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Operator ideal of s-type operators using weighted mean sequence space

EZGI ERDOĞAN and VATAN KARAKAYA

ABSTRACT. We introduce a different class of s-type operators by using the generalized weighted mean sequence space $c_0(u,v)$, then it is shown that this new class of operators is a quasi-Banach operator ideal. Moreover, their injectivity and surjectivity are investigated according to sort of s-number. Finally, we proof that it is a closed operator ideal under some conditions.

1. INTRODUCTION

Operator ideal is a natural generalization of the well-known ring theoretical notion. Operator ideal studied on Hilbert spaces in the beginning was introduced on the class of Banach spaces by Pietsch about 1969. s-number concept first used in the theory of the non-selfadjoint integral equations characterizes degree of approximation or compactness of a bounded linear operator and plays an important role in determining the new ideals. The main examples of s-numbers are approximation numbers, Kolmogorov numbers, Gel'fand numbers, Weyl numbers and Chang numbers. Various papers of operator ideals defined by using s-numbers of bounded linear operators can be found in the literature (see [2, 3, 9, 4]).

Initials of these studies are classes of ℓ_p $(1<p<\infty)$ type and c_0 type operators introduced by Pietsch [8], [5]. ℓ_p $(1<p<\infty)$ type and c_0 type operators are the operators having s-numbers in $p-$summable sequence space ℓ_p $(1<p<\infty)$ and null sequence space c_0, respectively. Next, Constantin [2], and Maji and Srivastava [4] generalized the class of ℓ_p $(1<p<\infty)$ type operators to classes of $ces-p$ type and $s-type\ ces\,(p,q)$ operators by using the Cesaro sequence space and weighted Cesaro sequence space, respectively. Finally, Şimşek et al. [9] have studied the ideal of all bounded linear operators whose sequence of approximation numbers belong to the generalized modular spaces of Cesaro type, and Kara and İlkhan [3] have studied more general class of ℓ_p type operators by using the generalized weighted mean sequence space.

Altay and Başar [1] introduced the sequence space $c_0(u,v)$ which is the set of all sequences having a range in c_0 under the generalized weighted mean transform. That is;

$$c_0(u,v)=\left\{x\in w:\ (y_n)=\left(\sum_{i=1}^{n}u_nv_ix_i\right)\in c_0\right\}$$

where $(u_n),(v_k)$ sequences of positive real numbers such that $u_n,v_k\neq 0$ for all $n,k\in\mathbb{N}$.

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Corresponding author: Vatan Karakaya; vkkaya@yahoo.com.

The sequence space $c_0(u,v)$ is called generalized weighted mean sequence space and a complete normed linear space with respect to the norm defined by

$$\|x\|_{c_0(u,v)} = \sup_n \left|\sum_{i=1}^{n} u_n v_i x_i\right|$$

We introduce a different class of s-type operators by using the generalized weighted mean sequence space $c_0(u,v)$, and then it is shown that this new class of operators is a quasi-Banach operator ideal. Moreover, their injectivity and surjectivity are investigated according to sort of s-number. Finally, we prove that it is a closed operator ideal.

2. Definitions and background

In this section, we give some definitions and terminologies with basic notations used throughout this paper.

Let E and F be Banach spaces. We denote by $\mathcal{L}(E,F)$ the set of all bounded linear operators acting between E and F, and by $\mathcal{L}$ the ring of all bounded linear operators acting between arbitrary Banach spaces. Throughout this paper E' and a will show dual of E and a continuous function on E, respectively. For $a \in E'$ and $y \in F$, the map $a \otimes y : E \to F$ is defined by $(a \otimes y)(x) = (a(x)y)$.

Definition 2.1. (**Finite-Rank Operator**) A finite-rank operator is a bounded linear operator between Banach spaces whose range is finite-dimensional.

Definition 2.2. (**Operator Ideal**) [5] An operator ideal $\mathcal{U}$ is a subclass of $\mathcal{L}$ such that the components

$$\mathcal{U}(E,F) := \mathcal{U} \cap \mathcal{L}(E,F)$$

satisfy the following conditions:

(OI_1) $a \otimes y \in \mathcal{U}(E,F)$ for $a \in E'$ and $y \in F$.

(OI_2) $S+T \in \mathcal{U}(E,F)$ for $S,T \in \mathcal{U}(E,F)$.

