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A forward-backward iterative method for zero points of sum of two accretive operators

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ABSTRACT. In this paper, we study a zero point problem of the sum of two accretive operators based on a viscosity forward-backward iterative algorithm with computational errors. Strong convergence results are established in the framework of q-uniformly smooth Banach spaces. We also apply the strong convergence results to solve variational inequality problems, convex minimization problems and fixed point problems.

1. INTRODUCTION

Fixed point theory of nonexpansive mappings has been applied to the variational inclusion problem of finding a point $z \in H$ such that $0 \in Az$, where H is a Hilbert space and $A : H \to 2^H$ is a maximal monotone operator. One of the most popular techniques for solving the inclusion problems goes back to the work of Browder [3]. The basic idea is to reduce the above inclusion problem to a fixed point problem of the resolvent operator $J_r^A := (I + rA)^{-1} : H \to 2^H$, where r is a positive real number. If A has some monotonicity condition, then the resolvent of A is firmly nonexpnsive, that is,

$$\langle J_r^A x - J_r^A y, x - y\rangle \geq \|J_r^A x - J_r^A y\|^2, \quad \forall x, y \in H.$$

The property of the resolvent ensures that the Picard iterative algorithm $x_{n+1} = J_r^A x_n$, converges weakly to a fixed point of J_r^A, which is necessarily a zero point of A.

Rockafellar [14] introduced the following proximal point algorithm: For any initial point $x_0 \in H$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}^A (x_n + e_n), \quad \forall n \geq 0,$$

where $\{r_n\}$ is a sequence of positive real numbers and $\{e_n\}$ is an error sequence in H. He proved the weak convergence of sequence $\{x_n\}$ under appropriate restrictions imposed on $\{r_n\}$. To find the strong convergence, Bruck [5] proposed the following algorithm: For any initial point $x_0 \in H$ and fixed point $u \in H$, $x_{n+1} = J_{r_n}^A u$, $\forall n \geq 0$. He proved the strong convergence of sequence $\{x_n\}$ under appropriate restrictions imposed on $\{r_n\}$. In the case of $A = S + T$, where S and T are monotone operators, splitting algorithms have recently been investigated for solving the inclusion problem; see [2, 6, 9, 11, 17] and the references therein. These algorithms in the framework of Hilbert spaces are based on the good properties of resolvent operators, but these properties are not available in the framework of general Banach spaces; see, for example, [13] and the references therein.

In this paper, we are interested in finding iteratively a zero point x^* of the sum of two accretive operators A and B, that is,

$$x^* \in (A + B)^{-1}(0). \tag{1.1}$$

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This inclusion problem, which includes variational inequality problems, equilibrium problems, complementary problems, minimization problems, fixed point problems as special cases, is quite general. Indeed, many nonlinear problems arising in applied areas such as signal processing, image recovery, and machine learning can be mathematically modeled as (1.1); see [7] and the references therein.

In this paper, we propose a viscosity forward-backward iterative algorithm with computational errors for zero points of the sum of two accretive operators. Strong convergence of the proposed algorithm is obtained in the framework of q-uniformly smooth Banach spaces. The organization of this paper is as follows. In Section 2, we provide some necessary mathematical preliminaries. In Section 3, the main strong convergence theorems are established in the framework of q-uniformly smooth Banach spaces without any compact restriction. Numerical experiments are also provided to support the main results.

2. Preliminaries

Throughout the paper, unless otherwise specified, we always assume that E is a Banach space with its dual E^*. Let $\mathfrak{J}_q : E \to 2^{E^*}$ be the generalized duality mapping defined by

$$\mathfrak{J}_q(x) := \left\{ y \in E^* : \|y\| = \|x\|^{q-1}, \langle y, x \rangle = \|x\|^q \right\}, \quad \forall x \in E.$$

Let $\varepsilon^E : [0,\infty) \to [0,\infty)$ be defined by

$$\varepsilon^E(t) = \sup\left\{ \frac{\|x+y\| - \|x-y\| - 2}{2} : t \geq \|y\|, \|x\| = 1 \right\}.$$

Then ε^E is said to be modulus of smoothness of E. Let $q > 1$ be some real number. E is said to be q-uniformly smooth if there exists a fixed constant $\kappa > 0$ such that $\varepsilon^E(t) \leq \kappa t^q$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth; see [16] and the references therein.

