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Robust vector optimization with a variable domination structure

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ABSTRACT. In this paper we propose a new definition of robustness for uncertain vector-valued optimization problems equipped with a variable domination structure, derive scalarization results and present algorithms for computing robust solutions.

1. INTRODUCTION

Uncertain data contaminate most optimization problems in various applications ranging from science and engineering to industry and thus represent an essential component in optimization. From a mathematical point of view, many problems can be modeled as an optimization problem and be solved, but in real life, having exact data is very rare and seems almost impossible. Due to a lack of complete information, uncertain data can highly affect solutions and thus influence the decision making process. Hence, it is crucial to address this important issue in optimization theory.

As was recently observed in [11, 12], robust multiobjective optimization is an important application of set optimization. In case uncertainties are present during an optimization process, the decision maker generally has two options: Using stochastic optimization approaches, solutions are desired that are likely to satisfy the given requirements (optimality and constraints). Alternatively, robust optimization searches for solutions, which are of good quality in the worst-case scenarios, regardless of how likely this event may be. Robust multiobjective optimization with a fixed domination structure was examined in [11, 12]. In this paper we develop corresponding results in case the ordering set depends on the decision variable.

First, variable domination structures were introduced by Yu in [19]. An interesting overview on recent developments in vector optimization with variable domination structure is given by Eichfelder in [6]. Motivated by applications in medical image registration [5, 6], variable domination structures in vector optimization gained recognition as they allow to introduce a specification of the decision-maker's preferences into the model. Due to these important applications, variable domination structures have gained increasing interest, compare [1, 2]. More recently, variable domination structures were introduced in set optimization, see [3, 8, 9, 15].

The focus of this paper lies in combining robust approaches to vector optimization with variable domination structures, which is a completely new concept in uncertain vector optimization. Our approach enables the decision maker to specify his / her preferences with regard to the domination structure rather than relying on a given optimality concept. Our analysis is, for a fixed domination structure, closely related to the approaches conducted in [11, 12]. We introduce a new definition of robust solutions of uncertain multiobjective

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optimization problems, where the domination structure is equipped with a variable domination structure. Then we develop a scalarization approach to obtain robust solutions of uncertain multiobjective optimization problems equipped with the mentioned variable domination structure. Moreover, we present two algorithms for computing robust solutions of uncertain vector-valued optimization problems.

2. PRELIMINARIES

We recall some notation of uncertain multiobjective optimization introduced in Ehrgott et al. [4] (see also [12]) used in this paper. Throughout this work, let Y be a linear topological space, X is a linear space and let an uncertainty set $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^N$ be given. For $\emptyset \neq \mathcal{X} \subseteq X$, let $f : \mathcal{X} \times \mathcal{U} \to Y$ be the function that is to be minimized. The uncertainty set $\mathcal{U}$ contains all the possible parameter values that the uncertain parameter may attain. Our goal is to obtain solutions that are *robust*, i.e., that perform well even in the worst-case scenario. For the scalar case $Y = \mathbb{R}$, this would mean to minimize the functional $\sup_{\xi \in \mathcal{U}} f(x, \xi)$ on $\mathcal{X}$. Of course, if f is vector-valued, this scalar approach cannot be easily transferred to vector optimization. Due to the absence of a total order on Y, we need to define the meaning of an optimal solution.

We define for $x \in \mathcal{X}$

$$f_{\mathcal{U}}(x) := \{f(x, \xi) \mid \xi \in \mathcal{U}\}$$

the image of f under $\mathcal{U}$. For a fixed $\xi \in \mathcal{U}$, the vector optimization problem is denoted by

$$(P(\xi)) \qquad \min_{x \in \mathcal{X}} f(x, \xi).$$

The family of all problems $\bigcup_{\xi \in \mathcal{U}} (P(\xi))$ is denoted by $P(\mathcal{U})$.

Consider the following figure (where $Y = \mathbb{R}^2$), which shows three feasible solution sets. Our objective is to attain a robust solution, i.e., a solution set $f_{\mathcal{U}}(x^0)$, which is not dominated by any other solution set. If we compare sets with respect to the natural ordering cone $\mathbb{R}^2_{\geqq} := \{x \in \mathbb{R}^2 | x_i \geq 0,\ i = 1, 2\}$, this means that for a robust solution x^0, there does not exist another element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}^2_{\geqq}$. In the figure below we see that x_1 is not robust since it is dominated by x^3. x^2 and x^3 are robust, as their solution sets are not dominated by any other set.