(OI_3) $RST \in \mathcal{U}(E_0,F_0)$ for $T \in \mathcal{L}(E_0,E)$, $S \in \mathcal{U}(E,F)$, $R \in \mathcal{L}(F,F_0)$, where E_0, F_0 are arbitrary Banach spaces.

From the condition (OI_2), it seems that $\lambda S \in \mathcal{U}(E,F)$ for $S \in \mathcal{U}(E,F)$ and $\lambda \in \mathbb{C}$, then every component $\mathcal{U}(E,F)$ is a linear subset of $\mathcal{L}(E,F)$.

Definition 2.3. (**Quasi Norm**) [5] A function α which assigns to every operator $T \in \mathcal{U}$ a non-negative number $\alpha(T)$ is called a quasi-norm on the operator ideal $\mathcal{U}$ if it has the following properties:

(QN_1) $\alpha(a \otimes y) = \|a\|_{E'} \|y\|_F$ for $a \in E'$ and $y \in F$.

(QN_2) $\alpha(S+T) \leq A[\alpha(S) + \alpha(T)]$ for $S,T \in \mathcal{U}(E,F)$, here $A \geq 1$ is a constant.

(QN_3) $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$ for $T \in \mathcal{L}(E_0,E)$, $S \in \mathcal{U}(E,F)$, $R \in \mathcal{L}(F,F_0)$, where E_0, F_0 are arbitrary Banach spaces.

When $C = 1$, then α becomes a norm on the operator ideal $\mathcal{U}$.

An ideal $\mathcal{U}$ with a quasi norm α, denoted by $[\mathcal{U},\alpha]$ and all components $\mathcal{U}(E,F)$ are linear topological Hausdorff spaces. A *quasi-Banach operator ideal* $[\mathcal{U},\alpha]$ is an operator ideal such that all components $\mathcal{U}(E,F)$ are complete under the quasi norm.

Definition 2.4. (**s-number**) [6] A map s which assigns to every operator S an unique sequence $(s_n(S))$ is called an $s-function$ if the following conditions are satisfied:

(OS_1)(monotonicity) $\|S\| = s_1(S) \geq s_2(S) \geq ... \geq 0$ for $S \in \mathcal{L}(E,F)$.

(OS_2)(additivity) $s_{n+m-1}(S+T) \leq s_n(S) + s_m(T)$ for $S,T \in \mathcal{L}(E,F)$.

(OS_3)(ideal property) $s_n(RST) \leq \|R\| s_n(S) \|T\|$ for $T \in \mathcal{L}(E_0, E)$, $S \in \mathcal{L}(E, F)$, $R \in \mathcal{L}(F, F_0)$.

(OS_4)(rank property) If $S \in \mathcal{F}(E, F)$ and $rank(S) < n$, then $s_n(S) = 0$.

(OS_5)(norm property) $s_n(I_n) = 1$, *where* I_n *is the identity map on the n-dimensional Hilbert space* ℓ_n^2.

$s_n(S)$ is called the n-th $s-number$ of the operator S.

Regard that if $m = 1$ in (OS_2), then $s_n(S + T) \leq s_n(S) + \|T\|$.
Some defined s-numbers are given below (see [5]):

The nth approximation number is defined as
$a_n(S) = \inf\{\|S - L\| : \ L \in \mathcal{L}(E, F), \ \text{rank}\ L < n\}$.

The nth Gel'fand number is defined as
$c_n(S) = \inf\{\|SJ_M\| : M \subset E, \ \text{codim}(M) < n\}$, where J_M is the natural embedding from subspace M of E into E.

The nth Kolmogorov number is defined as
$d_n(S) = \inf\{\|Q_N S\| : N \subset F, \ \dim(N) < n\}$, where Q_N is the quotient map from F onto F/N.

The nth Weyl number is defined as
$x_n(S) = \inf\{a_n(SA) : \|A\| \leq 1, \ \text{where}\ A : \ell_2 \to E\}$, where $a_n(SA)$ is the nth approximation number of the operator SA.

The nth Chang number is defined as
$y_n(S) = \inf\{a_n(BS) : \|B\| \leq 1, \ \text{where}\ B : F \to \ell_2\}$, where $a_n(BS)$ is the nth approximation number of the operator BS.

The nth Hilbert number is defined as
$h_n(S) = \inf\{a_n(BSA) : \|A\| \leq 1, \|B\| \leq 1, \ \text{where}\ A : \ell_2 \to E, \ B : F \to \ell_2\}$.

