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Existence and convergence for a new multivalued hybrid mapping in CAT(κ) spaces

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ABSTRACT. Most of the studies about hybrid mappings are carried out for single-valued mappings in Hilbert spaces. We define a new class of multivalued mappings in CAT (k) spaces which contains the multivalued generalization of (α, β) - hybrid mappings defined on Hilbert spaces. In this paper, we prove existence and convergence results for a new class of multivalued hybrid mappings on CAT(κ) spaces which are more general than Hilbert spaces and CAT(0) spaces.

1. INTRODUCTION

Although many real-life events and their scientific modellings have nonlinear structure, the fixed point studies for single-valued or multivalued mappings have been developed mostly on linear spaces, such as Hilbert spaces and Banach spaces. Therefore, it is very important to study fixed point theory on nonlinear spaces. The CAT(κ) spaces (for $\kappa \geq$0) form very good example of non-linear spaces which allow to develop fixed point theory on it due to their convex structure and rich properties similar to Banach and Hilbert spaces.

So far, fixed points studies on these spaces mainly focused on CAT(0) spaces for single and multivalued non-expansive mappings. However, not much is known on CAT (κ) spaces for multivalued mappings. We give definition of a new class of multivalued mappings in CAT(κ) spaces and this new class is general than a multivalued generalization of (α, β)-hybrid mappings in Hilbert spaces. Moreover, most of the studies about hybrid mappings are done for single-valued mappings in Hilbert spaces. In this paper, the results are given for multivalued hybrid mappings on general spaces than Hilbert spaces and CAT(0) spaces. Let H be a Hilbert space and $K \subseteq H, K \neq \emptyset$. Let us take T as a single valued mapping from K to H. If T satisfies

$$||Tx - Ty|| \leq ||x - y||, 2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2$$

and

$$3||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2 + ||Ty - x||^2$$

for all $x, y \in K$ then it called non-expansive, non-spreading [10] and hybrid [17], respectively. None of these classes of mappings is included in the other. In 2010, Aoyama et al. [1] defined λ-hybrid as follows;

$$(1 + \lambda)||Tx - Ty||^2 - \lambda||x - Ty||^2 \leq (1 - \lambda)||x - y||^2 + \lambda||Tx - y||^2$$

where $x, y \in K$ and λ is fixed real number. λ-hybrid mappings are general than non-expansive mappings, non-spreading mappings and hybrid mappings. In 2011, Aoyama

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and Kohsaka [2] introduced α-non-expansive mappings in Banach spaces as follows:

$$||Tx - Ty||^2 \leq (1-2\alpha)||x-y||^2 + \alpha||Tx-y||^2 + \alpha||x-Ty||^2$$

where $x, y \in K$ and $\alpha < 1$ is fixed. They showed that α-non-expansive and λ-hybrid are equivalent in Hilbert spaces for $\lambda < 2$. Kocourek et al. [9], introduced more general class of mappings than the above mappings in Hilbert spaces, called $(\alpha, \beta)-$generalized hybrid, as follows;

$$\alpha||Tx - Ty||^2 + (1-\alpha)||x - Ty||^2 \leq \beta||Tx - y||^2 + (1-\beta)||x-y||^2$$

where $x, y \in K$ and α, β are fixed real numbers.

Many iterative processes to find a fixed point of multivalued mappings have been introduced in metric and Banach spaces. The well known one is defined by Nadler as generalization of Picard as follows;

$$x_{n+1} \in Tx_n.$$

A multivalued version of Mann and Ishikawa fixed point procedures goes as follow;

$$x_{n+1} \in (1-\zeta_n)x_n + \zeta_n Tx_n$$

and

$$x_{n+1} \in (1-\zeta_n)x_n + \zeta_n Ty_n, \quad y_n \in (1-\varsigma_n)x_n + \varsigma_n Tx_n,$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0, 1]$.