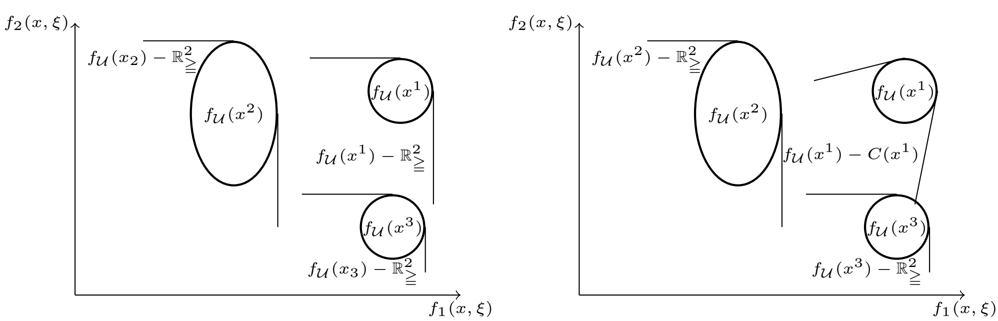

FIGURE 1. Left: x^1 is not robust, but x^2 and x^3 are robust. Right: x^1, x^2 and x^3 are robust.

The general idea now is that each solution $x \in \mathcal{X}$ is associated with a certain set $C(x)$, where $C : \mathcal{X} \rightrightarrows Y$ is a set-valued mapping. Consider, for example, the right illustration in Figure 1. The element x^1 now is equipped with a smaller cone, which means that x^1 is not dominated by any other solution anymore, thus x^1 is considered to be robust here. This

approach enables the decision-maker to specify his / her preferences directly when the uncertain optimization problem is modeled. There are various applications of variable domination structures in vector optimization available in the literature, for instance in the optimal treatment planning of intensity-modulated radiation therapy and consumer demand in economics (see [6]), to which our approach can directly be applied here. We will refer to this solution concept as *robust minimality* (see Definition 2.2). A second solution concept for robustness in vector optimization will be introduced in Definition 2.2.
Above we implicitly used the upper set less relation introduced by Kuroiwa [16, 17] to compare sets while focusing on their upper bounds.

Definition 2.1 (Upper Set Less Relation, [16, 17])**.** Let $C \subset Y$ be a proper pointed closed convex cone. Then the **upper set less relation** for two nonempty sets $A, B \subset Y$ is given as

$$A \preceq_C^u B :\Longleftrightarrow A \subseteq B - C.$$

In the following we will work with the variable domination structure given by the set-valued map $C : X \supseteq \mathcal{X} \rightrightarrows Y$, where we assume for each $x \in \mathcal{X}$ that $C(x)$ is a proper pointed closed convex cone with nonempty interior. Then we characterize the upper set less relation with respect to for two arbitrary nonempty sets $A, B \subset Y$ analogously to Definition 2.1 for fixed $x \in \mathcal{X}$ by

$$A \preceq_{Q(x)}^u B :\Longleftrightarrow A \subseteq B - Q(x), \tag{2.1}$$

where $Q(x) = C(x)$ ($Q(x) = C(x) \setminus \{0\}$, $Q(x) = \operatorname{int} C(x)$, respectively).

Related to the concept of variable domination structures, there are two main concepts known in the literature (see Eichfelder [6]). For a vector-valued function $f : X \to Y$ and a cone-valued map $C : X \rightrightarrows Y$, an element $f(x^0) = y^0$ is called **minimal** if there does not exist any $\overline{x} \in X$ with $f(\overline{x}) = \overline{y}$ such that

$$y^0 \in \overline{y} + C(x^0) \setminus \{0\}.$$

Moreover, an element x^0 is called **nondominated** if there does not exist $\overline{x} \in X$ with $f(\overline{x}) = \overline{y}$ such that

$$y^0 \in \overline{y} + C(\overline{x}) \setminus \{0\}.$$

Below we formally introduce two kinds of robust solutions of the family of uncertain problems $P(\mathcal{U})$ based on the concepts of minimal and nondominated solutions. The solution concept follows the line of the approach in [12], but here we consider variable domination structures.

