.14}$$

Now, we consider two cases.

Case I Suppose that there exists $n_0\in\mathbb{N}$, such that $\{D_f(p,x_n)\}$ is nonincreasing for all $n\geq n_0$. Then $\{D_f(p,x_n)\}$ is convergent and hence $\{D_f(p,x_n)\}-\{D_f(p,x_{n+1})\}\to 0$ as $n\to\infty$. From (3.14), we have

$$\lim_{n\to\infty}\beta_n^{(0)}\beta_n^{(i)}\rho_r^*(\|\nabla f(x_n)-\nabla f(u_n^{(i)})\|)=0,$$

which implies, by the property of ρ_r^* that

$$\nabla f(x_n)-\nabla f(u_n^{(i)}))\to 0 \text{ as } n\to\infty. \tag{3.15}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E and by Lemma 2.4, we have $f^*$is uniformly Fr$\acute{e}$chet differentiable on bounded subsets of E^*. Since f is Legendre by Lemma 2.5, we obtain that ∇f^* is uniformly continuous on bounded subsets of E^*. From (3.15) we get that

$$x_n - u_n^{(i)} \to 0 \text{ as } n \to \infty.$$

Since $d(x_n, T_i x_n) \leq \|x_n - u_n^{(i)}\|$, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \tag{3.16}$$

for each $i \in \{1, 2, ..., N\}$.

Since $\{x_n\}$ is bounded and E is reflexive, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Thus from (3.16) and the fact that each T_i is Bregman relatively nonexpansive, we obtain $z \in F(T_i)$, for each $i \in \{1, 2, ..., N\}$. This implies that $z \in \bigcap_{i=1}^N F(T_i)$.

We now show that $z \in F(S)$. By (3.10), we obtain that

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1 - \alpha_n) D_f(p, Sy_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle = D_f(p, Sy_n) \\ &-\alpha_n D_f(p, Sy_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle = D_f(p, Sy_n) - \alpha_n D_f(p, Sy_n) \\ &+\alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle - D_f(p, y_n) + D_f(p, y_n). \end{aligned} \tag{3.17}$$

By (3.9), we have

$$\begin{aligned} D_f(p, y_n) - D_f(p, Sy_n) &\leq D_f(p, y_n) - D_f(p, x_{n+1}) - \alpha_n D_f(p, Sy_n) \\ &+\alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p \leq D_f(p, x_n) - D_f(p, x_{n+1}) - \alpha_n D_f(p, Sy_n) \\ &+\alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle. \end{aligned} \tag{3.18}$$

Thus $D_f(p, y_n) - D_f(p, Sy_n) \to 0$ as $n \to \infty$. Since S is a Bregman strongly nonexpansive mapping, we have $\lim_{n\to\infty} D_f(y_n, Sy_n) = 0$. This implies that

$$\lim_{n\to\infty} \|Sy_n - y_n\| = 0. \tag{3.19}$$

We see that

$$\begin{aligned} D_f(x_n, y_n) &\leq D_f(x_n, \nabla f^*(\beta_n^{(0)} \nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)} \nabla f(u_n^{(i)})) \\ &\leq \beta_n^{(0)} D_f(x_n, x_n) + \sum_{i=1}^N \beta_n^{(i)} D_f(x_n, u_n^{(i)}). \end{aligned} \tag{3.20}$$

Since $\|x_n - u_n^{(i)}\| \to 0, n \to \infty$ and $\{u_n^i\}$ is a bounded sequence, by Lemma 2.9, we obtain that $\lim_{n\to\infty} \Delta_p(x_n, u_n^{(i)}) = 0$. From (3.20), it follows that $\lim_{n\to\infty} D_f(x_n, y_n) = 0$.
So that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since E is reflexive, $\{y_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we obtain that $y_{n_k} \rightharpoonup z$. Since $F(S) = \hat{F}(S)$ and (3.19), we have $z \in F(S)$. Thus $z \in F(S) \cap \bigcap_{i=1}^N F(T_i) = \mathcal{F}$.

Furthermore, we have that

$$D_f(y_n, z_n) \leq D_f(y_n, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(Sy_n))) \leq \alpha_n D_f(y_n, u) + (1-\alpha_n) D_f(y_n, Sy_n).$$

Therefore $\lim_{n\to\infty} D_f(y_n, z_n) = 0$. It follows that $\lim_{n\to\infty} \|y_n - z_n\| = 0$.