Gursoy and Karakaya [7] (see also [8]) introduced Picard-S iteration as follows:

$$x_{n+1} = Ty_n, \quad y_n = (1-\zeta_n)Tx_n + \zeta_n Tz_n, \quad z_n = (1-\varsigma_n)x_n + \varsigma_n Tx_n,$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0, 1]$. They have showed that it converges to fixed point of contraction mappings faster than Ishikawa, Noor, SP, CR, S and some other iterations. Also they use it to solve differential equations. Now, we define multivalued version of Picad-S iteration in CAT(κ) spaces as follows:

$$x_{n+1} = P_K(u_n), y_n = P_K((1-\zeta_n)w_n \oplus \zeta_n v_n), z_n = P_K((1-\varsigma_n)x_n \oplus \varsigma_n w_n) \tag{1.1}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0,1]$ with $\liminf_n(1-\varsigma_n)\varsigma_n > 0$, $u_n \in Ty_n$, $v_n \in Tz_n$ and $w_n \in Tx_n$.

2. Preliminaries

Let (X, d) be a metric space and $K \subseteq X, K \neq \emptyset$. In rest of this paper, we will use following notations; $C(X)$ for all nonempty, closed subsets of X, $CC(X)$ for all nonempty closed and convex subsets of X, $KC(X)$ for nonempty, compact and convex subsets of X and $CB(X)$ for all nonempty, closed and convex subsets of X. Let H_d be Pompeiu-Hausdorff metric on $CB(X)$, defined by

$$H_d(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

where $d(x, B) = \inf\{d(x, y); y \in B\}$. A point p is called fixed point of multivalued mapping T if $p \in Tp$ and the set of all fixed points of T is denoted by $F(T)$.

Let (X, d) be bounded metric space and take $x, y \in X$ and $K \subseteq X, K \neq \emptyset$. A geodesic path (or shortly a geodesic) joining x and y is a map $c : [0, t] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r), c(s)) = |r - s|$ for all $r, s \in [0, t]$.In fact, c is an isometry and $d(c(0), c(t)) = t$. The image of c, $c([o, t])$ is called geodesic segment from x to y and it is not necessarily be unique. It is unique then it is denoted by $[x, y]$. $z \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = (1-t)d(x, y)$ and $d(z, y) = td(x, y)$. The point z is denoted by $z = (1-t)x \oplus ty$. For fixed $r > 0$, the space (X, d) is called r-geodesic space if any two point $x, y \in X$ with $d(x, y) < r$ there is a geodesic joining x to y. if for every

$x, y \in X$, there is a geodesic path then (X, d) called geodesic space and uniquely geodesic space if that geodesic path is unique for any pair x, y. We call a subset $K \subseteq X$ as a convex subset if it contains all geodesic segment joining any pair of points in it.

Definition 2.1. (see:[3]) Let $\kappa \in \mathbb{R}$.

i) if $\kappa = 0$, *then* M_κ^n *is Euclidean space* $\mathbb{E}^n$,
ii) if $\kappa > 0$, *then* M_κ^n *is obtained from the sphere* $\mathbb{S}^n$ *by multiplying distance function by* $\frac{1}{\sqrt{\kappa}}$,
iii) if $\kappa < 0$, *then* M_κ^n *is obtained from hyperbolic space* $\mathbb{H}^n$ *by multiplying distance function by* $\frac{1}{\sqrt{-\kappa}}$.

