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Approximating common fixed point of asymptotically nonexpansive mappings without convergence condition

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ABSTRACT. In the context of a hyperbolic space, we introduce and study convergence of an implicit iterative scheme of a finite family of asymptotically nonexpansive mappings without convergence condition. The results presented substantially improve and extend several well-known resullts in uniformly convex Banach spaces.

1. INTRODUCTION

Let C be a nonempty subset of a metric space X and $T : C \to C$ be a mapping. Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$ and the set $\{1, 2, 3, ...N\}$ by I. The mapping T is : (i) asymptotically nonexpansive[7] if there is a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for $x, y \in C$ and $n \geq 1$; (ii) asymptotically quasi-nonexpansive[13] if $F(T) \neq \phi$ and there is a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for $x \in C,\ p \in F(T)$ and $n \geq 1$; (iii) uniformly $L-$Lipschitzian[14] if for some $L > 0$, $d(T^n x, T^n y) \leq Ld(x, y)$ for $x, y \in C$ and $n \geq 1$;(iv) semi-compact[1] if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$ as $i \to \infty$.

Recently, many papers have appeared on iterative approximation of fixed points and common fixed points of asymptotically (quasi-) nonexpansive mappings through different explicit and implicit iterative schemes in Banach spaces and some classes of metric space [2, 3, 4, 5, 9, 10, 11, 14, 19].

We enlist some important results for a ready reference

Theorem 1.1. ([17, Theorem 3.1]) *Let C a bounded, closed and convex subset of a uniformly convex Banach space with a Fréchet differentiable norm and $T : C \to C$ be an asymptotically nonexpansive mapping such that $u_n \geq 0$ and $\sum_{n=1}^{\infty} u_n < \infty$. Then for each $x_1 \in C$, the sequence $\{x_n\}$ defined by: $x_{n+1} = \alpha_n T^n + (1 - \alpha_n) x_n$ where $\{\alpha_n\}$ is sequence of real numbers bounded away from 0 and 1, converges weakly to a fixed point of T.*

Theorem 1.2. ([19, Theorem 3.4]) *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $T_1, T_2 : C \to C$ be two asymptotically nonexpansive mappings with a sequence $\{u_n\}$ of real numbers satisfying $u_n \geq 0$ and $\sum_{n=1}^{\infty} u_n < \infty$ and $F(T_1) \cap F(T_2) \neq \phi$. Let $\{x_n\}$ be the sequence defined by: $x_1 \in C, x_{n+1} = T_2^n \left[(1 - \alpha_n) T_1^n x_n + \alpha_n T_2^n x_n\right]$, where α_n is a sequence in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$ satisfying:$\|x_n - T_2^n x_n\| \leq \lambda \|T_1^n x_n - T_2^n x_n\|$ for all*

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$x, y \in C, \lambda > 1$.If $I - T_i\,(i = 1, 2)$ are demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.

Theorem 1.3. ([15, Theroem 3.1]) *Let C be a nonempty closed and convex subset of a Banach space. Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive mappings of C with $u_{in} \in [0, \infty)$(i.e., $\|T_i^n x - q_i\| \leq (1 + u_{in})\|x - q_i\|$ for all $x \in C, q_i \in F(T_i), i \in I$), such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$ and $F = \cap_{i=1}^{n} T_i \neq \phi$. Suppose $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then $\{x_n\}$ defined by the implicit iterative scheme :$x_0 \in C$, $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n$, $n \geq 1$ where $n = (k - 1)N + i$, $i = i(n) \in I$ and $k = k(n) \geq 1$ is a positive integer such that $k(n) \to \infty$ as $n \to \infty$, converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

Theorem 1.4. ([15, Theorem 3.4]) *Let C be a nonempty bounded, closed and convex subset of a Banach space. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitizian asymptotically quasi-nonexpansive mappings of C with $u_{in} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$ and $F \neq \phi$. Suppose that there exists one semi-compact member T in $\{T_i : i \in I\}$ and $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then $\{x_n\}$ in Theorem 1.3 converges strongly to a point in F.*

In Theorems 1.1−1.4, the convergence condition (CC), $\sum_{n=1}^{\infty} u_{in} < \infty$ is crucial in their proofs. Moreover, Tan and Xu[17] remarked: We do not know whether Theorem 1.1 remains valid if u_n is allowed to approach 0 slowly enough so that $\sum\limits_{n=1}^{\infty} u_n < \infty$.