Let $p = P_{\mathcal{F}}^f(u)$. We next show that $\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), z_n - p\rangle \leq 0$. Since $\lim_{n\to\infty} \|y_n - z_n\| = 0$, so we obtain that $z_{n_k} \rightharpoonup z$, it follows that

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), z_n - p\rangle = \lim_{n\to\infty} \langle \nabla f(u) - \nabla f(p), z_{n_k} - p\rangle.$$

Moreover, by Lemma 2.3, we have

$$\limsup_{n\to\infty}\langle \nabla f(u)-\nabla f(p), z_n-p\rangle = \langle f(u)-\nabla f(p), z-p\rangle \leq 0.$$

Now using the above inequality and (3.12), we obtain $D(p,x_n)\to 0$ as $n\to\infty$. Hence $x_n\to p$ as $n\to\infty$.

Case II Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(p,x_{n_i})\leq D_f(p,x_{n_i+1}),\ \text{ for all } i\in\mathbb{N}.$$

Then, by Lemma 2.11, there exists a non-decreasing sequence $m_k\subset\mathbb{N}$ such that $m_k\to\infty$, and $D_f(p,x_{m_k})\leq D_f(p,x_{m_k+1})$ and $D_f(p,x_k)\leq D_f(p,x_{m_k+1})$ for all $k\in\mathbb{N}$. Thus, we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty}(D_f(p,x_{m_k+1})-D_f(p,x_{m_k}))\\
&\leq \limsup_{n\to\infty}(D_f(p,x_{n+1})-D_f(p,x_n))\\
&\leq \limsup_{n\to\infty}(\alpha_n D_f(p,u)+(1-\alpha_n)D_f(p,Sy_n)-D_f(p,x_n))\\
&\leq \limsup_{n\to\infty}(\alpha_n D_f(p,u)+(1-\alpha_n)D_f(p,y_n)-D_f(p,x_n))\\
&\leq \limsup_{n\to\infty}(\alpha_n D_f(p,u)+(1-\alpha_n)D_f(p,x_n)-D_f(p,x_n))\\
&= \limsup_{n\to\infty}\alpha_n(D_f(p,u)-D_f(p,x_n))\\
&= 0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}(D_f(p,x_{m_k+1})-D_f(p,x_{m_k}))=0. \tag{3.21}$$

By (3.14), and $\alpha_n\to 0$, we obtain that

$$\rho_r^*(\|\nabla f(x_{m_k})-\nabla f(u_{m_k}^{(i)})\|)\to 0 \text{ as } k\to\infty$$

for each $i\in\{1,2,...,N\}$. By following the method of proof of case I, we obtain that $d(x_{m_k},T_i x_{m_k})\to 0$ as $k\to\infty$. As the proof in case I and (3.18), we obtain that $\lim_{n\to\infty}\|Sy_{m_k}-y_{m_k}\|=0$ and

$$\limsup_{n\to\infty}\langle \nabla f(u)-\nabla f(p), z_{m_k}-p\rangle\leq 0. \tag{3.22}$$

Then, from (3.12), we get that

$$D_f(p,x_{m_k+1})\leq (1-\alpha_{m_k})D_f(p,x_{m_k})+\alpha_{m_k}\langle\nabla f(u)-\nabla f(p), z_{m_k}-p\rangle.$$

Since $D_f(p,x_{m_k})\leq D_f(p,x_{m_k+1})$, the above inequality implies that

$$\begin{aligned}
\alpha_{m_k}D_f(p,x_{m_k}) &\leq D_f(p,x_{m_k})-D_f(p,x_{m_k+1})+\alpha_{m_k}\langle\nabla f(u)-\nabla f(p), z_{m_k}-p\rangle\\
&\leq \alpha_{m_k}\langle\nabla f(u)-\nabla f(p), z_{m_k}-p\rangle.
\end{aligned}$$

In particular, since $\alpha_{m_k}>0$, we have

$$D_f(p,x_{m_k})\leq\langle\nabla f(u)-\nabla f(p), z_{m_k}-p\rangle.$$

Hence, by the above inequality, we have $\lim_{k\to\infty}D_f(p,x_{m_k})=0$.