Definition 2.2 (Robust Minimal Solutions / Robust Nondominated Solutions)**.** A solution $x^0 \in \mathcal{X}$ of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively) **robust minimal** if there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for $Q = C(x^0)$ ($Q = \operatorname{int} C(x^0)$, $Q = C(x^0) \setminus \{0\}$): $f_{\mathcal{U}}(\overline{x}) \preceq_Q^u f_{\mathcal{U}}(x^0)$, or equivalently,

$$\nexists \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q.$$

A solution $x^0 \in \mathcal{X}$ of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively) **robust nondominated** if there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for $Q = C(\overline{x})$ ($Q = \operatorname{int} C(\overline{x})$, $Q = C(\overline{x}) \setminus \{0\}$): $f_{\mathcal{U}}(\overline{x}) \preceq_Q^u f_{\mathcal{U}}(x^0)$.

We establish that the upper set less relation is suitable for a risk-averse decision maker, since robust solutions are not dominated by other elements with respect to their upper bounds. Hence, there are no solutions whose worst cases are smaller with respect to an ordering set than the worst cases of said robust solution.

Below we provide a motivating example for the choice of the ordering cones in Definition 2.2 in the context of portfolio optimization, where the data describing the returns and risks are uncertain.

Example 2.1 (Portfolio Optimization)**.** A shareholder would like to invest in a portfolio consisting of n shares that maximizes his wins and minimizes the risk associated with the shares at the same time. Let $x = (x_1, \ldots, x_n)^T$ be the vector of shares and $r = (r_1, \ldots, r_n)^T$ be the vector of returns of the respective shares. The return of the whole portfolio is then $r_p := \langle r, x\rangle = r^T \cdot x$. Of course, the return vector r is subject to uncertainties such that r is a vector of random variables, such that we write $\mu := \mathbb{E}(r)$ (here $\mathbb{E}(\cdot)$ denotes the expected values of r) and $\mu_i := \mathbb{E}(r_i)$, $i = 1, 2, \ldots, n$. The covariance matrix that represents the risk is denoted by $D = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{1n} & \cdots & d_{nn} \end{pmatrix}$, which is assumed to be positive definite. The entries in the covariance matrix C can be computed by means of $d_{ij} = \mathbb{E}[(r_i - \mathbb{E}(r_i)) \cdot (r_j - \mathbb{E}(r_j))]$ for $i, j = 1, 2, \ldots, n$. The values in the main diagonal of D are the variances of the respective shares. The risk of a portfolio x can then be described by $x^T \cdot D \cdot x$. Moreover, it is assumed that no short sales are allowed, i.e., $x_i \geq 0, i = 1, 2, \ldots, n$, and all the available capital shall be used and normed to one, i.e., $\sum_{i=1}^{n} x_i = 1$. Now, the goal is to minimize the risk and maximize the returns simultaneously. Both objectives are contradictory, because higher returns are usually accompanied by higher risk. As is described in Fliege and Werner [10], the vector μ and the covariance matrix D are themselves perturbed by uncertainty, namely, by the given distribution function. In case that the probability distribution on the returns r_i is not (or just partly) known, we can assume that both μ as well as D depend additionally on some uncertain parameter ξ which is assumed to belong to a so-called uncertainty set $\mathcal{U}$. Then, we suppose that $\mu : \mathcal{U} \to \mathbb{R}$ and $D : \mathcal{U} \to \mathbb{R}^{n,n}$. Consequently, we obtain the following uncertain optimization problem for $\xi \in \mathcal{U}$

$$\min_{x \in K} \big(f_1(x, \xi), f_2(x, \xi)\big)^T, \tag{2.2}$$

with $f_1(x, \xi) := -\mu(\xi)^T \cdot x$, $f_2(x, \xi) := x^T \cdot D(\xi) \cdot x$, and $K := \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \ldots, n, \sum_{i=1}^{n} x_i = 1\}$, where $\mathcal{U}$ denotes the set of possible values of the uncertain vector ξ. If the decision-maker would like to follow a robust approach, all possible returns and risk outcomes can be allocated in a set $f_{\mathcal{U}}(x) := \{(f_1(x, \xi), f_2(x, \xi))^T \mid \xi \in \mathcal{U}\}$. The selection of the ordering cone $C(x)$ in the notion of *robust minimal* solutions can be an indicator for the relevance of the particular solution x. For example, if a solution x is known to be less relevant for the decision-maker, then the corresponding cone $C(x)$ can be chosen large enough, while more desired solutions can be equipped with a smaller cone. In that sense, $C(x)$ can be chosen to model the importance of each x. Therefore, possible portfolios can be ranked by a decision maker according to their preference. Given a finite number of portfolios $x^1, \ldots, x^m$, a decision-maker ranks these portfolios such that, after re-numbering, $x^1 \preccurlyeq \ldots \preccurlyeq x^m$, where $\preccurlyeq$ denotes a preference relation. Then the variable domination structure should be chosen such that $C(x^m) \subseteq \ldots \subseteq C(x^1)$, then portfolios with a higher ranking are equipped with a smaller ordering cone.