A mapping $T : E \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in E$. Let D be a nonempty subset of set C. A mapping $Q_D : C \to D$ is said to be a contraction if $Q_D^2 = Q_D$. It is called sunny if for each $x \in C$ and $t \in (0,1)$, we have $Q_D x = Q_D\big(tx + (1-t)Q_D x\big)$. Q_D is said to be a sunny nonexpansive retractction if Q_D is sunny, nonexpansive and a contraction. D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. It is known that Q_C is sunny nonexpansive if and only if $\langle x - Q_C x, \mathfrak{J}_q(y - Q_C x)\rangle \leq 0$, $\forall x \in E, y \in C$.

From now onward, we always assume that E is a q-uniformly smooth Banach space. Let I denote the identity operator on E. An operator $A \subset E \times E$ with range $R(A) = \cup\{Az : z \in D(A)\}$ and domain $D(A) = \{z \in E : Az \neq \emptyset\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, such that $\langle y_1 - y_2, \mathfrak{J}_q(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to be m-accretive if and only if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. Throughout this paper, we denote by $A^{-1}(0)$ the set of zero points of A. For an m-accretive operator A, we can define a nonexpansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A. An operator $A : C \to E$ is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, \mathfrak{J}_q(x - y)\rangle \geq \alpha \|Ax - Ay\|^q, \quad \forall x, y \in C.$$

In addition, we also need the following lemmas which play an important role in this paper.

Lemma 2.1. [16] *Let E be a q-uniformly smooth Banach space. Then*

$$q\langle y, \mathfrak{J}_q(x)\rangle + \|x\|^q + K_q\|y\|^q \geq \|x+y\|^q, \quad \forall x, y \in E,$$

where K_q is some fixed positive real constant.

Lemma 2.2. [8] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence of real numbers. Assume that

(a) $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$.
(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [1] *Let E be a Banach space and let A be an m-accretive operator on E. For $\mu_1 > 0$, $\mu_2 > 0$ and $x \in E$, we have*

$$J_{\mu_1}\Big(\Big(1 - \frac{\mu_1}{\mu_2}\Big)J_{\mu_2}x + \frac{\mu_1}{\mu_2}x\Big) = J_{\mu_2}x,$$

where $J_{\mu_2}^A = (I + \mu_2 A)^{-1}$ and $J_{\mu_1}^A = (I + \mu_1 A)^{-1}$.

Lemma 2.4. [12] *Let E be a real uniformly smooth Banach space, C be a nonempty convex closed subset of E, $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set and $f : C \to C$ be a contractive mapping. For each $t \in (0,1)$, let x_t be the unique solution of equation $tf(x) + (1-t)Tx = x$. Then $\{x_t\}$ converges in norm to a fixed point $Q_{F(T)}f(\bar{x}) = \bar{x}$, where $Q_{F(T)}$ is the unique sunny nonexpansive retraction from C onto $F(T)$, as $t \to 0$.*

Lemma 2.5. [10] *For a real number $\kappa > 1$, we have*

$$a_1^{\kappa} + (\kappa - 1)a_2^{\frac{\kappa}{\kappa-1}} \geq a_1 a_2 \kappa,$$

for arbitrary positive real numbers a_1 and a_2.

Lemma 2.6. *Let E be a Banach space, C be a nonempty convex and closed subset of E, $B : E \to 2^E$ be an m-accretive operator and $A : C \to E$ be a single valued operator. Then*

$$(A+B)^{-1}(0) = F(J_a^A(I - aA)),$$

where $J_a^A(I - aA)$ is the resolvent of A for $a > 0$.

Proof. Notice that

$$r \in (A+B)^{-1}(0) \Leftrightarrow r - aAr \in r + aBr \Leftrightarrow r \in F(J_a^A(I - aA)).$$

This completes the proof. □

3. Main results

Now, we are in a position to give the main results of this paper.