Remark 2.1. [5] For $S \in \mathcal{L}(E, F)$, the following inequalities hold;

$$h_n(S) \leq x_n(S) \leq c_n(S) \leq a_n(S), h_n(S) \leq y_n(S) \leq d_n(S) \leq a_n(S).$$

Lemma 2.1. [7] *Let* $S, T \in \mathcal{L}(E, F)$. *Then* $|s_n(T) - s_n(S)| \leq \|T - S\|$ *for* $n = 1, 2, ...$

Definition 2.5. (**injective**) [5] An s-number sequence $s = (s_n)$ is called injective if, given any metric injection $J \in \mathcal{L}(F, F_0)$, $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(E, F)$.

A quasi-normed operator ideal [$\mathcal{U}$, α] is called injective if $T \in \mathcal{U}(E, F)$ and $\alpha(T) = \alpha(JT)$ as $JT \in \mathcal{U}(E, F_0)$, where $T \in \mathcal{L}(E, F)$ and $J \in \mathcal{L}(F, F_0)$ is a metric injection.

Definition 2.6. (**surjective**) [5] An s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $Q \in \mathcal{L}(E_0, E)$, $s_n(T) = s_n(TQ)$ for all $T \in \mathcal{L}(E, F)$.

A quasi-normed operator ideal [$\mathcal{U}$, α] is called surjective if $T \in \mathcal{U}(E, F)$ and $\alpha(T) = \alpha(TQ)$ as $TQ \in \mathcal{U}(E_0, F)$, where $T \in \mathcal{L}(E, F)$ and $Q \in \mathcal{L}(E_0, E)$ is a metric surjection.

Proposition 2.1. [5] *Kolmogorov numbers and Weyl numbers are injective, Gel'fand numbers and Chang numbers are surjective.*

Definition 2.7. (**closed ideal**) [5] An operator ideal $\mathcal{U}$ is closed if all components $\mathcal{U}(E, F)$ are closed linear subsets of $\mathcal{L}(E, F)$. This means that $\mathcal{U}$ becomes a Banach operator ideal by using the ordinary operator norm.

3. S-TYPE WEIGHTED MEAN NULL OPERATOR IDEAL

We call an operator $T \in \mathcal{L}(E, F)$ is of *s-type generalized weighted mean null operator* if $\sum_{i=1}^{n} u_n v_i(s_i(T)) \in c_0$. We denote by $\mathcal{U}_{c_0}^{(s)}$ class of all s-type generalized weighted mean null operators between any two Banach spaces. i.e.

$$\mathcal{U}_{c_0}^{(s)} = \left\{ T \in \mathcal{L}(E,F) : \lim_{n\to\infty} u_n \sum_{i=1}^{n} v_i \left(s_i\left(T\right)\right) = 0 \right\}$$

Let $(u_n), (v_k)$ be bounded sequences of positive real numbers such that

$$v_{2k-1} + v_{2k} \le C v_k, \tag{3.1}$$

$$(u_n) \in c_0, \tag{3.2}$$

where $C > 1$ is independent of k.

Theorem 3.1. *Let* $(u_n), (v_k)$ *be sequences that provide the conditions* (3.1) *and* (3.2), *then* $\mathcal{U}_{c_0}^{(s)}$ *is an operator ideal.*

Proof. To prove $\mathcal{U}_{c_0}^{(s)}$ is an operator ideal, it will be shown the conditions (OI_1), (OI_2), (OI_3) are provided.

Let $a \in E'$ and $y \in F$, then $a \otimes y$ is a finite rank operator with rank one and so by definition of the s-number, we have $s_n(a \otimes y) = 0$, for all $n \ge 2$. We need to show that $a \otimes y \in \mathcal{U}_{c_0}^{(s)}$.

$$\left(\sum_{i=1}^{n} u_n v_i \left(s_i\left(a \otimes y\right)\right) \right)_{n=1}^{\infty} = \left(u_n v_1 s_1 \left(a \otimes y\right)\right)_{n=1}^{\infty}$$

Since $(u_n) \in c_0$, then $\lim_{n\to\infty} u_n v_1 s_1(a \otimes y) = 0$ and we obtain $a \otimes y \in \mathcal{U}_{c_0}^{(s)}(E \to F)$.