Since the involved data μ and C are assumed to be uncertain, it seems likely that there exist undesired elements in the objective space that are located far from where most uncertain data is found. If there exists such an element $f(\overline{x}, \widetilde{\xi})$ belonging to the set $f_{\mathcal{U}}(\overline{x})$ as illustrated in Figure 2 and the standard fixed ordering cone $\mathbb{R}^2_+$ is used, then another solution x might become *robust nondominated*, although most of the data is located worse than $\overline{x}$, due to single elements or "outliers". Clearly, the portfolio $\overline{x}$ is the preferred solution, and the portfolio x should not be considered. When one uses a variable domination structure, the cone $C(\overline{x})$ which is associated with a set $f_{\mathcal{U}}(\overline{x})$ could be adapted accordingly to deal which such a situation. If real-world data is available, problem (2.2) can be solved by using Algorithms 1 and 2.

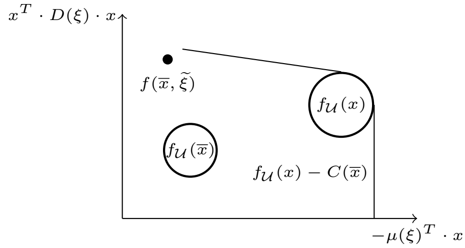

FIGURE 2. If isolated elements exists that are located far from where most data is found, minimality of other elements, which otherwise would be considered "worse", is affected (see Example 2.1). Such a situation can be resolved using variable domination structures.

3. MAIN RESULTS

In this section, we provide some scalarizing methods for finding strictly (weakly, ·, respectively) robust minimal / nondominated solutions of the uncertain problem $P(\mathcal{U})$. Recall that the dual cone of a cone C is defined by $C^* := \{y^* \in Y^* | \, \forall y \in C : \; y^*(y) \geq 0\}$, where Y^* denotes the dual space of Y. The quasi-interior of C^* is given as $C^{\#} := \{y^* \in Y^* | \, \forall y \in C \setminus \{0\} : \; y^*(y) > 0\}$. For fixed $x \in \mathcal{X}$, we denote the dual cone of $C(x)$ by $C^*(x)$, and the quasi-interior of $C^*(x)$ is written as $C^{\#}(x)$. Furthermore, we define $\overline{C}^* := \cap_{x \in \mathcal{X}} C^*(x)$ and $\overline{C}^{\#} := \cap_{x \in \mathcal{X}} C^{\#}(x)$.

Theorem 3.1. *Consider the minimization problem*

$$(P\mathcal{U}_{y^*}) \qquad \min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} \; y^*(f(x,\xi))$$

for some $y^* \in Y^* \setminus \{0\}$. The following statements hold.

(a) *If $x^0 \in \mathcal{X}$ is the unique optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in C^*(x^0)\setminus\{0\}$, then x^0 is strictly robust minimal.*
(b) *If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in C^*(x^0)\setminus\{0\}$ and $\max_{\xi \in \mathcal{U}} \; y^*(f(x,\xi))$ exists for $y^* \in C^*(x^0) \setminus \{0\}$ and each $x \in \mathcal{X}$, then x^0 is weakly robust minimal.*
(c) *If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in C^{\#}(x^0)$ and $\max_{\xi \in \mathcal{U}} \; y^*(f(x,\xi))$ exists for each $x \in \mathcal{X}$ with $y^* \in C^{\#}(x^0)$, then x^0 is robust minimal.*
(d) *If $x^0 \in \mathcal{X}$ is the unique optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in \overline{C}^*\setminus\{0\}$, then x^0 is strictly robust nondominated and strictly robust minimal.*
(e) *If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in \overline{C}^* \setminus \{0\}$ and $\max_{\xi \in \mathcal{U}} \; y^*(f(x,\xi))$ exists for $y^* \in \overline{C}^* \setminus \{0\}$ and each $x \in \mathcal{X}$, then x^0 is weakly robust nondominated and weakly robust minimal.*
(f) *If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}_{y^*})$ with $y^* \in \overline{C}^{\#}$ and $\max_{\xi \in \mathcal{U}} \; y^*(f(x,\xi))$ exists for each $x \in \mathcal{X}$ with $y^* \in \overline{C}^{\#}$, then x^0 is robust nondominated and robust minimal.*