Recently, Guo and Cho [6] have established strong convergence of an extended form of $\{x_n\}$ in Theorem 1.3 to a common fixed point of the family $\{T_i : i \in I\}$ of asymptotically nonexpansive mappings satisfying (CC) and the following condition:

(C1): There exists a T_i , $1 \leq i \leq N$, and $s > 0$ such that $d(x, T_i x) \geq s d(x, F)$ for all $x \in C$.

It is remarked that the authors in ([2, 6, 15]) have employed complicated techniques to establish convergence results in uniformly convex Banach spaces even in the presence of the Opial property and (CC).

Fixed point theory on linear domains is very rich. So their extension to nonlinear domain (metric space) becomes essential. As the iterative schemes in Theorems 1.1−1.4 involve general convex combinations, we need some convex structure in a metric space to investigate their convergence on a nonlinear domain.

A metric space (X, d) is convex [12] if it satisfies the the following properties:
(i) There exists a family F of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x, y] \in F$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$);
(ii) The point $z = \alpha x \oplus (1 - \alpha)y$ is unique in $[x, y]$ such that

$$d(x, z) = (1 - \alpha)d(x, y) \text{ and } d(z, y) = \alpha d(x, y) \text{ for } \alpha \in [0, 1].$$

A convex metric space X is hyperbolic if

$$d(\alpha x \oplus (1 - \alpha)y, \alpha z \oplus (1 - \alpha)w) \leq \alpha d(x, z) + (1 - \alpha) d(y, w) \tag{1.1}$$

for all $x, y, z, w \in X$ and $\alpha \in [0, 1]$.

For $z = w$, the hyperbolic inequality (1.1) reduces to:

$$d(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d(x, z) + (1 - \alpha) d(y, z).$$

The above inequality corresponds to convex structure of Takahashi [16].

A subset C of a hyperbolic space X is convex if $\alpha x \oplus (1-\alpha)y \in C$ for all $x, y \in C$ and $\alpha \in [0,1]$.

A hyperbolic space X is uniformly convex if

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r} d\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right) : d(a,x) \le r, d(a,y) \le r, d(x,y) \ge r\varepsilon\right\} > 0,$$

for any $a, x, y \in X$, $r > 0$ and $\varepsilon > 0$.

From now onwards, we assume that X is a uniformly convex hyperbolic space with the property that for every $s \ge 0, \varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ depending on s and ε such that $\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0$ for any $r > s$.

Let $\{x_n\}$ be a bounded sequence in X. We define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n) \text{ for all } x \in X.$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined as

$$r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\}).$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{y \in X : r(y, \{x_n\}) = r(\{x_n\})\}.$$

A sequence $\{x_n\}$, $\triangle-$converges to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_{n\to\infty} x_n = x$.

A subset C of a metric space X is a Chebyshev set if for each point $x \in X$, there corresponds a unique point $z \in C$ such that $d(x,z) = d(x,C)$. If C is a Chebyshev set, one can define the nearest point projection $P : X \to C$ by assigning z to x.

Xu [18], extensively used the concept of $p-$uniform convexity; its nonlinear version for $p = 2$ has been introduced by Khamsi and Khan [8] as follows:

For a fixed $a \in X, r > 0, \varepsilon > 0$, define

$$\Psi(r,\varepsilon) = \inf\left(\frac{1}{2}d(a,x)^2 + \frac{1}{2}d(a,y)^2 - d\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2\right)$$

where the infimum is taken over all $x, y \in X$ such that $d(a,x) \le r, d(a,y) \le r$ and $d(x,y) \ge r\varepsilon$.