Theorem 3.1. *Let E be a q-uniformly smooth Banach space with the constant K_q and C be a nonempty convex closed subset of E. Let $B : E \to 2^E$ be an m-accretive operator such that $D(B) \subset C$ and $A : C \to E$ be an α-inverse strongly accretive operator. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Let $f : C \to C$ be a κ-contraction mapping. Let $\{r_n\}$ be a positive number sequence and $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $J_{r_n}^B = (I + r_n B)^{-1}$ and $\{e_n\}$ be an error sequence in E. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}^B(y_n - r_n A y_n + e_n),$$

where $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$, $n \geq 0$. *Assume that* $\{\alpha_n\}$, $\{e_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(b) $0 < a \leq r_n \leq b < \left(\frac{q\alpha}{K_q}\right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two constants.*

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ *converges strongly to* $\bar{x} = Proj_{(A+B)^{-1}(0)} f(\bar{x})$, *where* $Proj_{(A+B)^{-1}(0)}$ *is the unique sunny nonexpansive retraction of* C *onto* $(A+B)^{-1}(0)$.

Proof. First, we show that the sequence $I - r_nA$ is nonexpansive. In view of Lemma 2.1, we find that

$$\begin{aligned}
&\|(I-r_nA)x-(I-r_nA)y\|^q \\
&\leq \|x-y\|^q + K_q r_n^q \|Ax-Ay\|^q - qr_n\langle Ax-Ay, \mathfrak{J}_q(x-y)\rangle \\
&\leq \|x-y\|^q + K_q r_n^q \|Ax-Ay\|^q - qr_n\alpha\|Ax-Ay\|^q \\
&= \|x-y\|^q - (\alpha q - K_q r_n^{q-1}) r_n \|Ax-Ay\|^q, \quad \forall x,y \in C.
\end{aligned}$$

By assumption (b), $I - r_nA$ is nonexpansive. Fix $p \in (A+B)^{-1}(0)$, we find that

$$\begin{aligned}
\|y_n - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|f(x_n) - p\| \\
&\leq \alpha_n\|f(p) - p\| + \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|(y_n - r_nAy_n + e_n) - p\| \\
&\leq \|e_n\| + \|(I-r_nA)p - (I-r_nA)y_n\| \\
&\leq \|e_n\| + \alpha_n\|f(p)-p\| + \big(1-\alpha_n(1-\kappa)\big)\|x_n - p\| \\
&\leq \|e_n\| + \max\left\{\frac{\|f(p)-p\|}{1-\kappa} + \|x_n - p\|\right\} \\
&\leq \|e_{n-1}\| + \|e_n\| + \max\left\{\frac{\|p-f(p)\|}{1-\kappa}, \|x_{n-1} - p\|\right\} \\
&\vdots \\
&\leq \sum_{i=0}^{n}\|e_i\| + \max\left\{\frac{\|p-f(p)\|}{1-\kappa}, \|x_0 - p\|\right\} \\
&\leq \sum_{i=0}^{\infty}\|e_i\| + \max\left\{\frac{\|p-f(p)\|}{1-\kappa}, \|x_0 - p\|\right\} < \infty.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is a bounded sequence, so is $\{y_n\}$. Notice that

$$\|y_n - y_{n-1}\| \leq |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\| + \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\|.$$

Putting $z_n = y_n - r_nAy_n + e_n$, we find that

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|e_n\| + \|e_{n-1}\| + \|y_n - y_{n-1}\| + \|r_n - r_{n-1}\|\|Ay_{n-1}\| \\
&\leq \|e_n\| + \|e_{n-1}\| + \big(1-\alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\| \\
&\quad + |r_n - r_{n-1}|\|Ay_{n-1}\|.
\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}z_n+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}z_n\right)-J_{r_{n-1}}z_{n-1}\right\| \\
&\le \left\|\left(1-\frac{r_{n-1}}{r_n}\right)(J_{r_n}z_n-z_{n-1})+\frac{r_{n-1}}{r_n}(z_n-z_{n-1})\right\| \\
&\le \left\|(z_n-z_{n-1})+\left(1-\frac{r_{n-1}}{r_n}\right)(J_{r_n}z_n-z_n)\right\| \\
&\le \frac{|r_n-r_{n-1}|}{a}\|J_{r_n}z_n-z_n\|+\|z_n-z_{n-1}\| \\
&\le f_n+\big(1-\alpha_n(1-\kappa)\big)\|x_n-x_{n-1}\|,
\end{aligned}$$

where

$$f_n=|\alpha_n-\alpha_{n-1}|\|x_{n-1}-f(x_{n-1})\|+|r_n-r_{n-1}|\left(\|Ay_{n-1}\|+\frac{\|z_n-J^B_{r_n}z_n\|}{a}\right)+\|e_n\|+\|e_{n-1}\|.$$