Proof. We first show assertions (a)–(c). Suppose that $x^0 \in \mathcal{X}$ is not strictly (weakly, $\cdot$, respectively) robust minimal. Consequently, there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq^u_Q f_{\mathcal{U}}(x^0)$, which is equivalent to $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$, where $Q = C(x^0)$ ($Q = \operatorname{int} C(x^0)$, $Q = C(x^0) \setminus \{0\}$, respectively). This implies

$$\forall \xi \in \mathcal{U} \; \exists \eta \in \mathcal{U} : \; f(\overline{x}, \xi) \in f(x^0, \eta) - Q.$$

Due to the definition of the dual cone $C^*(x^0)$ and the quasi-interior of $C^*(x^0)$, we obtain for $y^* \in C^*(x^0)$ ($y^* \in C^*(x^0) \setminus \{0\}$, $y^* \in C^{\#}(x^0)$, respectively)

$$y^*(f(\overline{x}, \xi)) \, [\leq / < / <] \, y^*(f(x^0, \eta(\xi)))$$

for every $\xi \in \mathcal{U}$. We arrive at

$$\sup_{\xi \in \mathcal{U}} \; y^*(f(\overline{x}, \xi)) \, [\leq / < / <] \, \sup_{\xi \in \mathcal{U}} \; y^*(f(x^0, \eta(\xi))) \leq \sup_{\eta \in \mathcal{U}} \; y^*(f(x^0, \eta)). \tag{3.3}$$

Note that the strict inequalities in (3.3) hold because the existence of $\max_{\xi \in \mathcal{U}} \; y^*(f(x, \xi))$ was assumed. But this is a contradiction to the assumption. The assertions (d)–(f) can be proven in a similar manner, bearing in mind that $y^* \in \overline{C}^* \setminus \{0\}$ implies that $y^* \in C^*(x) \setminus \{0\}$ for every $x \in \mathcal{X}$, and in particular for $x = x^0$. □

At this point it is interesting to investigate whether it is possible to provide assumptions that ensure the inverse direction in Theorem 3.1 to hold true. To this end, we follow an approach by Jahn [14, Lemma 2.1], which we adapt to a variable domination setting.

Theorem 3.2. *Consider the minimization problem ($P\mathcal{U}_{y^*}$), and assume that the objective space Y is locally convex. Suppose that x^0 is strictly robust minimal and that the set $f_{\mathcal{U}}(x^0) - C(x^0)$ is closed and convex. Then there does not exist an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for every $y^* \in C^*(x^0)$*

$$\sup_{\xi \in \mathcal{U}} y^*(f(\overline{x}, \xi)) \leq \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)).$$

Proof. Assume that $x^0 \in \mathcal{X}$ is strictly robust minimal. This is equivalent to

$$\begin{aligned} &\nexists \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - C(x^0) \\ \iff &\forall \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) \nsubseteq f_{\mathcal{U}}(x^0) - C(x^0) \\ \iff &\forall \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \exists \xi_{\overline{x}} \in \mathcal{U} : \; f(\overline{x}, \xi_{\overline{x}}) \notin f_{\mathcal{U}}(x^0) - C(x^0). \end{aligned}$$

Since $f_{\mathcal{U}}(x^0) - C(x^0)$ is closed and convex, we use a classical separation argument (see, for example, [13, Theorem 3.18.]) such that we get

$$\begin{gathered} \forall \overline{x} \in \mathcal{X} \setminus \{x^0\} \; \exists \xi_{\overline{x}} \in \mathcal{U}, \; \exists y^* \in Y^* \setminus \{0\}, \; \alpha \in \mathbb{R} : \\ y^*(f(\overline{x}, \xi_{\overline{x}})) > \alpha \geq y^*(y) \; \forall y \in f_{\mathcal{U}}(x^0) - C(x^0), \end{gathered} \tag{3.4}$$

and this yields

$$\forall \overline{x} \in \mathcal{X} \setminus \{x^0\} \; \exists y^* \in Y^* \setminus \{0\}, \; \alpha \in \mathbb{R} : \sup_{\xi \in \mathcal{U}} y^*(f(\overline{x}, \xi)) > \alpha \geq \sup_{y \in f_{\mathcal{U}}(x^0) - C(x^0)} y^*(y).$$

