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Fixed point theorems for a pair of single-valued operators in a metric space endowed with a reflexive relation

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ABSTRACT. In this paper we provide some existence and uniqueness theorems for coupled fixed points for a pair of contractive operators satisfying a mixed monotone property, in a metric space endowed with a reflexive relation. An application to a first-order differential system equation with PBV conditions is also given to illustrate the utility of our results.

1. INTRODUCTION

Coupled fixed points were first introduced by Opoitsev [9], [10], then studied and improved by Guo and Lakshmikantham in [7] in the context of partially ordered subsets of a metric space. Bhaskar and Lakshmikantham also obtained important, inspiring results in [6] regarding the existence and uniqueness of coupled fixed points of a mapping in metric spaces endowed with partial order using a weak contractivity type of assumption. In the next decade, the partial order was replaced by many other relations (reflexive [1], transitive[17], [3]) and the tendency is to replace not only this, but also the initial contractive condition with more general, symmetrical ones(see [4], [13], [5], etc.).

Decades after first being mentioned, Petruşel et. al. come with a fresh approach on coupled fixed points, using two operators, instead of one (see [11], [18], [19]), obtaining exciting results regarding a couple of mixed monotone operators in partially ordered metric spaces. Thus, they obtain a generalization of the classical concept of coupled fixed point which comes to widen the class of problems solvable by fixed point results.

The aim of this paper is to present some existence and uniqueness results for the coupled fixed point problem associated to a pair of mixed-monotone singlevalued operators in a metric space endowed with a reflexive relation, based on the approach of Urs and Petruşel in [11] and Asgari and Mousavi in [1]. In order to prove the effectiveness of the results presented, in the last part of our paper we provide an application to a first-order differential system with PVB conditions.

2. PRELIMINARIES

The purpose of this section is to summarize some of the result that lead to the ones presented in section 3.

The following definition presents the coupled fixed point of a pair of mappings, as considered in [11], [18], [19].

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Definition 2.1. [11] Let X be a nonempty set and $T : X \times X \to X \times X$ be an operator defined by

$$T(x,y) := \begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix},$$

where $T_1, T_2 : X \times X \to X$.

- By definition, a solution (x, y) for the system
$$\begin{cases} T_1(x,y) = x \\ T_2(x,y) = y \end{cases}$$
is called a coupled fixed point for the operator T, respectively, for the pair (T_1, T_2).
- The cartesian product of T and T is denoted by $T \times T$ and it is defined in the following way: $T \times T : Z \times Z \to Z \times Z, \quad (T \times T)(z,w) := (T(z), t(w))$, where $Z := X \times X$ and $z := (x, y), w := (u, v)$ are two arbitrary elements in Z.

Remark 2.1. In the definition above, if $T_1 = T_2$, we obtain the classical definition of coupled fixed point of an operator.

The following results is one of the main results in [11] and establishes the existence of a unique coupled fixed point for the pair of mappings considered.

Theorem 2.1. [11] Let $(X, d, \leq)$ be an ordered complete metric space and let $T_1, T_2 : X \times X \to X$ be two operators. We suppose:

(1) for each $z = (x, y)$, $w = (u, v) \in X \times X$ which are not comparable with respect to the partial ordering $\leq$ on $X \times X$, there exists $t := (t_1, t_2) \in X \times X$ such that t is comparable with both z and w, i.e.,
$$((x \geq t_1 \quad and \quad y \leq t_2) \quad or \quad (x \leq t_1 \quad and \quad y \geq t_2)) \quad and$$
$$((u \geq t_1 \quad and \quad v \leq t_2) \quad or \quad (u \leq t_1 \quad and \quad v \geq t_2));$$

(2) for all $((x \geq u \quad and \quad y \leq v)$ or $(u \geq x \quad and \quad v \leq y)$ we have
$$\begin{cases} T_1(x,y) \geq T_1(u,v) \\ T_2(x,y) \leq T_2(u,v) \end{cases}$$
or
$$\begin{cases} T_1(u,v) \geq T_1(x,y) \\ T_2(u,v) \leq T_2(x,y) \end{cases}$$

(3) $T_1, T_2 : X \times X \to X$ are continuous;