In a geodesic metric space (X, d), a geodesic triangle, $\Delta(x, y, z)$ consist of three point x, y, z as vertices and three geodesic segments of any pair of these points, that is, $q \in \Delta(x, y, z)$ means that $q \in [x, y] \cup [x, z] \cup [y, z]$. The triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ in M_κ^2 is called comparison triangle for the triangle $\Delta(x, y, z)$ such that $d(x, y) = d(\overline{x}, \overline{y}), d(x, z) = d(\overline{x}, \overline{z})$ and $d(y, z) = d(\overline{y}, \overline{z})$ and such a comparison triangle always exists provided that the perimeter $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa (D_\kappa = \frac{\pi}{\sqrt{K}}$ if $\kappa > 0$ and ∞ otherwise) in M_κ^2 (Lemma 2.14 in [3]). A point point $\overline{z} \in [\overline{x}, \overline{y}]$ called comparison point for $z \in [x, y]$ if $d(x, z) = d(\overline{x}, \overline{z})$. A geodesic triangle $\Delta(x, y, z)$ in X with perimeter less than $2D_\kappa$ (and given a comparison triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ for $\Delta(x, y, z)$ in M_κ^2) satisfies $CAT(\kappa)$ inequality if $d(p, q) \leq d(\overline{p}, \overline{q})$ for all $p, q \in \Delta(x, y, z)$ where $\overline{p}, \overline{q} \in \overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ are the comparison points of p, q respectively. The D_κ-geodesic metric space (X, d) is called $CAT(\kappa)$ space if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

If for every $x, y, z \in X$, there is an $R \in (0, 2]$ satisfying

$$d^2(x, (1-\lambda)y \oplus \lambda z) \leq (1-\lambda)d^2(x, y) + \lambda d^2(x, z) - \frac{R}{2}\lambda(1-\lambda)d^2(y, z),$$

then (X, d) is called $R-$*convex* [13]. Hence, (X, d) is a $CAT(0)$ space if and only if it is a $2-$*convex space*.

Lemma 2.1. (see:[14]) *Let* $\kappa > 0$ *and* (X, d) *be a* $CAT(\kappa)$ *space with* $diam(X) < \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \frac{\pi}{2})$. *Then* (X, d) *is a* $R-$*convex space for* $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.

Proposition 2.1. (see:[3]) *Let* X *be* $CAT(\kappa)$ *space. Then any ball of radius smaller than* $\frac{\pi}{2\sqrt{\kappa}}$ *is convex.*

Proposition 2.2. (Exercise 2.3 (1) in [3]) *Let* $\kappa > 0$ *and* (X, d) *be a* $CAT(\kappa)$ *space with* $diam(X) < \frac{D_\kappa}{2} = \frac{\pi}{2\sqrt{\kappa}}$. *Then, for any* $x, y, z \in X$ *and* $t \in [0, 1]$, *we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

Proposition 2.3. (see:[15]) *The modulus of convexity for* $CAT(\kappa)$ *space* X *(of dimension* ≥ 2*) and number* $r < \frac{\pi}{2\sqrt{\kappa}}$ *and let* m *denote the midpoint of the segment* $[x, y]$ *joining* x *and* y *defined by the modulus* δ_r *by sitting*

$$\delta(r, \varepsilon) = \inf\{1 - \frac{1}{r}d(a, m)\}$$

where the infimum is taken over all points $a, x, y \in X$ *satisfying* $d(a, x) \leq r$, $d(a, y) \leq r$ *and* $\varepsilon \leq d(x, y) < \frac{\pi}{2\sqrt{\kappa}}$.

Lemma 2.2. (see:[15]) *Let* X *be a complete* $CAT(\kappa)$*space with modulus of convexity* $\delta(r, \varepsilon)$ *and let* $x \in E$. *Suppose that* $\delta(r, \varepsilon)$ *increases with* r *(for a fixed* ϵ*) and suppose* $\{t_n\}$ *is a sequence in* $[b, c]$ *for some* $b, c \in (0, 1)$, $\{x_n\}$ *and* $\{y_n\}$ *are the sequences in* X *such that* $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ *and* $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ *for some* $r \geq 0$. *Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Let $\{x_n\}$ be a bounded sequence in a $CAT(\kappa)$ space X, $x \in X$ and

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}); x \in X.\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$ is defined by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}); x \in K.\},$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$$

and let $\omega_w(x_n) := \cup A(\{x_n\})$ where union is taken on all subsequences of $\{x_n\}$.