Furthermore,

$$\sup_{y \in f_{\mathcal{U}}(x^0) - C(x^0)} y^*(y) = \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)) + \sup_{c \in -C(x^0)} y^*(c) = \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)).$$

To show that $y^* \in C^*(x^0)$, suppose that $y^* \notin C^*(x^0)$, which means that there are $c \in C(x^0)$ such that $y^*(c) < 0$. With (3.4), we obtain for any $\xi \in \mathcal{U}$, $c \in C(x^0)$ and some $\lambda \geq 0$

$$\alpha \geq y^*(f(x^0, \xi) - \lambda c) = y^*(f(x, \xi)) - \lambda y^*(c) \stackrel{\lambda \to +\infty}{\to} +\infty,$$

a contradiction. Altogether, we conclude with

$$\forall \overline{x} \in \mathcal{X} \setminus \{x^0\}\ \exists y^* \in C^*(x^0) : \sup_{\xi \in \mathcal{U}} y^*(f(\overline{x}, \xi)) > \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)),$$

which is equivalent to

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}\ \forall y^* \in C^*(x^0) : \sup_{\xi \in \mathcal{U}} y^*(f(\overline{x}, \xi)) \leq \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)).$$

□

For the following theorem, we define $\overline{C} := \cap_{x \in \mathcal{X}} C(x)$, and the dual cone of $\overline{C}$ will be denoted by $(\overline{C})^*$ (as opposed to $\overline{C}^*$, which was the intersection of all dual cones $C^*(x)$, $x \in \mathcal{X}$, in Theorem 3.1). We skip the proof of the following result, as it can be derived similarly to the proof the Theorem 3.2.

Theorem 3.3. *Consider the minimization problem ($P\mathcal{U}_{y^*}$), and assume that the objective space Y is locally convex. Suppose that x^0 is strictly robust nondominated and that the set $f_{\mathcal{U}}(x^0) - \overline{C}$ is closed and convex. Then there does not exist an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for every $y^* \in (\overline{C})^*$*

$$\sup_{\xi \in \mathcal{U}} y^*(f(\overline{x}, \xi)) \leq \sup_{\xi \in \mathcal{U}} y^*(f(x^0, \xi)).$$

Using the above scalarization results, we are now able to formulate a first algorithm for finding strictly (·, weakly) robust minimal and nondominated solutions of $P(\mathcal{U})$. In the following, we assume that $\bigcap_{x \in \mathcal{X}} C^*(x) \neq \{0\}$. For brevity, solutions that are strictly (weakly, ·) robust minimal *and* nondominated will be called **strictly (weakly, ·) robust**.

Algorithm 1 for deriving strictly (weakly, ·) robust solutions to $P(\mathcal{U})$

Input: Uncertain multiobjective problem $P(\mathcal{U})$, solution sets $\text{Opt}_C = \text{Opt}_{\text{int } C} = \text{Opt}_{C\setminus\{0\}} = \emptyset$.

Step 1: Compute the set $C := \bigcap_{x \in X} C^*(x)$. Set $\overline{C} := C \setminus \{0\}$. Denote $\overline{C}^{\#} = \cap_{x \in \mathcal{X}} C^{\#}$.

Step 2: If $\overline{C} = \emptyset$: **STOP. Output:** Set of strictly robust solutions Opt_C, set of weakly robust solutions $\text{Opt}_{\text{int } C}$, set of robust solutions $\text{Opt}_{C\setminus\{0\}}$.

Step 3: Choose $y^* \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*\}$.

Step 4: Find an optimal solution x^0 of $(P\mathcal{U}_{y^*})$.

a): If x^0 is a unique optimal solution of $(P\mathcal{U}_{y^*})$, then

$$\text{Opt}_C := \text{Opt}_C \cup \{x^0\}.$$

b): If $\max_{\xi \in \mathcal{U}}\ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$, then

$$\text{Opt}_{\text{int } C} := \text{Opt}_{\text{int } C} \cup \{x^0\}.$$

c): If $\max_{\xi \in \mathcal{U}}\ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$ and $y^* \in \overline{C}^{\#}$, then

$$\text{Opt}_{C\setminus\{0\}} := \text{Opt}_{C\setminus\{0\}} \cup \{x^0\}.$$

Step 5: Go to Step 2.