(4) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ such that
$$\begin{cases} z_0^1 \geq T_1(z_0^1, z_0^2) \\ z_0^2 \leq T_2(z_0^1, z_0^2) \end{cases}$$
or
$$\begin{cases} T_1(z_0^1, z_0^2) \geq z_0^1 \\ T_2(z_0^1, z_0^2) \leq z_0^2 \end{cases}$$

(5) there exists a matrix $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in M_2(\mathbb{R}_+)$ convergent toward zero such that
$$d(T_1(x,y), T_1(u,v)) \leq k_1 d(x,u) + k_2 d(y,v)$$
$$d(T_2(x,y), T_2(u,v)) \leq k_3 d(x,u) + k_4 d(y,v)$$

for all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$).
Then there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$x^* = T_1(x^*, y^*) \quad and \quad y^* = T_2(x^*, y^*)$$

and the sequence of the successive approximations $(T_1^n(w_0^1, w_0^2), T_2^n(w_0^1, w_0^2))$ converges to (x^*, y^*) as $n \to \infty$, for all $w_0 = (w_0^1, w_0^2) \in X \times X$

Next, we will recall some of the definitions from [1], for coupled fixed points, mixed monotony in the case of metric spaces endowed with a reflexive relation.

Definition 2.2. ([1]) The mapping f is called orbitally continuous if $(x, y), (a, b) \in X \times X$ and $f^{n_k}(x, y) \to a$ and $f^{n_k}(y, x) \to b$, when $k \to \infty$, implies $f^{n_k+1}(x, y) \to f(a, b)$ and $f^{n_k+1}(y, x) \to f(b, a)$, when $k \to \infty$.

Definition 2.3. ([1]) Let X be a nonempty set and let R be a reflexive relation on X, $f : X \times X \to X$. The mapping f has the mixed $R-$**monotone property** on X if $f \times f(X_R(x, y)) \subseteq X_R(f \times f(x, y))$, for all $(x, y) \in X \times X$, where $X_R(x, y) = \{(z, t) \in X \times X : zRx \wedge yRt\}$, $\forall (x, y) \in X \times X$.

Definition 2.4. [1] Let X be a topological space and let $F : X^2 \to X$ be a mapping.

- Then an element $(x, y) \in X^2$ is called a coupled attractor basin element of F with respect to $(\overline{x}, \overline{y}) \in X^2$ if $F^n(x, y) \to \overline{x}$ and $F^n(y, x) \to \overline{y}$, as $n \to \infty$ and an element $x \in X$ is called an attractor basin element of F with respect to $\overline{x} \in X$, if $F^n(x, x) \to \overline{x}$, as $n \to \infty$. The set of all coupled attractor basin elements of F with respect to $(\overline{x}, \overline{y})$ will be denoted by $A_f(\overline{x}, \overline{y})$ and the set of all attractor basin elements of F with respect to $\overline{x} \in X$, by $A_f(\overline{x})$.
- The mapping F is called a Picard operator, if there exists $\overline{x} \in X$ such that $F_f = \{\overline{x}\}$ and $A_f(\overline{x}) = X$.

The next theorem is the main uniqueness result in [1].An error estimate for the described method is also provided.

Theorem 2.2. ([1]) Let (X, d) be a metric space and R a reflexive relation on X. If $f : X \times X \to X$ is a mapping such that:

- f has the mixed $R-$monotone property on X.
- (X, d) is a complete metric space.
- f has an $R-$coupled fixed point, i.e. there exists $(x_0, y_0) \in X \times X$ such that $f \times f(x_0, y_0) \in X_R(x_0, y_0)$.
- there exists a constant $k \in [0, 1)$ such that:

$$d(f(x, y), f(z, t)) \leq \frac{k}{2}[d(x, z) + d(y, t)], \forall (x, y) \in X_R(z, t).$$

- f is orbitally continuous.

Then:

- There exists $x^*, y^* \in X$ such that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.
- The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = f(y_n, x_n)$ converge respectively to x^* and y^*.
- The error estimation is given by :

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)].$$

In [12] Rus and Petruşel note that, when working with a relation and a metric we have to include in the hypothesis a condition of compatibility between the two structures. Thus, in this case, the reflexive relation R and the metric d are compatible if x_nRy_n implies $\lim_{n\to\infty} x_n R \lim_{n\to\infty} y_n, \forall n \in \mathbb{N}$.