We say that X is $2-$uniformly convex if

$$c_X = \inf\left\{\frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2} : r > 0, \varepsilon > 0\right\} > 0. \tag{1.2}$$

It has been shown in [8] that any $CAT(0)$ space is $2-$uniformly convex with $c_X = \frac{1}{4}$.

Using the uniqueness of $\alpha x \oplus (1-\alpha)y$ in the metric segment $[x,y]$, a generalization of Sun's implicit iterative scheme in Theorem 1.3 becomes:

$$x_0 \in C,\ x_n = \alpha_n x_{n-1} \oplus (1-\alpha_n) T_{i(n)}^{k(n)} x_n,\ \ n \ge 1. \tag{1.3}$$

We now show that $\{x_n\}$ in (1.3) is well-defined for a family $\{T_i : i \in I\}$ of N uniformly $L-$Lipschitizian mappings. Choose $x_0 \in C$ and define $S : C \to C$ by $Sx = \alpha_1 x_0 \oplus (1-\alpha_1) T_1 x$.The existence of x_1 is guaranteed if S has a fixed point. Using the hyperbolic inequality for $x, y \in C$,we have

$$\begin{aligned} d(Sx, Sy) &= d(\alpha_1 x_0 \oplus (1-\alpha_1) T_1 x, \alpha_1 x_0 \oplus (1-\alpha_1) T_1 x) \\ &\le (1-\alpha_1) d(T_1 x, T_1 y) \\ &\le (1-\alpha_1) L d(x,y). \end{aligned}$$

Thus S is a contraction if $L \in \left(0, \frac{1}{1-\alpha_1}\right)$. By the Banach contraction principle, S has a unique fixed point. Thus, the existence of x_1 is established. Similarly, we can establish the existence of $x_2, x_3, x_4, ...$ Thus, the implicit iterative scheme (1.3) is well-defined.

In this paper, we prove convergence results for the implicit iterative scheme (1.3) for a finite family of asymptotically nonexpansive mappings not admitting (CC) in a uniformly convex hyperbolic space setting. Our results generalize and compliment some important known results on a linear domain.

We need the following lemmas in our convergence analysis.

Lemma 1.1. [4] *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 1.2. [11] *Every closed and convex subset of a complete uniformly convex hyperbolic space is a Chebyshev set.*

Lemma 1.3. [4] *Let C be a nonempty closed and convex subset of a uniformly convex hyperbolic space. Let ρ be the asymptotic radius of a bounded sequence $\{x_n\}$ in C and $A_C(\{x_n\}) = \{x\}$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$ (a real number), then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.4. [8] *Suppose that X is $2-$uniformly convex hyperbolic space with $c_X > 0$ (cf.(1.2)). Then for any $\alpha \in [0,1]$, we have:*

$$\begin{aligned} d(u, \alpha x \oplus (1-\alpha) y)^2 &\le \alpha d(u,x)^2 + (1-\alpha) d(u,y)^2 \\ &\quad -4c_X \min\left\{\alpha^2, (1-\alpha)^2\right\} d(x,y)^2 \end{aligned}$$

for any $u, x, y \in X$.

2. Convergence results

First, we establish a pair of lemmas.