A sufficient condition for the existence of an optimal solution of $(P\mathcal{U}_{y^*})$ in Step 4 of the preceding algorithm is given by the theorem of Weierstrass: If $\mathcal{U}$ is compact and $f(x, \cdot)$ is continuous in $\xi \in \mathcal{U}$, then an optimal solution of $(P\mathcal{U}_{y^*})$ exists. Furthermore, we present an interactive algorithm for finding a single robust minimal solution to the uncertain multiobjective optimization problem $P(\mathcal{U})$. This algorithm uses the input of the decision maker, who either accepts the calculated solution or not. In the interactive procedure in

Algorithm 2 we use a surrogate one-parametric optimization problem. Therefore, a systematic generation of solutions is possible.

Algorithm 2 for deriving a single accepted strictly (weakly, ·) robust solution to $P(\mathcal{U})$:

Input: Uncertain multiobjective problem $P(\mathcal{U})$.

Step 1: Compute the set $C := \bigcap_{x\in X} C^*(x)$. Set $\overline{C} := C\setminus\{0\}$. Denote $\overline{C}^{\#} = \cap_{x\in\mathcal{X}} C^{\#}$.

Step 2: Choose $y^* \in \overline{C}$.

Step 3: Find an optimal solution x^0 to $(P\mathcal{U}_{y^*})$.

- **a):** If x^0 is a unique optimal solution of $(P\mathcal{U}_{y^*})$, then x^0 is strictly robust robust for $P(\mathcal{U})$.
- **b):** If $\max_{\xi\in\mathcal{U}}\ y^*\circ f(x,\xi)$ exists for all $x\in\mathcal{X}$, then x^0 is weakly robust for $P(\mathcal{U})$.
- **c):** If $\max_{\xi\in\mathcal{U}}\ y^*\circ f(x,\xi)$ exists for all $x\in\mathcal{X}$ and $y^*\in\overline{C}^{\#}$, then x^0 is robust for $P(\mathcal{U})$.

If x^0 is accepted by the decision-maker: **STOP. Output:** x^0. Otherwise, go to Step 4.

Step 4: Set $l := 0$, $t^0 := 0$. Choose $\hat{y}^* \in \overline{C}$, such that $\bar{y}^* \neq y^*$. Go to Step 5.

Step 5: Choose t^{l+1} with $0 \le t^l < t^{l+1} \le 1$ and compute an optimal solution x^{l+1} to

$$(P(\mathcal{U})_{\hat{y}^j+t^{l+1}(\hat{y}^{j+1}-y^j)}) \qquad \min_{x\in\mathcal{X}}\sup_{\xi\in\mathcal{U}}\ \left(\hat{y}^j + t^{l+1}(\hat{y}^{j+1}-y^j)\right)\circ f(x,\xi).$$

Let $y^* = \hat{y}^j + t^{l+1}(\hat{y}^{j+1}-y^j)$.

- **a):** If x^{l+1} is a unique optimal solution of $(P\mathcal{U}_{y^*})$, then x^0 is strictly robust robust for $P(\mathcal{U})$.
- **b):** If $\max_{\xi\in\mathcal{U}}\ y^*\circ f(x,\xi)$ exists for all $x\in\mathcal{X}$, then x^{l+1} is weakly robust for $P(\mathcal{U})$.
- **c):** If $\max_{\xi\in\mathcal{U}}\ y^*\circ f(x,\xi)$ exists for all $x\in\mathcal{X}$ and $y^*\in\overline{C}^{\#}$, then x^{l+1} is robust for $P(\mathcal{U})$.

If no optimal solution to $(P(\mathcal{U})_{\bar{y}^j+t^{l+1}(\bar{y}^{j+1}-\bar{y}^j)})$ can be found for all $t > t^l$, go to Step 2. Otherwise, go to Step 6.

Step 6: If x^{l+1} is accepted by the decision maker: **STOP. Output:** x^{l+1}. Otherwise, go to Step 7.

Step 7: If $t^{l+1} = 1$, then set $l := l+1$ and go to Step 4. Otherwise, set $l := l+1$ and go to Step 5.