Let $S, T \in \mathcal{U}_{c_0}^{(s)}(E \to F)$. It follows from definition (2.2), s-number forms positive terms and it is non increasing. Using (OS_2),

$$\sum_{i=1}^{n} u_n v_i \left(s_i\left(S+T\right)\right) \le \sum_{i=1}^{n} u_n v_{2i-1} s_{2i-1}(S+T) + \sum_{i=1}^{n} u_n v_{2i} s_{2i}(S+T)$$

$$\le \sum_{i=1}^{n} u_n \left(v_{2i-1} + v_{2i}\right) s_{2i-1}(S+T) \le C \sum_{i=1}^{n} u_n v_i s_{2i-1}(S+T) \le C \sum_{i=1}^{n} u_n v_i \left(s_i(S) + s_i(T)\right).$$

$$\sum_{i=1}^{n} u_n v_i \left(s_i\left(S+T\right)\right) \le C \left[\sum_{i=1}^{n} u_n v_i s_i(S) + \sum_{i=1}^{n} u_n v_i s_i(T) \right]. \tag{3.3}$$

Since $S, T \in \mathcal{U}_{c_0}^{(s)}(E \to F)$, then we get $S + T \in \mathcal{U}_{c_0}^{(s)}(E \to F)$.

To provide (OI_3), we apply the condition (OS_3).

For $T \in \mathcal{L}(E_0, E)$, $R \in \mathcal{L}(F, F_0)$,

$$\sum_{i=1}^{n} u_n v_i \left(s_i\left(RST\right)\right) \le \sum_{i=1}^{n} u_n v_i \left\|R\right\| s_n(S) \left\|T\right\| = \left\|R\right\| \left\|T\right\| \sum_{i=1}^{n} u_n v_i s_n(S).$$

Thus $RST \in \mathcal{U}_{c_0}^{(s)}(E \to F)$, since $S \in \mathcal{U}_{c_0}^{(s)}(E \to F)$.

It is shown that $\mathcal{U}_{c_0}^{(s)}$ is an operator ideal. □

Let us define the function $\mathfrak{N}_{c_0}^{(s)} : \mathcal{U}_{c_0}^{(s)} \to \mathbb{R}^+$ as follows:

$$\mathfrak{N}_{c_0}^{(s)}(S) = \frac{\sup_n \left| \sum_{i=1}^{n} u_n v_i \left(s_i\left(S\right)\right) \right|}{\sup_n \left|u_n v_1\right|},$$ where $(u_n), (v_k)$ are sequences that provide the conditions (3.1) and (3.2).

Theorem 3.2. *The operator ideal* $\mathcal{U}_{c_0}^{(s)}$ *is a quasi-normed operator ideal with the quasi norm* $\mathfrak{N}_{c_0}^{(s)}$.

Proof. To prove $\mathfrak{N}_{c_0}^{(s)}$ is a quasi norm on the operator ideal $\mathcal{U}_{c_0}^{(s)}$, it must be shown that the function $\mathfrak{N}_{c_0}^{(s)}$ satisfies (QN_1), (QN_2), (QN_3).

i) Let $a \in E'$ and $y \in F$, then $a \otimes y$ is a finite rank operator with rank one and so by definition of the s-number, we have $s_n(a \otimes y) = 0$, for all $n \geq 2$.

$$\mathfrak{N}_{c_0}^{(s)}(a \otimes y) = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(a \otimes y))\right|}{\sup_n |u_n v_1|} = \frac{\sup_n |u_n v_1 (s_1(a \otimes y))|}{\sup_n |u_n v_1|}$$
$$= \frac{\sup_n |u_n v_1 \|a \otimes y\||}{\sup_n |u_n v_1|} = \|a \otimes y\|$$

Since $\|a \otimes y\| = \sup_{\|x\|=1} \|a(x) y\| = \|a\| \|y\|$, we get $\mathfrak{N}_{c_0}^{(s)}(a \otimes y) = \|a\| \|y\|$

ii) Using the inequality (3.3), we get

$$\mathfrak{N}_{c_0}^{(s)}(S+T) \leq \frac{\sup_n \left|C\left[\sum_{i=1}^{n} u_n v_i s_i(S) + \sum_{i=1}^{n} u_n v_i s_i(T)\right]\right|}{\sup_n |u_n v_1|}$$
$$\leq C\left[\frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i s_i(S)\right|}{\sup_n |u_n v_1|} + \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i s_i(T)\right|}{\sup_n |u_n v_1|}\right] = C\left[\mathfrak{N}_{c_0}^{(s)}(S) + \mathfrak{N}_{c_0}^{(s)}(T)\right].$$

iii) Now, we show the last condition (QN_3). With the third condition (OS_3) in the definition (2.4), we obtain

$$\mathfrak{N}_{c_0}^{(s)}(RST) \leq \frac{\sup_n \left|\|R\| \|T\| \sum_{i=1}^{n} u_n v_i (s_i(S))\right|}{\sup_n |u_n v_1|} = \|R\| \|T\| \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(S))\right|}{\sup_n |u_n v_1|}.$$