This together with (3.21), gives $\lim_{k\to\infty}D_f(p,x_{m_k+1})=0$. By $D_f(p,x_k)\leq D_f(p,x_{m_k+1})$ for all $k\in\mathbb{N}$, we conclude that $\lim_{k\to\infty}D_f(p,x_k)=0$. Hence $x_k\to p$ as $k\to\infty$.

Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p=P_c^f(u)$ and the proof is complete. □

If we assume that T_i $(i = 1, 2, ..., N)$ to be a Bregman relative quasi-nonexpansive multi-valued mapping in Theorem 3.1, then we get the following corollary:

Corollary 3.1. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and subset of $int(domf)$ and $S : C \to C$ be Bregman strongly nonexpansive mapping with respect to f. Let $\{T_i\}_{i=1}^N$ be a finite family of Bregman relative quasi-nonexpansive multi-valued mappings of C into $CB(C)$. Assume that $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_c^f \nabla f^*[\beta_n^{(0)} \nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)} \nabla f(u_n^{(i)})], \;\; u_n^{(i)} \in T_i x_n, \\ x_{n+1} = P_c^f \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(S y_n)], \;\; n \in \mathbb{N}, \end{cases} \tag{3.23}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{\beta_n^{(i)}\}_{i=1}^N \subset [a, b] \subset (0, 1)$ and $\sum_{i=1}^N \beta_n^{(i)} = 1$. Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

If we assume that $T_i = T$ for each $i = 1, 2, ...N$ in Theorem 3.1, then we get the following corollary:

Corollary 3.2. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and subset of $int(domf)$ and $S : C \to C$ be Bregman strongly nonexpansive mapping with respect to f. Let T be a Bregman relative nonexpansive multi-valued mappings of C into $CB(C)$. Assume that $\mathcal{F} = F(S) \cap F(T)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_c^f \nabla f^*[\beta \nabla f(x_n) + (1 - \beta) \nabla f(u_n)], \;\; u_n \in T x_n, \\ x_{n+1} = P_c^f \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(S y_n)], \;\; n \in \mathbb{N}, \end{cases} \tag{3.24}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\beta \subset (0, 1)$. Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

If we assume that each T_i $(i = 1, 2, ..., N)$ is a Bregman relative nonexpansive single-valued mapping and S is an identity mapping on C, we get the following corollary:

Corollary 3.3. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and subset of $int(domf)$. Let $\{T_i\}_{i=1}^N$ be a finite family of Bregman relative nonexpansive mappings of C into C. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_c^f \nabla f^*[\beta_n^{(0)} \nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)} \nabla f(T_i x_n)], \\ x_{n+1} = P_c^f \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n)], \;\; n \in \mathbb{N}, \end{cases} \tag{3.25}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{\beta_n^{(i)}\}_{i=1}^N \subset [a, b] \subset (0, 1)$ and $\sum_{i=1}^N \beta_n^{(i)} = 1$. Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

4. Application

In this section, we give an application of Theorem 3.1, which is the variational inequality problems and the zeros of maximal monotone operator in the framework of reflexive Banach spaces.

4.1 Variational Inequality Problems

Definition 4.5. ([17]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. A mapping $A : E \to 2^{E^*}$ satisfying the range condition, i.e., $ran(\nabla f - A) \subset ran(\nabla f)$ is called Bregman inverse strongly monotone if $domA \cap int(domf) \neq \emptyset$ and for any $x, y \in int(domf)$ and each $u \in Ax$, $v \in Ay$,

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0.$$

Let $A : C \to E^*$ be a Bregman inverse strongly monotone operator, and let C be a nonempty, closed and convex subset of $domA$. The variational inequality problem corresponding to A is to find $x^* \in C$, such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \ \ \forall y \in C. \tag{4.26}$$

The set of solutions of equation (4.26) is denoted by $VI(C, A)$.

Definition 4.6. ([17]) Let $A : E \to 2^{E^*}$ be an any operator; the anti-resolvent $A^f : E \to E^f$ of A is defined by:

$$A^f = \nabla f^* \circ (\nabla f - A).$$

We see that $domA^f \subset domA \cap int(domf)$ and $ranA^f \subset int(domf)$. Moreover, an operator A is Bregman inverse strongly monotone if and only if anti-resolvent A^f is a single-valued Bregman firmly nonexpansive mapping (see Lemma 3.4(c) and (d) in [7]).