3. MAIN RESULTS

The next theorem extends the results of Urs [19] in the case of metric space endowed with a reflexive relation.

Theorem 3.3. Let (X,d) be a metric space and R a reflexive relation on X such that R and d are compatible. If $f_1, f_2 : X \times X \to X$ two mappings such that:

- f_1, f_2 have the mixed $R-$monotone property on X.
- (X,d) is a complete metric space.
- there exists $(x_0, y_0) \in X \times X$ such that $f_1 \times f_2(x_0, y_0) \in X_R(x_0, y_0)$.
- there exists a constant $k \in [0,1)$ such that:
  $d(f_1(x,y), f_1(z,t)) + d(f_2(x,y), f_2(z,t)) \leq k \cdot [d(x,z) + d(y,t)], \forall (x,y) \in X_R(z,t)$.
- for $(x,y), (a,b) \in X \times X$ such that $f_1^{n_k}(x,y) \to a$ and $f_2^{n_k}(x,y) \to b$, we have $f_1^{n_k+1}(x,y) \to f_1(a,b)$ and $f_2^{n_k+1}(x,y) \to f_2(a,b)$, when $k \to \infty$.

Then:

- There exists $x^*, y^* \in X$ such that $f_1(x^*, y^*) = x^*$ and $f_2(x^*, y^*) = y^*$.
- The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = f_1(x_n, y_n)$ and $y_{n+1} = f_2(x_n, y_n)$ converge respectively to x^* and y^*.
- The error estimation is given by :

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)}[d(f_1(x_0,y_0), x_0) + d(f_2(x_0,y_0), y_0)].$$

Proof. From the third assumption in the hypothesis , we have that the pair (f_1, f_2) admits an R-coupled fixed point; let $(x_0, y_0) \in X \times X$ be it, we have $f_1 \times f_2(x_0, y_0) \in X_R(x_0, y_0)$. Further, using the mixed R-monotone property of f_1, f_2, we have $f_1 \times f_2(x_0, y_0) \in X_R(F(x_0, y_0), F(y_0, x_0))$. Using the induction, we can easily prove that:

$$(f_1^n(x_0,y_0), f_2(x_0,y_0)) \in X_R(f_1^{n-1}(x_0,y_0), f_2^{n-1}(x_0,y_0)). \tag{3.1}$$

We define $d_2 : X^2 \times X^2 \to \mathbb{R}_+$ $d_2(Y,Z) = \frac{1}{2}[d(x,z) + d(y,t)], \forall Y = (x,y), Z = (z,t) \in X^2$.
d_2 is a metric on X^2 because:

- $d_2(Y,Z) = 0 \Leftrightarrow Y = Z$ is a simple task to check , using the definition of d_2 and the fact that d is a metric.
- $d_2(Y,Z) = d_2(Z,Y), \forall Y, Z \in X^2$ holds, because d is a metric, and the sum in d_2's definition is commutative.
- $d_2(Y,Z) \leq d_2(Y,T) + d_2(T,Z), \forall Y, T, Z \in X^2$ can also be easily checked.

Therefore (X^2, d_2) is a complete metric space.
We consider the operator:
$T : X^2 \to X^2$ defined by $T(Y) = (f_1(x,y), f_2(x,y)), \forall Y = (x,y) \in X^2$.
For $Y = (x,y), Z = (z,t) \in X^2$, considering the definition for d_2, we have:

$$d_2(T(Y), T(Z)) = \frac{d(f_1(x,y), f_1(z,t)) + d(f_2(x,y), f_2(z,t))}{2}$$

and

$$d_2(Y,Z) = \frac{d(x,z) + d(y,t)}{2}.$$

By the contractivity condition (3.3) we have

$$d_2(T(Y), T(Z)) \leq k \cdot d_2(Y,Z), \forall Y, Z \in X^2, Y \geq Z. \tag{3.2}$$

From (3.1), we have the monotony of t and that $\{Z_n\}_{n\geq 0}-$ is nondecreasing, we denote $Z_n = (f_1^{n-1}(x_0,y_0), f_2^{n-1}(x_0,y_0))$.