Lemma 2.5. *Let C be a nonempty bounded, closed and convex subset of a $2-$uniformly convex hyperbolic space. Let $\{T_i : i \in I\}$ be N uniformly $L-$Lipschitzian asymptotically quasi-nonexpansive of C. Then, for the sequence $\{x_n\}$ in (1.3) with $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$, we have $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for all $l \in I$.*

Proof. Define $u_n = \max_{1\le i\le N} u_{in}$ and fix $q \in F$. Then $d\left(T_{i(n)}^{k(n)} x, q\right) \le (1+u_n) d(x,q)$ for all $x \in C, q \in F, i \in I$ and $u_n \to 0$ as $n \to \infty$. Set $\sigma_n = d\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right), n = (k-1)N + i$ where $i = i(n) \in I, k = k(n) \ge 1$ is a positive integer such that $k(n) \to \infty$ as $n \to \infty$. Boundedness of C assures that $\{d(x_n, q), d(T_i^k x_n, q)\}$ is a bounded set. Fix $r > 0$. Then $\max\{d(x_n, q), d(T_i^k x_n, q)\} \le r$. By Lemma 1.4 and the scheme (1.3), we get

$$\begin{aligned} d(x_n, q)^2 &= d\left(\alpha_n x_{n-1} \oplus (1-\alpha_n) T_{i(n)}^{k(n)} x_n, q\right)^2 \\ &\le \alpha_n d(x_{n-1}, q)^2 + (1-\alpha_n)(1+v_k) d(x_n, q)^2 - 4c_M \alpha_n (1-\alpha_n) \sigma_n \end{aligned}$$

where $v_k = 2u_k + u_k^2$. Using the information $s < \alpha_n < 1-s$ in the above inequality, we have:

$$d(x_n, q)^2 \le d(x_{n-1}, q)^2 + \frac{r}{s} v_k - 4sc_M \sigma_n. \tag{2.4}$$

Assume that $\limsup_{n\to\infty} \sigma_n > 0$. Then there exists a $\sigma > 0$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\sigma_{n_j} \geq \sigma > 0$. Therefore, inequality (2.4) reduces to

$$\begin{aligned} d\left(x_{n_j}, q\right)^2 &\leq d\left(x_{n_j-1}, q\right)^2 + \frac{r}{s} v_{k(n_j)} - 4sc_M\sigma \\ &\leq d\left(x_{n_j-1}, q\right)^2 + \frac{r}{s}\left(v_{k(n_j)} - \frac{2s^2c_M\sigma}{r}\right) - 2sc_M\sigma. \end{aligned}$$

As $v_{k(n_j)} \to 0$, so there exists $k_0 \in N$ such that $v_{k(n_j)} < \dfrac{2s^2c_M\sigma}{r}$ for all $k(n_j) \geq k_0$.

Hence, for all $n_j \geq (n_j)_0 = (k_0 - 1)N + i$, the above inequality becomes

$$2sc_M\sigma \leq d\left(x_{n_j-1}, q\right)^2 - d\left(x_{n_j}, q\right)^2.$$

For $m \geq (n_j)_0$, we have:

$$\begin{aligned} \sum\nolimits_{n_j=(n_j)_0}^{m} 2sc_M\sigma &\leq d\left(x_{(n_j)_0-1}, q\right)^2 - d\left(x_m, q\right)^2 \\ &\leq d\left(x_{(n_j)_0-1}, q\right)^2. \end{aligned}$$

When $m \to \infty$ in the above inequality, we have

$$\infty = d\left(x_{(n_j)_0-1}, q\right)^2 < \infty$$

a contradiction. Hence $\lim_{n\to\infty} d\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right) = 0$.

Further,

$$\begin{aligned} d\left(x_n, x_{n-1}\right) &\leq (1-\alpha_n) d\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right) \\ &\leq (1-s) d\left(T_{i(n)}^{k(n)} x_n, x_{n-1}\right), \end{aligned}$$

gives that $\lim_{n\to\infty} d\left(x_n, x_{n-1}\right) = 0$.

For a fixed $j \in I$, we have

$$d\left(x_{n+j}, x_n\right) \leq d\left(x_{n+j}, x_{n+j-1}\right) + \ldots + d\left(x_n, x_{n-1}\right)$$

and hence

$$\lim_{n\to\infty} d\left(x_{n+j}, x_n\right) = 0 \text{ for all } j \in I. \tag{2.5}$$

For $n > N$, $n = (n - N)(\text{mod} N)$. Also $n = (k(n) - 1)N + i(n)$. Hence $n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N)N + i(n - N)$.