It follows from (a), (b) and (c) that $\sum_{n=1}^{\infty} f_n<\infty$. From Lemma 2.2, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. In view of $y_n-x_n=\alpha_n(f(x_n)-x_n)$, we find from the above that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Hence

(3.2) $$\lim_{n\to\infty}\|y_n-x_{n+1}\|=0.$$

Notice that

$$\begin{aligned}
\|J^B_{r_n}(y_n-r_nAy_n)-y_n\| &\le \|J^B_{r_n}(y_n-r_nAy_n)-J^B_{r_n}(y_n-r_nAy_n+e_n)\|+\|x_{n+1}-y_n\| \\
&\le \|e_n\|+\|x_{n+1}-y_n\|.
\end{aligned}$$

This implies from (3.2) that

(3.3) $$\lim_{n\to\infty}\|y_n-J^B_{r_n}(y_n-r_nAy_n)\|=0.$$

Notice that

$$\left\langle\frac{y_n-J^B_a(I-aA)y_n}{a}-\frac{y_n-J^B_{r_n}(I-r_nA)y_n}{r_n},\mathfrak{J}_q\big(J^B_a(I-aA)y_n-J^B_{r_n}(I-r_nA)y_n\big)\right\rangle\ge 0.$$

Hence, we find that

$$\begin{aligned}
&\|J^B_a(I-aA)y_n-J^B_{r_n}(I-r_nA)y_n\|^q \\
&\le \frac{r_n-a}{r_n}\left\langle y_n-J^B_{r_n}(I-r_nA)y_n,\mathfrak{J}_q\left(J^B_a(I-aA)y_n-J_{r_n}(I-r_nA)y_n\right)\right\rangle \\
&\le \|y_n-J^B_{r_n}(I-r_nA)y_n\|\|J^B_a(I-aA)y_n-J^B_{r_n}(I-r_nA)y_n\|^{q-1}.
\end{aligned}$$

This implies that

$$\|J^B_a(I-aA)y_n-J^B_{r_n}(I-r_nA)y_n\|\le\|y_n-J^B_{r_n}(I-r_nA)y_n\|.$$

It follows that

$$\begin{aligned}
\|J^B_a(I-aA)y_n-y_n\| &\le \|J^B_a(I-aA)y_n-J_{r_n}(I-r_nA)y_n\|+\|J^B_{r_n}(I-r_nA)y_n-y_n\| \\
&\le 2\|J^B_{r_n}(I-r_nA)y_n-y_n\|.
\end{aligned}$$

From (3.3), we have

(3.4) $$\lim_{n\to\infty}\|J^B_a(y_n-aAy_n)-y_n\|=0.$$

Notice that $J^B_a(I-aA)$ is a nonexpansive mapping. Put $\bar{x}=\lim_{t\to 0}x_t$, where x_t solves the fixed point equation $x_t=(1-t)J^B_a(I-aA)x_t+tf(x_t)$, $\forall t\in(0,1)$. Furthermore, one has $\bar{x}=$

$Proj_{(A+B)^{-1}(0)}f(\bar{x})$, where $Proj_{(A+B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto zero point $(A+B)^{-1}(0)$.

Now, we are in a position to claim that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle \le 0$. It follows that

$$\begin{aligned}\|x_t-y_n\|^q &\le t\langle f(x_t)-y_n,\mathfrak{J}_q(x_t-y_n)\rangle+(1-t)\langle J_a^B(I-aA)x_t-y_n,\mathfrak{J}_q(x_t-y_n)\rangle\\ &\le t\langle f(x_t)-x_t,\mathfrak{J}_q(x_t-y_n)\rangle+t\langle x_t-y_n,\mathfrak{J}_q(x_t-y_n)\rangle\\ &\quad+(1-t)\big(\langle J_a^B(I-aA)x_t-J_a^B(I-aA)y_n,\mathfrak{J}_q(x_t-y_n)\rangle+\langle J_a^B(I-aA)y_n-y_n,\mathfrak{J}_q(x_t-y_n)\rangle\big)\\ &\le t\langle f(x_t)-x_t,\mathfrak{J}_q(x_t-y_n)\rangle+\|x_t-y_n\|^q+(1-t)\|J_a^B(I-aA)y_n-y_n\|\|x_t-y_n\|^{q-1},\end{aligned}$$