We denote $Y = Z_n \geq Z_{n-1} = V$.
We replace this in (3.2), obtaining:

$$d_2(T(Z_n), T(Z_{n-1})) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1 \Leftrightarrow$$

$$\Leftrightarrow d_2(Z_{n+1}, Z_n) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1.$$

Using the induction, we have:

$$d_2(Z_{n+1}, Z_n) \leq k^n \cdot d_2(Z_1, Z_0), n \geq 1.$$

Let $i < j$. We get:

$$d_2(Z_i, Z_j) \leq \sum_{l=i+1}^{j} d_2(Z_l, Z_{l-1}) \leq (k^i + k^{i+1} + ... + k^{j-i-1}) \cdot d_2(Z_1, Z_0) \leq$$

$$\leq k^i \frac{1 - k^{j-i-1}}{1-k} \cdot d_2(Z_1, Z_0) \tag{3.3}$$

$\Rightarrow \{Z_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space $(X^2, d_2) \Rightarrow$

$$\Rightarrow \lim_{n \to \infty} Z^n = Z^*.$$

We now use (3.1): $t(Z^*) = Z^* \Leftrightarrow (f_1(x^*, y^*), f_2(y^*, x^*)) = (x^*, y^*) \Leftrightarrow f_1(x^*, y^*) = x^*, f_2(x^*, y^*) = y^* \Leftrightarrow (x^*, y^*)$ is the coupled fixed point for the pair (f_1, f_2).
Since (X, d) is a complete metric space, $\exists \quad x^*, y^* \in X$ such that $f_1^n(x_0, y_0) \to x^*$, $f_2^n(x_0, y_0) \to y^*, \quad n \to \infty$. Using the last assumption in the hypothesis, we have:

$$\{x_n\}_{n \in \mathbb{N}} \to x^*, x_{n+1} = f_1(x_n, y_n)$$

$$\{y_n\}_{n \in \mathbb{N}} \to y^*, y_{n+1} = f_2(x_n, y_n)$$

So, by (3.3) we have:

$$d_2((x_n, y_n), (x^*, y^*)) \leq \frac{k^n}{1-k} \cdot d_2((x_1, y_1), (x_0, y_0)), n \geq 0.$$

We return to the original metric d:

$$\frac{d(x_n, x^*) + d(y_n, y^*)}{2} \leq \frac{k^n}{1-k} \cdot \frac{d(x_1, x_0) + d(y_1, y_0)}{2} \Leftrightarrow$$

$$\Leftrightarrow d(x_n, x^*) + d(y_n, y^*) \leq \max_{n \in \mathbb{N}} \{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{1-k} \cdot [d(x_1, x_0) + d(y_1, y_0)].$$

But $x_{n+1} = f_1(x_n, y_n)$ and $y_{n+1} = f_2(x_n, y_n)$. We get:

$$\max_{n \in \mathbb{N}} \{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{1-k} \cdot [d(f_1(x_0, y_0), x_0) + d(f_2(x_0, y_0), y_0)].$$

□

Theorem 3.4. In addition to the hypothesis of Theorem 3.3, we suppose that there exists $(r, s) \in X^2$ such that $(x, y), (x_0, y_0) \in X_R(r, s), \forall (x, y), (x_0, y_0) \in X^2$. Then, the pair (f_1, f_2) admits a unique fixed point.

Proof. From Theorem 3.3 it follows that there exists $x^*, y^* \in X$ such that $f_1(x^*, y^*) = x^*, f_2(x^*, y^*) = y^*$.
The next step is to show that $A_f(x^*, y^*) = X \times X$.
Let $(x, y) \in X^2$. Since f_1, f_2 have the mixed $R-$monotone property on X, then there exists