Then, we get $\mathfrak{N}_{c_0}^{(s)}(RST) \leq \|R\| \mathfrak{N}_{c_0}^{(s)}(S) \|T\|$. □

Theorem 3.3. *The operator ideal* $\left[\mathcal{U}_{c_0}^{(s)}, \mathfrak{N}_{c_0}^{(s)}\right]$ *is a quasi-Banach operator ideal under the quasi-norm* $\mathfrak{N}_{c_0}^{(s)}$.

Proof. It must shown that each component $\mathcal{U}_{c_0}^{(s)}(E, F)$ of $\mathcal{U}_{c_0}^{(s)}$ is complete under the quasi-norm $\mathfrak{N}_{c_0}^{(s)}$. For $T \in \mathcal{U}_{c_0}^{(s)}$, we have

$$\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T))\right| \geq \sup_n |u_n v_1 s_1(T)| = \|T\| \sup_n |u_n v_1|.$$

Then,

$$\mathfrak{N}_{c_0}^{(s)}(T) \geq \|.T\| \tag{3.4}$$

Let (T_r) be a Cauchy sequence in $\mathcal{U}_{c_0}^{(s)}$. Then for arbitrary ε positive number, there exists $n_0 \in \mathbb{N}$ such that

$$\mathfrak{N}_{c_0}^{(s)}(T_r - T_m) < \varepsilon \text{ for every } r, m \geq n_0. \tag{3.5}$$

It follows from (3.4) that,

$$\|T_r - T_m\| \leq \mathfrak{N}_{c_0}^{(s)}(T_r - T_m) < \varepsilon \ \forall r, m \geq n_0.$$

Then (T_r) is a Cauchy sequence in $\mathcal{L}(E,F)$. It is well known that $\mathcal{L}(E,F)$ is a Banach space, when F is Banach space. So, $\|T_r - T\| \to 0$, as $r \to \infty$, for $T \in \mathcal{L}(E,F)$.

We need to show that $T_r \to T$ as $r \to \infty$ in $\mathcal{U}_{c_0}^{(s)}(E,F)$.

Applying Lemma 2.1, we get that

$$|s_n(T_m - T_r) - s_n(T - T_r)| \le \|(T_m - T_r) - (T - T_r)\| = \|T_m - T\|$$

Since $T_m \to T$, as $m \to \infty$, we obtain

$$s_n(T_m - T_r) \to s_n(T - T_r) \text{ as } m \to \infty. \tag{3.6}$$

Using (3.5), for $r, m \ge n_0$ the following holds

$$\mathfrak{N}_{c_0}^{(s)}(T_m - T_r) = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T_m - T_r))\right|}{\sup_n |u_n v_1|} < \varepsilon$$

or, equally,

$$\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T_m - T_r))\right| < \varepsilon \sup_n |u_n v_1|$$

From (3.6) and choosing a fixed $r \ge n_0$, as $m \to \infty$

$$\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T - T_r))\right| < \varepsilon \sup_n |u_n v_1| \tag{3.7}$$

then

$$\mathfrak{N}_{c_0}^{(s)}(T_r - T) < \varepsilon \ \ \forall r \ge n_{0.}$$

This shows that, $T_r \to T$ under the quasi-norm $\mathfrak{N}_{c_0}^{(s)}$.