which implies that

$$t\langle f(x_t)-x_t,\mathfrak{J}_q(y_n-x_t)\rangle \le \|J_a^B(I-aA)y_n-y_n\|\|x_t-y_n\|^{q-1}.$$

Fixing t and letting $n\to\infty$, we find from (3.4) that

$$\limsup_{n\to\infty}\langle f(x_t)-x_t,\mathfrak{J}_q(y_n-x_t)\rangle \le 0.$$

Since $\mathfrak{J}_q: E\to E^*$ is uniformly continuous on any bounded sets of E, which ensures that limits $\limsup_{n\to\infty}$ and $\limsup_{t\to 0}$ are interchangeable, we find that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle \le 0. \tag{3.5}$$

Finally, we prove $x_n\to\bar{x}$ as $n\to\infty$. In view of Lemma 2.5, we find that

$$\begin{aligned}\|y_n-\bar{x}\|^q &\le (1-\alpha_n)\langle x_n-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle+\alpha_n\langle f(x_n)-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle\\ &=\alpha_n\langle f(x_n)-f(\bar{x}),\mathfrak{J}_q(y_n-\bar{x})\rangle+(1-\alpha_n)\langle x_n-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle+\alpha_n\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle\\ &\le \alpha_n\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle+\big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|\|y_n-\bar{x}\|^{q-1}\\ &\le \big(1-\alpha_n(1-\kappa)\big)\left(\frac{1}{q}\|x_n-\bar{x}\|^q+\frac{q-1}{q}\|y_n-\bar{x}\|^q\right)+\alpha_n\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle.\end{aligned}$$

This implies that

$$\|y_n-\bar{x}\|^q \le q\alpha_n\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle+\big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|^q. \tag{3.6}$$

It also follows from Lemma 2.5 that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^q &\le \|J_{r_n}(y_n-r_nAy_n+e_n)-\bar{x}\|\|x_{n+1}-\bar{x}\|^{q-1}\\ &\le \|(I-r_nA)y_n-(I-r_nA)\bar{x}+e_n\|\|x_{n+1}-\bar{x}\|^{q-1}\\ &\le \|e_n\|\|x_{n+1}-\bar{x}\|^{q-1}+\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\|^{q-1}\\ &\le \|e_n\|\|x_{n+1}-\bar{x}\|^{q-1}+\frac{1}{q}\|y_n-\bar{x}\|^q+\frac{q-1}{q}\|x_{n+1}-\bar{x}\|^q.\end{aligned}$$

This implies from (3.6) that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^q &\le \|y_n-\bar{x}\|^q+q\|e_n\|\|x_{n+1}-\bar{x}\|^{q-1}\\ &\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|^q+q\alpha_n\langle f(\bar{x})-\bar{x},\mathfrak{J}_q(y_n-\bar{x})\rangle+q\|e_n\|\|x_{n+1}-\bar{x}\|^{q-1}.\end{aligned}$$

By using Lemma 2.2, we find that $\{x_n\}$ converges strongly to $\bar{x}$. □

From Theorem 3.1, we have the following result on a finite family of inverse strongly accretive operators.

Corollary 3.1. *Let E be a q-uniformly smooth Banach space with the constant K_q and let C be a nonempty closed and convex subset of E. Let $\mu \geq 1$ be some positive integer and $A_i : C \to E$ be an λ_i-inverse strongly accretive operator, for each $1 \leq i \leq \mu$. Let $B : E \to 2^E$ be an m-accretive operator such that $D(B) \subset C$. Assume that $(\sum_{i=1}^{\mu} A_i + B)^{-1}(0) \neq \emptyset$. Let $f : C \to C$ be a κ-contraction mapping. Let $\{r_n\}$ be a positive number sequence and $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $J_{r_n}^B = (I + r_n B)^{-1}$ and let $\{e_n\}$ be an error sequence in E. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}^B \left(y_n - r_n \sum_{i=1}^{\mu} A_i y_n + e_n \right),$$

where $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$, $\forall n \geq 0$. Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(b) $0 < a \leq r_n \leq b < \left(\frac{q \sum_{i=1}^{\mu} \lambda_i}{K_q} \right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two constants.*