Definition 2.2. (see:[6]) A sequence $\{x_n\} \subset X$ is said to be $\Delta-$ convergent to $x \in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the $\Delta-$limit of $\{x_n\}$.

Proposition 2.4. (see:[6]) *Let X be a complete $CAT(\kappa)$ space, $K \subseteq X$ nonempty, closed and convex, $\{x_n\}$ is a sequence in X. If $r_K(\{x_n\}) < \frac{\pi}{2\sqrt{\kappa}}$, then $A_K(\{x_n\})$ consists of exactly one point.*

Lemma 2.3. (see:[5])

i) Every bounded sequence in X has a Δ-convergent subsequence,
ii) If K is a closed and convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.

Lemma 2.4. (see:[5]) *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = u$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5. (see:[6]) *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, closed and convex subset of X. Then*

i) the metric projection $P_K(x)$ of x onto K is a singleton,
ii) if $x \notin K$ and $y \in K$ with $u \neq P_K(x)$, then $\angle_{P_K(x)}(x, y) \geq \frac{\pi}{2}$,
iii) for each $y \in K$, $d(P_K(x), P_K(y)) \leq d(x, y)$.

Definition 2.3. Let (X, d) be a metric space. T is called $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping from X to 2^X if

$$a_1(x)d^2(u, v) + a_2(x)d^2(u, y) \leq b_1(x)d^2(x, v) + b_2(x)d^2(x, y)$$

is satisfied for all $x, y \in X$, $u \in Tx$ and $v \in Ty$ where $a_1, a_2 : X \to \mathbb{R}$ and $b_1, b_2 : X \to [0, 1]$ with $a_1(x) + a_2(x) \geq 1$, $a_1(x) \leq 0$ or $a_2(x) \leq 0$ and $b_1(x) + b_2(x) \leq 1$.

In the rest of the paper, X will be a complete $CAT(\kappa)$ space ($\kappa > 0$).

3. EXISTENCE RESULTS

Proposition 3.5. *Let K be a nonempty, closed and convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be $(a_1, a_2, b_1, b_2)-$ multivalued hybrid mapping from K to 2^X with $F(T) \neq \emptyset$. Then $F(T)$ closed and $Tp = \{p\}$ for all $p \in F(T)$.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \to x \in X$. Then for any $u \in Tx$, we have

$d^2(u, x_n) \leq a_1(x)d^2(u, x_n) + a_2(x)d^2(u, x_n) \leq b_1(x)d^2(x, x_n) + b_2(x)d^2(x, x_n) \leq d^2(x, x_n)$.

Taking limit on n, we have

$$d(u, x) \leq 0$$

and so $u = x \in Tx = \{x\}$. □

Theorem 3.1. *(Demiclosed principle) Let K be a nonempty, closed convex subset of X where $diam(X) < \frac{\pi}{2\sqrt{\kappa}}$, and $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping. Let $\{x_n\}$ be a sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 2.3, $z \in K$. We can find a sequence $\{y_n\}$ such that $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Because T is $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping, we have

$$a_1(z)d^2(u, y_n) + a_2(z)d^2(u, x_n) \leq b_1(z)d^2(z, y_n) + b_2(z)d^2(z, x_n).$$

for all $u \in Tz$. Then, by triangular inequality, we have $d(x_n, u) \leq d(x_n, y_n) + d(y_n, u)$. So, we have, $\limsup_{n\to\infty} d(x_n, u) \leq \limsup_{n\to\infty} d(y_n, u)$ and again since $d(y_n, u) \leq d(y_n, x_n) + d(x_n, u)$, we have $\limsup_{n\to\infty} d(y_n, u) \leq \limsup_{n\to\infty} d(x_n, u)$, combining these, we have that $\limsup_{n\to\infty} d(x_n, u) = \limsup_{n\to\infty} d(y_n, u)$. Then we get

$$\begin{aligned} a_1(z)d^2(u, y_n) + a_2(z)d^2(u, x_n) &\leq b_1(z)d^2(z, y_n) + b_2(z)d^2(z, x_n) \\ &\leq b_1(z)[d(z, x_n) + d(x_n, y_n)]^2 + b_2(z)d^2(x_n, z) \end{aligned}$$

which implies that $\limsup_{n\to\infty} d(u, x_n) \leq \limsup_{n\to\infty} d(z, x_n)$. Hence, $z = u \in Tz$. □