$(r,s) \in X^2$ such that $(x,y),(x_0,y_0) \in X_R(r,s)$. From $(x_0,y_0) \in X_R(r,s)$ and the fact that (X,d) is a complete metric space, it follows that for $n \in \mathbb{N}$

$$(f_1^n(x_0,y_0), f_2^n(x_0,y_0)) \in X_R(f_1^n(r,s), f_2^n(r,s)).$$

From the sixth assumption of 3.3 (i.e. for $(x,y),(a,b) \in X \times X$ such that $f_1^{n_k}(x,y) \to a$ and $f_2^{n_k}(x,y) \to b$, we have $f_1^{n_k+1}(x,y) \to f_1(a,b)$ and $f_2^{n_k+1}(x,y) \to f_2(a,b)$, when $k \to \infty$) we have:

$$d(f_1^n(x_0,y_0), F^n(r,s)) \leq k^n \cdot [d(x_0,r) + d(y_0,s)],$$

and

$$d(f_2^n(x_0,y_0), F^n(r,s)) \leq k^n \cdot [d(x_0,r) + d(y_0,s)].$$

Now, using the fact that $(x_0,y_0) \in A_f(x^*,y^*)$, it follows that $(r,s) \in A_f(x^*,y^*)$. Thus, $A_f(x^*,y^*) = X^2$.
Therefore the pair(f_1,f_2) admits a unique fixed point.

□

It is important to note that the results presented are extensions of important results in the field.

Corollary 3.1. (1) If, in Theorems 3.3 and 3.4, we have $f_1 = f_2$, we get the results of Dobrican presented in [8].

(2) If, in Theorems 3.3 and 3.4, we have $f_1 = f_2$ and we replace the contractive condition with $d(f(x,y),f(z,t)) \leq \frac{k}{2}[d(x,z)+d(y,t)], \forall (x,y) \in X_R(z,t)$, we get the results of Dobrican presented in [1].

(3) If, in Theorems 3.3 and 3.4, we endow the metric space with a relation of partial order (instead of the reflexive relation), we obtain similar results to the ones obtained by Urs, Petruşel and Petruşel in [11], [18] and [19].

(4) If, in Theorems 3.3 and 3.4, we have $f_1 = f_2$ and we replace the relation R with $''\leq''$ we obtain the results of Berinde [4]. In addition to this, if we replace the contractive condition involved with $d(f(x,y),f(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)], \forall x \geq u, y \leq v$, we obtain the results of Bhaskar and Lakshmikantham [6].

4. An application

In this section we will study the existence and uniqueness of the solution of a first-order periodic boundary value system, as an application to the results presented in the previous section.
In a similar context, Berinde in [4], Bhaskar and Lakshmikantham [6], Urs [19] also studied the existence and uniqueness of solutions for a periodic boundary value problem, in the framework of a partially ordered metric space. In this case, we will endow the metric space with a reflexive relation.
Let's denote the reflexive relation by $''R''$ on $C(I) \times C(I)$ and let there be $z := (x,y)$ and $w := (u,v)$ two arbitrary elements of $C(I) \times C(I)$. Then, by definition, $z \in X_R(w) \Leftrightarrow x \leq u$ and $y \geq v$.
It can easily be checked that $(x,x) \in X_R(x,x)$ and if $(x,y) \in X_R(u,v)(i.e. z \in X_R(w))$ and $(u,v) \in X_R(x,y)(i.e. w \in X_R(z))$, we have $z = w$, but the relation of transitivity (necessary for R to be a relation of order) does not hold in this case.
Let's consider the same periodic boundary value system studied in [19]:

$$(4.4)\qquad \begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)), \quad \forall t \in I := [0, T] \\ x(0) = x(T) \\ y(0) = y(T) \end{cases}$$

where $T > 0$ and $f_1, f_2 : I \times \mathbb{R}^2 \to \mathbb{R}$.We also suppose that :

(C1.) there exist $\lambda, \mu_1, \mu_2, \mu_3, \mu_4 > 0$, $\dfrac{\mu_1 + \mu_2}{1 - \mu_3 - \mu_4} < 1$ such that

$0 \le [f_1(t, x, y) + \lambda x] - [f_1(t, u, v) + \lambda u] \le \lambda[\mu_1(x - u) + \mu_2(y - v)] - \lambda[\mu_3(x - u) + \mu_2(y - v)] \le [f_2(t, x, y) + \lambda x] - [f_2(t, u, v) + \lambda u] \le 0$,

$\forall t \in I$ and $x, y, u, v \in \mathbb{R}$, where f_1, f_2 are two continuous functions.