To complete the proof, we shall show that $T \in \mathcal{U}_{c_0}^{(s)}$.

$$\begin{aligned}\sum_{i=1}^{n} u_n v_i (s_i(T)) &\le \sum_{i=1}^{n} u_n v_{2i-1}(s_{2i-1}(T)) + \sum_{i=1}^{n} u_n v_{2i}(s_{2i}(T)) \le \sum_{i=1}^{n} u_n (v_{2i-1} + v_{2i})(s_{2i-1}(T)) \\ &\le C\sum_{i=1}^{n} u_n v_i (s_{2i-1}(T)) = C\sum_{i=1}^{n} u_n v_i (s_{2i-1}(T - T_m + T_m)) \\ &\le C\left[\sum_{i=1}^{n} u_n v_i (s_i(T - T_m)) + \sum_{i=1}^{n} u_n v_i (s_i(T_m))\right].\end{aligned}$$

From (3.7) and $(T_m) \in \mathcal{U}_{c_0}^{(s)}$, we get $0 \le \sum_{i=1}^{n} u_n v_i (s_i(T)) \le C(1 + \sup_n |u_n v_1|)\varepsilon$ and then $\sum_{i=1}^{n} u_n v_i (s_i(T)) \in c_0$. Hence, $T \in \mathcal{U}_{c_0}^{(s)}$. This completes the proof. □

Theorem 3.4. *If s-number sequence is injective, then the quasi-Banach operator ideal* $\left[\mathcal{U}_{c_0}^{(s)}, \mathfrak{N}_{c_0}^{(s)}\right]$ *is injective.*

Proof. Suppose that $JT \in \mathcal{U}_{c_0}^{(s)}(E, F_0)$ for an arbitrary $T \in \mathcal{L}(E,F)$ and a metric injection $J \in \mathcal{L}(F, F_0)$. We shall show that $\mathfrak{N}_{c_0}^{(s)}(JT) = \mathfrak{N}_{c_0}^{(s)}(T)$ and $T \in \mathcal{U}_{c_0}^{(s)}(E,F)$.

By hypothesis, we have $s_n(JT) = s_n(T)$ for every $T \in \mathcal{L}(E,F)$ and $n \in \mathbb{N}$. Thus,

$$\mathfrak{N}_{c_0}^{(s)}(JT) = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(JT))\right|}{\sup_n |u_n v_1|} = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T))\right|}{\sup_n |u_n v_1|} = \mathfrak{N}_{c_0}^{(s)}(T)$$

Since $JT \in \mathcal{U}_{c_0}^{(s)}(E, F_0)$, we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} u_n v_i (s_i(T)) = \lim_{n\to\infty} \sum_{i=1}^{n} u_n v_i (s_i(JT)) = 0.$$

Therefore, $\mathfrak{N}_{c_0}^{(s)}(JT) = \mathfrak{N}_{c_0}^{(s)}(T)$ and $T \in \mathcal{U}_{c_0}^{(s)}(E,F)$ hold. □

Remark 3.2. The quasi-Banach operator ideals $\left[\mathcal{U}_{c_0}^{(c)}, \mathfrak{N}_{c_0}^{(c)}\right]$ and $\left[\mathcal{U}_{c_0}^{(x)}, \mathfrak{N}_{c_0}^{(x)}\right]$ corresponding to the Gel'fand numbers $c = (c_n)$ and the Weyl numbers $x = (x_n)$, respectively, are injective quasi-Banach operator ideals.

Theorem 3.5. *The quasi-Banach operator ideal* $\left[\mathcal{U}_{c_0}^{(s)}, \mathfrak{N}_{c_0}^{(s)}\right]$ *is surjective, whenever s-number sequence is surjective.*

Proof. Suppose that $TQ \in \mathcal{U}_{c_0}^{(s)}(E_0, F)$ for an arbitrary $T \in \mathcal{L}(E,F)$ and a metric surjection $Q \in \mathcal{L}(E_0, E)$. We shall show that $\mathfrak{N}_{c_0}^{(s)}(TQ) = \mathfrak{N}_{c_0}^{(s)}(T)$ and $T \in \mathcal{U}_{c_0}^{(s)}(E,F)$.

By hypothesis, s-number sequence is surjective, then $s_n(TQ) = s_n(T)$ for every $T \in \mathcal{L}(E,F)$ and $n \in \mathbb{N}$. Thus,

$$\mathfrak{N}_{c_0}^{(s)}(TQ) = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(TQ))\right|}{\sup_n |u_n v_1|} = \frac{\sup_n \left|\sum_{i=1}^{n} u_n v_i (s_i(T))\right|}{\sup_n |u_n v_1|} = \mathfrak{N}_{c_0}^{(s)}(T)$$