Corollary 3.1. *Let K be a nonempty, closed and convex subset of X where $diam(X) < \frac{\pi}{2\sqrt{\kappa}}$, and $T : K \to X$ be a $(a_1, a_2, b_1, b_2)-$ hybrid mapping. Let$\{x_n\}$ be a sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $Tz = z$.*

Corollary 3.2. *Let X be a complete $CAT(0)$ space and K be a nonempty, closed and convex subset of X, and $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping. Let$\{x_n\}$ be a bounded sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $z \in Tz$.*

Corollary 3.3. *Let X be a complete $CAT(0)$ space and K be a nonempty, closed and convex subset of X, and $T : K \to X$ be a $(a_1, a_2, b_1, b_2)-$ hybrid mapping. Let $\{x_n\}$ be a bounded sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in K$ and $Tz = z$.*

Theorem 3.2. *Let K be a nonempty, closed and convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be $(a_1, a_2, b_1, b_2)-$ multivalued hybrid mapping from K to $C(K)$. Then $F(T) \neq \emptyset$.*

Proof. Let $x_0 \in K$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$. Assume that $A_K\{x_n\} = \{z\}$. Then $z \in K$ by Lemma 2.3 , for all $n \in \mathbb{N}$ and for any $u \in Tz$, we have

$$a_1(z)d^2(u, x_n) + a_2(z)d^2(u, x_{n-1}) \leq b_1(z)d^2(z, x_n) + b_2(z)d^2(z, x_{n-1})$$

and taking limit superior on both side implies that

$$\limsup_{n\to\infty} d^2(u, x_n) \leq \limsup_{n\to\infty} d^2(z, x_n);$$

hence $z = u \in Tz = \{u\}$. □

Corollary 3.4. *Let K be a nonempty, closed and convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and T be $(a_1, a_2, b_1, b_2)-$ hybrid mapping from K to K. Then $F(T) \neq \emptyset$.*

Proof. Let $F = \{T(x)\}$. Then F is $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping from K to $CC(K)$. Hence F has at least one fixed point and so does T by Theorem 3.2. □

Since for any $\kappa > \kappa'$, $CAT(\kappa')$ space is $CAT(\kappa)$, therefore following corollaries holds.

Corollary 3.5. *Let X be a complete $CAT(0)$ space, K be a nonempty, closed and convex subset of X and T be $(a_1, a_2, b_1, b_2)-$ multivalued hybrid mapping from K to $CC(K)$. Then there is an $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$.*

Corollary 3.6. *Let X be a complete $CAT(0)$ space, K be a nonempty, closed and convex subset of X and T be $(a_1, a_2, b_1, b_2)-$hybrid mapping from K to K. Then there is an $x_0 \in K$ such that the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is bounded if and only if $F(T) \neq \emptyset$.*

Example 3.1. Let $X = [1, 7]$ with usual metric and $T : X \to C(X)$ be multivalued mapping defined by

$$Tx = \begin{cases} \{1\}, & x \in [1, 4]; \\ [1, \frac{2x^2+1}{x^2+1}], & x \in (4, 7]. \end{cases}$$