That is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$.

Therefore, we have

$$\begin{aligned} d\left(x_{n-1}, T_n x_n\right) &\leq d\left(x_{n-1}, T_{i(n)}^{k(n)} x_n\right) + d\left(T_{i(n)}^{k(n)} x_n, T x_n\right) \\ &\leq \sigma_n + L d\left(T_{i(n)}^{k(n)-1} x_n, x_n\right) \\ &= \sigma_n + L^2 d\left(x_n, x_{n-N}\right) + L d\left(T_{i(n-N)}^{k(n-N)} x_{n-N}, x_{(n-N)-1}\right) \\ &\quad + L d\left(x_{(n-N)-1}, x_n\right) \\ &= \sigma_n + L^2 d\left(x_n, x_{n-N}\right) + L\sigma_{n-N} + L d\left(x_{(n-N)-1}, x_n\right) \end{aligned}$$

which together with (2.5) yields, $\lim_{n\to\infty} d\left(x_{n-1}, T x_n\right) = 0$.

Since $d\left(x_n, T x_n\right) \leq d\left(x_n, x_{n-1}\right) + d\left(x_{n-1}, T_n x_n\right)$, so we have

$$\lim_{n\to\infty} d\left(x_n, T_n x_n\right) = 0. \tag{2.6}$$

Hence, for all $l \in I$, the inequality

$$\begin{aligned} d(x_n, T_{n+l}x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}) + d(T_{n+l}x_{n+l}, T_{n+l}x_n) \\ &\leq (1+L)d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}) \end{aligned}$$

together with (2.5) and (2.6) implies that

$$\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0 \text{ for all } l \in I.$$

Now for each $l \in I$, the sequence $\{d(x_n, T_l x_n)\}$ is a subsequence of $\cup_{i\in I} \{d(x_n, T_{n+i}x_n)\}$ and $\lim_{n\to\infty} d(x_n, T_{n+i}x_n) = 0$ for each $i \in I$.

Thus

$$\lim_{n\to\infty} d(x_n, T_l x_n) = 0 \text{ for all } l \in I. \tag{2.7}$$

□

Lemma 2.6. *If C is a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ lies in C.*

Proof. Let $x \in A(\{x_n\})$ .We show that $x \in C$. Suppose $x \notin C$. By Lemma 1.2, C is a Chebyshev set , so we can define the nearest point projection $P : X \to C$. Now $d(Px, x_n) < d(x, x_n) \Longrightarrow r(Px, \{x_n\}) < r(x, \{x_n\}) \Longrightarrow x \notin A(\{x_n\})$, a contradiction. Hence $x \in C$. □

Our convergence results to follow are independent of both (CC) and (C1).

Theorem 2.5. *Let C be a nonempty bounded, closed and convex subset of a complete $2-$uniformly convex hyperbolic space X. Let $\{T_i : i \in I\}$ be N asymptotically nonexpansive mappings of C. Let $\{x_n\}$ be as in (1.3) with $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. If $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$, then $\{x_n\}$, $\triangle-$converges to a point in F.*

Proof. Obviously, $\{x_n\}$ is bounded. By Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{z_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{z_n\}) = \{z\}$. Then by Lemma 2.6, $z \in C$. Next, we show that $z \in F$. By Lemma 2.5, $\lim_{n\to\infty} d(z_n, T_l z_n) = 0$ for all $l \in I$.