Since $TQ \in \mathcal{U}_{c_0}^{(s)}(E_0, F)$, we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} u_n v_i (s_i(T)) = \lim_{n\to\infty} \sum_{i=1}^{n} u_n v_i (s_i(TQ)) = 0.$$

Therefore, $\left[\mathcal{U}_{c_0}^{(s)}, \mathfrak{N}_{c_0}^{(s)}\right]$ is surjective. □

Remark 3.3. The quasi-Banach operator ideals $\left[\mathcal{U}_{c_0}^{(d)}, \mathfrak{N}_{c_0}^{(d)}\right]$ and $\left[\mathcal{U}_{c_0}^{(y)}, \mathfrak{N}_{c_0}^{(y)}\right]$ corresponding to the Kolmogorov numbers $d = (d_n)$ and the Chang numbers $y = (y_n)$, respectively, are surjective quasi-Banach operator ideals.

Theorem 3.6. *The following inclusions hold:*

$$I.\, \mathcal{U}_{c_0}^{(a)} \subseteq \mathcal{U}_{c_0}^{(c)} \subseteq \mathcal{U}_{c_0}^{(x)} \subseteq \mathcal{U}_{c_0}^{(h)};\, II.\, \mathcal{U}_{c_0}^{(a)} \subseteq \mathcal{U}_{c_0}^{(d)} \subseteq \mathcal{U}_{c_0}^{(y)} \subseteq \mathcal{U}_{c_0}^{(h)}.$$

Proof. I. Let assume that $S \in \mathcal{U}_{c_0}^{(a)}$. Then, $\left(\sum_{i=1}^{n} u_n v_i (a_i(S))\right)_{n=1}^{\infty} \in c_0$. Applying Remark 2.1, we obtain

$$0 \le \sum_{i=1}^{n} u_n v_i (h_i(S)) \le \sum_{i=1}^{n} u_n v_i (c_i(S)) \le \sum_{i=1}^{n} u_n v_i (x_i(S)) \le \sum_{i=1}^{n} u_n v_i (a_i(S))$$

II. The proof is similar to part I. □

Theorem 3.7. *Let* $(v_k) \in \ell_1$, *then* $\mathcal{U}_{c_0}^{(s)}$ *is a closed operator ideal.*

Proof. Let $(T_k) \in \mathcal{U}_{c_0}^{(s)}(E,F)$ and $\|T_k - T\| \to 0$ for $T \in \mathcal{L}(E,F)$. We need to show that $T \in \mathcal{U}_{c_0}^{(s)}(E,F)$. For given $\varepsilon > 0$, we fix k and n_0 such that

$$\|T_k - T\| < \varepsilon \text{ and } s_n(T_k) < \varepsilon \text{ for all } n \geq n_0$$

From the condition (OS$_2$) of definition 2.4, we get

$$s_n(T) \leq \|T_k - T\| + s_n(T_k) \leq 2\varepsilon \text{ for all } n \geq n_0$$

and

$$\sum_{i=1}^{n} u_n v_i s_i(T) \leq \sum_{i=1}^{n} u_n v_i \left[\|T_k - T\| + s_i(T_k)\right] \leq \sum_{i=1}^{n} u_n v_i (2\varepsilon)$$

Then

$$0 \leq \sum_{i=1}^{n} u_n v_i s_i(T) \leq \sum_{i=1}^{n} u_n v_i (2\varepsilon) \leq 2\|u_n\|_\infty \|v_k\|_1 \varepsilon$$

Thus $\left(\sum_{i=1}^{n} u_n v_i s_i(T)\right) \in c_0$ and $T \in \mathcal{U}_{c_0}^{(s)}(E,F)$. □

Remark 3.4.

- In particular if we take $v_n = 1$ and $u_n = \frac{1}{\sum_{i=1}^{n} v_i}$ for all $n \in \mathbb{N}$, then $\mathcal{U}_{c_0}^{(s)}$ is called as Cesaro null type operator ideal.
- Taking $u_n = \frac{1}{\sum_{i=1}^{n} v_i}$ for all $n \in \mathbb{N}$, then $\mathcal{U}_{c_0}^{(s)}$ is reduced Norlund null type operator ideal.

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Marmara University
Department of Mathematics
34722, Kadıköy, Istanbul, Turkey
E-mail address: ezgi.erdogan@marmara.edu.tr

Yildiz Technical University
Department of Mathematical Engineering
34220, Davutpaşa, Istanbul, Turkey
E-mail address: vkkaya@yahoo.com