We will show that T is a $(a_1, a_2, b_1, b_2)-$ multivalued hybrid mapping with $a_1(x) = \frac{2x+3}{x+2}$, $a_2(x) = \frac{-x-1}{x+2}, b_1(x) = \frac{x+1}{x+2}, b_2(x) = \frac{1}{x+2}$ for all $x \in X$.
Case 1: if $x, y \in [1, 4]$, it is obvious.
Case 2: if $x \in [1, 4], y \in (4, 7]$, then for all $u \in Tx$ and $v \in Ty$, we have that $d^2(u, v) \leq 1, 9 < d^2(Tx, y) \leq d^2(u, y), 0 < d^2(x, Ty) \leq d^2(x, v)$ and so

$$\begin{aligned} \frac{2x+3}{x+2} d^2(u, v) \leq \frac{2x+3}{x+2} &\leq \frac{9x+9}{x+2} + \frac{x+1}{x+2} d^2(x, v) + \frac{1}{x+2} d^2(x, y) \\ &\leq \frac{x+1}{x+2} d^2(u, y) + \frac{x+1}{x+2} d^2(x, v) + \frac{1}{x+2} d^2(x, y). \end{aligned}$$

Case 3: if $x, y \in (4, 7]$, then for all $u \in Tx$ and $v \in Ty$, we have that $d^2(u, v) \leq 1, 4 < d^2(Tx, y) \leq d^2(u, y)$, $4 < d^2(x, Ty) \leq d^2(x, v)$ and so

$$\begin{aligned} \frac{3x+3}{x+2} d^2(u, v) \leq \frac{3x+3}{x+2} &\leq \frac{4x+4}{x+2} + \frac{4x+4}{x+2} + \frac{1}{x+2} d^2(x, y) \\ &\leq \frac{x+1}{x+2} d^2(u, y) + \frac{x+1}{x+2} d^2(x, v) + \frac{1}{x+2} d^2(x, y). \end{aligned}$$

Thus T is a $(a_1, a_2, b_1, b_2)-$ multivalued hybrid mapping with fixed point 1, $T(1) = \{1\}$.

4. Convergence Results

Lemma 4.6. *Let K be a nonempty, closed and convex subset of X where $diam(X) \leq \frac{\pi - \varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, and $T : K \to CC(X)$ be $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping. Let$\{x_n\}$ be a sequence in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$. Then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ includes exactly one point.*

Proof. Take $u \in \omega_w(x_n)$. Then we can find a subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Then by Lemma 2.3, we can find a subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta - \lim_{n\to\infty} v_n = v \in K$ .By Theorem 3.1, we have $v \in F(T)$ and by Lemma 2.4, we conclude that $u = v$. Hence, we get $\omega_w(x_n) \subseteq F(T)$. Let us take subsequence $\{u_n\}$ of $\{x_n\}$with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges, by Lemma 2.4, we have $x = u$, this means that $\omega_w(x_n)$ includes exactly one point. □

Theorem 4.3. *Let K be a nonempty, closed and convex subset of X where $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, and $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping with $T(p) = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by (1.1) then, $\{x_n\}$ $\Delta-$convergent to an element of $F(T)$.*

Proof. Let $p \in F(T)$. Then $T(p) = \{p\}$. Since T is a $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping, then for all $x \in K, u \in Tx$, we have that

$$d^2(u, p) \leq a_1(x)d^2(u, p) + a_2(x)d^2(u, p) \leq b_1(x)d^2(x, p) + b_2(x)d^2(x, p) \leq d^2(x, p).$$

Hence, we get $d(u, p) \leq d(x, p)$. Then

$$d(z_n, p) = d(P_K((1 - \varsigma_n)x_n \oplus \varsigma_n w_n), p) \leq d((1 - \varsigma_n)x_n \oplus \varsigma_n w_n, p)$$
$$\leq (1 - \varsigma_n)d(x_n, p) + \varsigma_n d(w_n, p) \leq (1 - \varsigma_n)d(x_n, p) + \varsigma_n d(x_n, p) \leq d(x_n, p),$$
$$d(y_n, p) = d(P_K((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \leq d((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \leq (1-\zeta_n)d(w_n, p)+\zeta_n d(v_n, p)$$
$$\leq (1 - \zeta_n)d(x_n, p) + \zeta_n d(z_n, p) \leq d(x_n, p)$$