Lemma 4.12. ([16]) Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and $f : E \to (-\infty, +\infty]$ be a Legendre and totally convex function that satisfies the range condition. If C is a nonempty, closed and convex subset of $domA \cap int(domf)$, then:

(i) $P^f_C \circ A^f$ is Bregman relatively nonexpansive mapping, where $A^f = \nabla f^* \circ (\nabla f - A)$;
(ii) $F(P^f_C \circ A^f) = VI(C, A)$.

Theorem 4.2. *Let E be a real reflexive Banach space E, $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, which satisfies the range condition, C be a nonempty, closed and subset of $domA \cap int(domf)$ and $A_i : C \to E^*$ $(i = 1, ..., N)$ be a Bregman inverse strongly monotone function and $S : C \to C$ be Bregman strongly nonexpansive mapping with respect to f. Assume that $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N VI(C, A_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P^f_c \nabla f^*[\beta^{(0)}_n \nabla f(x_n) + \sum_{i=1}^N \beta^{(i)}_n \nabla f(P^f_C \circ A^f_i(x_n))], \\ x_{n+1} = P^f_c \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Sy_n)], \ \ n \in \mathbb{N}, \end{cases} \tag{4.27}$$

where $A^f_i = \nabla f^* \circ (\nabla f - A_i)$ for $i = 1, 2, ..., N$. Suppose that $\{\alpha_n\}$ and $\{\beta^{(i)}_n\}_{i=1}^N$ are as in Theorem 3.1. Then $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.

4.2 Zeros of Maximal Monotone Operator

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We denoted by $G(A)$ as the graph of A, i.e., $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. An operator A is called to be monotone if $\langle x^* - y^*, x - y\rangle > 0$ for each $(x, x^*), (y, y^*) \in G(A)$. We call monotone operator A a maximal if its graph is not contained in the graph of any other monotone operators on E. It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{x \in E : 0^* \in Ax\}$ is closed and convex. The resolvent of A, denoted by $Res^f_{\lambda A} : E \to 2^E$, is defined as follows:

$$Res^f_{\lambda A}(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x)$$

where $\lambda > 0$.It is known that $F(Res^f_{\lambda A}) = A^{-1}(0^*)$ and $Res^f_{\lambda A}$ is single-valued and Bregman firmly nonexpansive (see [2]).

In addition, Reich and Sabach [18] proved that if f is a Legendre function, which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then $\hat{F}(Res^f_{\lambda A}) =$

$F(Res^f_{\lambda A})$. And so that if $\hat{F}(Res^f_{\lambda A}) = F(Res^f_{\lambda A})$, then a Bregman that is firmly nonexpansive is a Bregman relatively nonexpansive mapping.

The Yosida approximation $A_\lambda(x) : E \to E$, $\lambda > 0$, is defined by:

$$A_\lambda(x) = \frac{1}{\lambda}(\nabla f(x) - \nabla f(Res^f_{\lambda A})) \text{ for all } x \in E \text{ and } \lambda > 0.$$

Proposition 4.1. ([17]) For any $\lambda > 0$ and for any $x \in X$, we have

(i) $\left(Res^f_{\lambda A}(x), A_\lambda(x)\right) \in G(A)$;
(ii) $0^* \in Ax$ if and only if $0^* \in A_\lambda(x)$.

We take $C = E$ and $T_i = Res^f_{\lambda A_i} > 0$ for all $i = 1, ..., N$ in Theorem 3.1, then we obtain that the following Theorem

Theorem 4.3. *Let E be a real reflexive Banach space E, $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, which satisfies the range condition, C be a nonempty, closed and subset of $domA \cap int(domf)$ and $A_i : E \to E^*(i = 1, 2, ..., N)$ be a finite collection of maximal monotone operators and $S : C \to C$ be Bregman strongly nonexpansive mapping with respect to f. Assume that $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P^f_c \nabla f^*[\beta_n^{(0)} \nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)} \nabla f(Res^f_{\lambda A}(x_n))], \\ x_{n+1} = P^f_c \nabla f^*[\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Sy_n)], \quad n \in \mathbb{N}, \end{cases} \tag{4.28}$$

where $\lambda > 0$. Suppose that $\{\alpha_n\}$ and $\{\beta_n^{(i)}\}_{i=1}^N$ are as in Theorem 3.1. Then $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.