4. Conclusions

This paper explores robust approaches to uncertain vector-valued optimization problems, where the ordering is equipped with a variable domination structure. In robust optimization, one traditionally hedges against perturbations in the worst-case scenarios. Robust solutions are then immunized against perturbations, and thus this approach is applicable if a decision maker acts risk averse. In uncertain vector optimization, this situation can be modeled by using the upper set less relation. By allowing the domination structure to vary depending on the risk-averseness of the practitioner, it is now possible to fully model his / her preferences in order to obtain robust solutions. This paper introduces this concept, gives some scalarization results along with new numerical algorithms that generate robust solutions. Our work leaves many avenues for future research. For instance, a deeper analysis of our proposed algorithms for finding robust solutions of uncertain vector optimization problems with variable domination structures is necessary. Moreover, it would be interesting to investigate different set relations in relation with robustness in vector optimization with variable domination structures.

REFERENCES

[1] Bao, T. Q. and Mordukhovich, B. S., *Necessary nondomination conditions in sets and vector optimization with variable ordering structures*, J. Optim. Theory Appl., **162** (2014), No. 2, 350–370

[2] Bao, T. Q., Mordukhovich, B. S. and Soubeyran, A., *Variational analysis in psychological modeling*, J. Optim. Theory Appl., **164** (2015), No. 1, 290-315

[3] Durea, M., Strugariu, R., and Tammer, Chr., *On set-valued optimization problems with variable ordering structure*, J. Global Optim., **61** (2015), No. 4, 745–767

[4] Ehrgott, M., Ide, J., and Schöbel, A., *Minmax robustness for multi-objective optimization problems*, Eur. J. Oper. Res., **239** (2014), No. 1, 17–31

[5] Eichfelder, G., *Optimal elements in vector optimization with a variable ordering structure*, J. Optim. Theory Appl., **151** (2011), No. 2, 217–240

[6] Eichfelder G., *Variable Ordering Structures in Vector Optimization*, Springer, 2014

[7] Eichfelder, G., Bao, T. Q., Soleimani, B. and Tammer, Chr., *Ekeland's variational principle for vector optimization with variable ordering structure*, Technical report, Preprint-Series of the Institute of Mathematics, Ilmenau University of Technology, Germany, 2014

[8] Eichfelder, G. and Pielecka, M., *Set approach for set optimization with variable ordering structures part i: Set relations and relationship to vector approach*, J. Optim. Theory Appl., **171** (2016), No. 3, 931–946

[9] Eichfelder, G. and Pielecka, M., *Set approach for set optimization with variable ordering structures part ii: Scalarization approaches*, J. Optim. Theory Appl., **171** (2016), No. 3, 947–963

[10] Fliege, J. and Werner, R., *Robust multiobjective optimization & applications in portfolio optimization*, Eur. J. Oper. Res., **234** (2014), No. 2, 422–433

[11] Ide, J., and Köbis, E., *Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations*, Math. Method Oper. Res., **80** (2014), No. 1, 99–127

[12] Ide, J., Köbis, E., Kuroiwa, D., Schöbel, A. and Tammer, Chr., *The relationship between multicriteria robustness concepts and set-valued optimization*, Fixed Point Theory Appl., DOI: 10.1186/1687-1812-2014-83, (2014)

[13] Jahn, J., *Vector Optimization - Introduction, Theory, and Extensions*, Springer, Berlin, Heidelberg, 2011

[14] Jahn, J., *Vectorization in set optimization*, J. Optim. Theory Appl., **167** (2015), No. 3, 783–795

[15] E. Köbis. *Set optimization by means of variable order relations*, Optimization, DOI: 10.1080/02331934.2016.1172226, (2016)

[16] Kuroiwa, D., Some duality theorems of set-valued optimization with natural criteria, In *Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis*. World Scientific, 221-228, 1999

[17] Kuroiwa, D., *The natural criteria in set-valued optimization*, Sūrikaisekikenkyūsho Kōkyūroku, (1031):85–90, Research on nonlinear analysis and convex analysis, Kyoto, 1997

[18] Kuroiwa, D. and Lee, G. M., *On robust multiobjective optimization*, Vietnam J. Math., **40** (2012) No. 2-3, 305–317

[19] Yu, P. L., *Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives*, J. Optim. Theory Appl., **14** (1974), 319–377

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