and

$$d(x_{n+1}, p) = d(P_K(u_n), P_K(p)) \leq d(u_n, p) \leq d(y_n, p).$$

So, $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ implies $\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(y_n, p)$ exists. Let us say, $\lim_{n\to\infty} d(x_n, p) = k$. Since $d(w_n, p) \leq d(x_n, p)$ and $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$, we have that $\limsup_{n\to\infty} d(w_n, p) \leq k, \limsup_{n\to\infty} d(v_n, p) \leq k$ and

$$\begin{aligned} d(y_n, p) &= d(P_K((1 - \zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq d((1 - \zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq (1 - \zeta_n)d(w_n, p) + \zeta_n d(v_n, p) \\ &\leq (1 - \zeta_n)d(x_n, p) + \zeta_n d(z_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

which implies that that $\lim_{n\to\infty} d((1 - \zeta_n)w_n \oplus \zeta_n v_n, p) = k$, so by Lemma 2.2, we have that $\lim_{n\to\infty} d(w_n, v_n) = 0$. And again from

$$\begin{aligned} d(y_n, p) &= d(P_K((1 - \zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq d((1 - \zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq (1 - \zeta_n)d(w_n, p) + \zeta_n d(v_n, p) \\ &\leq (1 - \zeta_n)(d(w_n, v_n) + d(v_n, p)) + \zeta_n d(v_n, p) \\ &\leq (1 - \zeta_n)d(w_n, v_n) + d(v_n, p) \end{aligned}$$

we have that $k \leq \liminf_{n\to\infty} d(v_n, p)$ and since $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$, we have that $\lim_{n\to\infty} d(z_n, p) = k$. By $R-$convexivity, we have

$$\begin{aligned} d^2(z_n, p) &= d^2(P_K((1 - \varsigma_n)x_n \oplus \varsigma_n w_n), P_p) \\ &\leq d^2((1 - \varsigma_n)x_n \oplus \varsigma_n w_n, p) \\ &\leq (1 - \varsigma_n)d^2(x_n, p) + \varsigma_n d^2(w_n, p) - \frac{R}{2}(1 - \varsigma_n)\varsigma_n d^2(x_n, w_n) \\ &\leq (1 - \varsigma_n)d^2(x_n, p) + \varsigma_n d^2(x_n, p) - \frac{R}{2}(1 - \varsigma_n)\varsigma_n d^2(x_n, w_n) \\ &\leq d^2(x_n, p) - \frac{R}{2}(1 - \varsigma_n)\varsigma_n d^2(x_n, w_n) \end{aligned}$$

which implies that

$$\frac{R}{2}(1 - \varsigma_n)\varsigma_n d^2(x_n, w_n) \leq d^2(x_n, p) - d^2(z_n, p).$$

Since $\lim_{n\to\infty}(d^2(x_n,p) - d^2(z_n,p)) = 0$ and $\liminf_n(1-\varsigma_n)\varsigma_n > 0$, therefore we get $\lim_{n\to\infty} d(x_n,w_n) = 0$ and hence $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. So, by Lemma 4.6, $\{x_n\}$ has $\Delta-$limit which in $F(T)$. □

Theorem 4.4. *Let K be a nonempty, compact and convex subset of X where $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$, and $T : K \to KC(X)$ be a continuous $(a_1,a_2,b_1,b_2)-$multivalued hybrid mapping with $T(p) = \{p\}$ for all $p \in F(T)$.If $\{x_n\}$ is a sequence in K defined by (1.1) then, $\{x_n\}$ strongly converges to an element of $F(T)$.*

Proof. By Theorem 4.3, we have that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$ and $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. Since K is compact, there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $\lim_{i\to\infty} x_{n_i} = z$. Then we have

$$d(z,Tz) \leq d(z,x_{n_i}) + d(x_{n_i},Tx_{n_i}) + H_d(Tx_{n_i},Tz)$$

and taking limit on i, continuity of T implies that $z \in Tz$. □

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