5. Numerical example

In this section, we present some numerical example for supporting Theorem 3.1 on a real line. Let $E = \mathbb{R}$, $C = [-1, 1]$, and $f(x) = \frac{2}{3}x^2$ (f is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E appeared in the numerical example of [22])

Furthermore, let $S = P_C$ (which is an example of a Bregman strongly nonexpansive mapping, see [11]), and

$$T_i x = \begin{cases} [(\frac{\phi_i}{2} - 1)x, (\phi_i - 1)x] & x < 0, \\ \{0\} & x = 0, \\ [(1 - \phi_i)x, (1 - \frac{\phi_i}{2})x] & x > 0, \end{cases}$$

where $\phi_i = \frac{i}{i+1}$ for all $i = 1, 2$. Next, we show that T_i is a Bregman relative nonexpansive multi-valued mapping of C into $CB(C)$ for all $i = 1, 2$. Clearly $F(T_i) = 0 = \hat{F}(T_i)$ for all $i = 1, 2$. Consider

$$\begin{aligned} D_f(0, x) &= f(0) - f(x) - \langle \nabla f(x), 0 - x \rangle \\ &= 0 - \frac{2}{3}x^2 - \langle \frac{4}{3}x, -x \rangle \\ &= 0 - \frac{2}{3}x^2 + \frac{4}{3}x^2 \\ &= \frac{2}{3}x^2. \end{aligned}$$

Since $u_i \in T_i x$, we have $u_i \in T_i x \leq x$ for all $i = 1, 2$. Thus $f(u_i) \leq f(x)$ for all $i = 1, 2$. Consider, for all $i = 1, 2$,

$$\begin{aligned} D_f(0,u_i) &= f(0) - f(u_i) - \langle \nabla f(u_i), 0 - u_i \rangle \\ &= 0 - \frac{2}{3}u_i^2 - \langle \frac{4}{3}u_i, -u_i \rangle \\ &= 0 - \frac{2}{3}u_i^2 + \frac{4}{3}u_i^2 \\ &= \frac{2}{3}u_i^2 \\ &= f(u_i) \leq f(x) = \frac{2}{3}x^2 = D_f(0,x). \end{aligned}$$

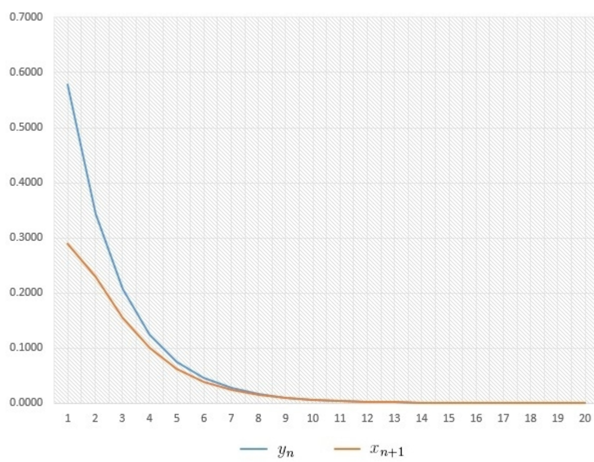
Thus T_i is a Bregman relative nonexpansive multi-valued mapping of C into $CB(C)$ for all $i = 1, 2$. Set $u = 0$ and $x_1 = 1$. Now take $u_n^{(1)} = \frac{1}{2}x_n$ and $u_n^{(2)} = \frac{2}{3}x_n$.

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = P_c^f \nabla f^*[\beta_n^{(0)} \nabla f(x_n) + \sum_{i=1}^N \beta_n^{(i)} \nabla f(u_n^{(i)})], \;\; u_n^{(i)} \in T_i x_n, \\ x_{n+1} = P_c^f \nabla f^*[\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Sy_n)], \;\; n \in \mathbb{N}, \end{cases} \tag{5.29}$$

where $\alpha_n = \frac{1}{n+1}$ and $\beta_n^{(0)} = \frac{1}{12n}$, $\beta_n^{(1)} = \frac{12n-1}{36n}$, $\beta_n^{(2)} = \frac{12n-1}{18n}$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $0 = P_{\mathcal{F}}^f(u)$.

Figure of the value of the sequences $\{x_n\}$ and $\{y_n\}$



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