Hence for a fixed $l \in I$ and for any $m, n \geq 1$,we have

$$\begin{aligned} d(T_l^m z, z_n) &\leq d(T_l^m z, T_l^m z_n) + d(T_l^m z_n, T_l^{m-1} z_n) + \dots + d(T_l^2 z_n, T_l z_n) + d(T_l z_n, z_n) \\ &\leq (1+u_{lm}) d(z, z_n) + d(T_l z_n, z_n) + \sum_{i=1}^{m-1} (1+u_{li}) d(Tz_n, z_n). \end{aligned}$$

By taking $\limsup_{n\to\infty}$ on both sides in the above inequality, we have

$$\limsup_{n\to\infty} d(T_l^m z, z_n) \leq (1+u_{lm}) r(z, \{z_n\}).$$

Therefore

$$\limsup_{m\to\infty} r(T_l^m z, \{z_n\}) \leq r(z, \{z_n\}).$$

By the definition of $A(\{z_n\})$, we have

$$r(z, \{z_n\}) \leq \liminf_{m\to\infty} r(T_l^m z, \{z_n\}).$$

That is,

$$\lim_{m\to\infty} r(T_l^m z, \{z_n\}) = r(z, \{z_n\})$$

By Lemma 1.3, $\lim_{m\to\infty} T_l^m z = z$. Since T_l is continuous, therefore $T_l z = T_l\left(\lim_{m\to\infty} T_l^m z\right) = \lim_{m\to\infty} T_l^{m+1} z = z$. This proves that $z \in F(T_l)$ and hence $z \in F$.

If $x \neq z$, then by the uniqueness of asymptotic centres, we have that

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z),
\end{aligned}$$

a contradiction. So $x = z$. Hence x is the unique asymptotic center of every subsequence $\{z_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$, $\triangle-$converges to $x \in F$. □

Theorem 2.6. *Let C be a nonempty bounded, closed and convex subset of a $2-$uniformly convex hyperbolic space X. Let $\{T_i : i \in I\}$ be N asymptotically nonexpansive mappings of C. Let $\{x_n\}$ be as in (1.3) with $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. If at least one member T in $\{T_i : i \in I\}$ is semi-compact, then $\{x_n\}$ converges strongly to a point in F.*

Proof. Without loss of generality, assume that T_1 is semi-compact. Since T_1 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$.

Using $x_n = x_{n_j}$ in (2.7), we get

$$d(p, T_l p) = \lim_{n_j\to\infty} d\left(x_{n_j}, T_l x_{n_j}\right) = 0.$$

This proves that $p \in F$.

The inequality

$$d\left(x_{n_j+1}, p\right) \leq d\left(x_{n_j+1}, x_{n_j}\right) + d\left(x_{n_j}, p\right)$$

gives that $x_{n_j+1} \to p$ as $d\left(x_{n_j+1}, x_{n_j}\right) \to 0$ and $x_{n_j} \to p$ as $j \to \infty$.Continuing in this way, for any $m \geq 0$,we get $x_{n_j+m} \to p$ as $j \to \infty$. By induction, one can obtain that $\bigcup_{m=0}^{\infty}\{x_{n_j+m}\}$ converges to p as $j \to \infty$; in fact $\{x_n\}_{n=n_1}^{\infty} = \bigcup_{m=0}^{\infty}\{x_{n_j+m}\}_{j=1}^{\infty}$ gives that $x_n \to p$ as $n \to \infty$. □

Remark 2.1. (1) Our strong convergence result(Theorem 2.6) on the one hand provides affirmative answer to the question posed by Tan and Xu [17] on a very general nonlinear domain, namely, hyperbolic space and on the other hand it generalizes Theorem 1.4.
(2) Theorem 2.5 sets an analogue of Theorems 1.1−1.2.
(3) Theorem 2.6 extends and improves([6], Theorems 3-4).
(4) All the results in this paper hold good in CAT(0) spaces.

Open Problems: (1) Translate the hybrid iterative scheme of Yolacan and Kiziltunc [20] on Banach spaces for hyperbolic metric spaces and establish their convergence results without (CC) on the new nonlinear domain.

(2) Carry out convergence analysis of iterative scheme (1.4) in [10] without (CC).

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