(C2.) for each $z = (x, y)$ and $w = (u, v) \in C(I) \times C(I)$, if $z \in X_R(w)$ or $w \in X_R(z)$, we have:

$$\begin{cases} f_2(t, x, y)] \le f_2(t, u, v) \\ f_1(t, x, y) \ge f_1(t, u, v) \end{cases}$$

or

$$\begin{cases} f_2(t, u, v) \le f_2(t, x, y) \\ f_1(t, u, v) \ge f_1(t, x, y) \end{cases}$$

(C3.) there exists $z_0 := (z_0^1, z_0^2) \in C(I) \times C(I)$ such that:

$$\begin{cases} z_0^1(t) \le f_1(t, z_0^1(t), z_0^2(t)) \\ z_0^2(t) \ge f_2(t, z_0^1(t), z_0^2(t)) \end{cases}$$

or

$$\begin{cases} f_1(t, z_0^1(t), z_0^2(t)) \le z_0^1(t) \\ f_2(t, z_0^1(t), z_0^2(t)) \ge z_0^2(t) \end{cases}$$

(C4.) the following inequalities hold:

$$\begin{cases} (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^1(s) ds \ge z_0^1(t) \\ (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^2(s) ds \le z_0^2(t), \forall t \in I. \end{cases}$$

We recall that the problem (see [19],[6], [16]),

$$\begin{cases} x'(t) = h(t) \\ x(0) = x(T), t \in I, \end{cases}$$

where $h \in C(I)$ and $x \in C^1(I)$, is equivalent, for some $\lambda \neq 0$ to

$$x(t) = \int_0^T G_\lambda(t, s)[h(s) + \lambda x(s)] ds, \forall t \in I,$$

where $G_\lambda(t,s)$ is defined like in [19]:

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, 0 \le s \le t \le T \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, 0 \le t \le s \le T \\ . \end{cases}$$

Thus, we have that the system [4.4] is equivalent to the coupled fixed point problem :

$$\begin{cases} x = F_1(x,y) \\ y = F_2(x,y), \end{cases}$$

where $F_1, F_2 : X^2 \to X, X = C(I)$,

$$F_1(x,y)(t) = \int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s)) + \lambda x(s)]ds$$

$$F_2(x,y)(t) = \int_0^T G_\lambda(t,s)[f_2(s,x(s),y(s)) + \lambda y(s)]ds$$

In order to apply the results presented in the previous section, we have to consider the complete metric space (X,d), where $X = C(I,\mathbb{R}$ and the metric d is induced by the sup-norm on X,

$$d(u,v) = \sup_{t\in I} |u(t) - v(t)|, \forall u,v \in C(I).$$

We also have to link the problem introduced above to the theoretical results recalled and presented; consequently, if $(x,y) \in X^2$ is a coupled point of F, then we have $x(t) = F_1(x,y)(t)$ and, similarly, $y(t) = F_2(x,y)(t), \forall t \in I$, where $F := (F_1,F_2)$.

Theorem 4.5. Consider the problem [4.4] under the assumptions (1)-(4). Then there exists a unique solution (x^*,y^*) of the BVP [4.4].

Proof. In order to reach the conclusion of this results, we will apply Theorem 3.4.For this, we have to verify all the assumptions of this Theorem:
We have that (X,d) is a complete metric space, so the second hypothesis of Theorem [3.4] is verified.
From the first condition (C1.), $0 \le [f_1(t,x,y) + \lambda x] - [f_1(t,u,v) + \lambda u] \le \lambda[\mu_1(x-u) + \mu_2(y-v)] - \lambda[\mu_3(x-u) + \mu_2(y-v)] \le [f_2(t,x,y) + \lambda x] - [f_2(t,u,v) + \lambda u] \le 0$, we have that

$$|[F_1(x,y)(t) - F_1(u,v)(t)| =$$

$$\left| \int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s)) + \lambda x(s)]ds - \int_0^T G_\lambda(t,s)[f_2(s,u(s),v(s)) + \lambda u(s)]ds \right|$$

$$= \left| \int_0^T G_\lambda[(t,s)[f_1(s,x(s),y(s)) - f_1(s,u(s),v(s)) + \lambda x(s) - \lambda u(s)]ds \right|$$

$$\le \lambda \int_0^T G_\lambda(t,s) \left|(\mu_1(x(s)-u(s))| + |\mu_2(y(s)-v(s))|\right)ds$$

$$\le \mu_1 d(x,u) + \mu_2 d(y,v).$$

Applying $\sup_{t\in I}$, we get:

$$d(F_1(x,y),F_1(u,v)) \le \mu_1 d(x,u) + \mu_2 d(y,v). \tag{4.5}$$

In a similar way, we get

$$d(F_2(x,y),F_2(u,v)) \le \mu_3 d(x,u) + \mu_4 d(y,v). \tag{4.6}$$

Summing up relations (4.5) and (4.6), we get:

$$d(F_1(x,y),F_1(u,v)) + d(F_2(x,y),F_2(u,v)) \le (\mu_1+\mu_3)d(x,u) + (\mu_2+\mu_4)d(y,v),$$

where $\mu_1+\mu_2+\mu_3+\mu_4 < 1$ follows from condition (C1.). Consequently, the fourth hypothesis of Theorem 3.3 is verified.
From the second condition (C2.) we have that $(f_1(t,x,y), f_2(t,x,y)) \in X_R(f_1(t,u,v), f_2(t,u,v)$, $\forall w \in X_R(z), z = (x,y), w = (u,v)$ which is equivalent to $f_1 \times f_2(t,x,y) \in X_R(f_1 \times f_2(t,u,v),(t,u,v))$. Thus f_1 and f_2 have the mixed $R-$monotone property on X, so the first hypothesis of Theorem 3.3 is also checked. In a similar way we prove the mixed $R-$monotone property of f_1 and f_2 using the other pair of assumptions in condition (C2.). Since f_1, f_2 have the mixed $R-$monotone property on X, then there exists $(r,s) \in X^2$ such that $(x,y),(x_0,y_0) \in X_R(r,s)$, so the additional assumption of 3.4 is verified.
Now, from the third condition (C3.), $z_0^1(t) \le f_1(t,z_0^1(t),z_0^2(t))$ and $z_0^2(t) \ge f_2(t,z_0^1(t),z_0^2(t))$, where $z_0 = (z_0^1, z_0^2)$ we obtain that $(f_1(t,z_0^1(t),z_0^2(t)) \in X_R(z_0^1,z_0^2) \leftrightarrow f_1 \times f_2(t,z_0^1(t),z_0^2(t)) \in X_R(z_0^1,z_0^2)$. It follows that there exists a coupled fixed point, namely $z_0 = (z_0^1,z_0^2) \in X \times X$, for the pair (f_1,f_2)(the third hypothesis of Theorem 3.3).
Further, it can be easily checked that, for any $n \in \mathbb{N}$,

$$(f_1^n(t,z_0^1(t),z_0^2(t)), f_2^n(t,z_0^1(t),z_0^2(t))) \in X_R(f_1^{n-1}(t,z_0^1(t),z_0^2(t)), f_2^{n-1}(t,z_0^1(t),z_0^2(t))).$$

Using this and the continuity of f_1 and f_2, it can be easily proved that $\left\{f_1^n(t,z_0^1(t),z_0^2(t))\right\}_{n\in\mathbb{N}}$ and $\left\{f_1^n(t,z_0^1(t),z_0^2(t))\right\}_{n\in\mathbb{N}}$ are Cauchy sequences in X, so the last hypothesis of Theorem 3.3 is also checked. Thus, we get that the periodic boundary problem 4.4 has a unique solution in $C(I) \times C(I)$. □

5. Conclusions

Our approach brings numerous new features to the coupled fixed point theory. First, the contractive condition is weaker than the one used in [1],[6]. Second, the reflexive relation R used in our results is more flexible than the relation of order used in most of the papers devoted to coupled fixed points appeared lately (see [7],[6],[19],[4]...). Third, the proof of existence and uniqueness of the solution to the PBV problem is essentially different from the proofs presented so far, because it takes advantage of the properties of the reflexive relation R, instead of using the relation of order(see [4],[6],[19],...).
In addition to this, Corollary 3.1 emphasizes the fact that our results are more general than some results in the field which could be considered, under certain conditions, particular cases of the theorems presented in Section 3.

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