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to $\bar{x} = Proj_{(\sum_{i=1}^{\mu} A_i + B)^{-1}(0)} f(\bar{x})$, where $Proj_{(\sum_{i=1}^{\mu} A_i + B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto $(\sum_{i=1}^{\mu} A_i + B)^{-1}(0)$.

Proof. Notice that $\sum_{i=1}^{\mu} A_i : C \to E$ is $\sum_{i=1}^{\mu} \lambda_i$-inverse strongly accretive. Indeed,

$$\begin{aligned}
&\left\langle \sum_{i=1}^{\mu} A_i x - \sum_{i=1}^{\mu} A_i y, \mathfrak{J}_q(x-y) \right\rangle \\
&= \langle A_1 x - A_1 y, \mathfrak{J}_q(x-y)\rangle + \langle A_2 x - A_2 y, \mathfrak{J}_q(x-y)\rangle + \cdots + \langle A_\mu x - A_\mu y, \mathfrak{J}_q(x-y)\rangle \\
&\geq \lambda_1 \|x-y\|^q + \lambda_2 \|x-y\|^q + \cdots + \lambda_\mu \|x-y\|^q \\
&\geq \sum_{i=1}^{\mu} \lambda_i \|x-y\|^q, \quad \forall x, y \in C.
\end{aligned}$$

We can get the desired conclusion from Theorem 3.1 easily. □

Next, we give a result on zero points of an *m*-accretive operator.

Corollary 3.2. *Let E be a q-uniformly smooth Banach space with the constant K_q and let C be a nonempty convex closed subset of E. Let $B : E \to 2^E$ be an m-accretive operator with a nonempty zero point set such that $D(B) \subset C$. Let $f : C \to C$ be a κ-contraction mapping. Let $\{r_n\}$ be a positive number sequence and $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $J_{r_n}^B = (I + r_n B)^{-1}$ and $\{e_n\}$ be an error sequence in E. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}^B \big((1-\alpha_n)x_n + \alpha_n f(x_n) + e_n \big).$$

Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(b) $0 < a \leq r_n$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a is a constant.*

(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to $\bar{x} = Proj_{B^{-1}(0)} f(\bar{x})$, where $Proj_{B^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto $B^{-1}(0)$.

In the framework of Hilbert spaces, the concept of monotonicity coincides with the concept of accretivity. From now onward, we always assume that C is a nonempty closed convex subset of a Hilbert space H. From Theorem 3.1, we have the following result immediately.

Corollary 3.3. *Let $A : C \to H$ be an α-inverse strongly monotone operator and $B : H \to 2^H$ be a maximal monotone operator such that $D(B) \subset C$. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Let $f : C \to C$ be a κ-contraction mapping. Let $\{r_n\}$ be a positive number sequence and $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $J_{r_n} = (I + r_n B)^{-1}$ and let $\{e_n\}$ be an error sequence in H. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J^B_{r_n}(y_n - r_n A y_n + e_n),$$

where $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$, $\forall n \geq 0$. Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.
(b) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two constants.*
(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to $\bar{x} = Proj_{(A+B)^{-1}(0)} f(\bar{x})$, where $Proj_{(A+B)^{-1}(0)}$ is the metric projection of C onto $(A+B)^{-1}(0)$.

Let $A : C \to H$ be a monotone operator. Recall that the classical variational inequality, denoted by $VI(C,A)$, is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in C. \tag{3.7}$$

One see that variational inequality (3.7) is equivalent to a fixed point problem. The element $u \in C$ is a solution of variational inequality (3.7) if and only if $u \in C$ satisfies equation $u = P_C(u - rAu)$, where $r > 0$ is a constant.

Let B^C be the indicator function of C, that is,

$$B^C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Since B^C is a proper lower and semicontinuous convex function on H, the subdifferential ∂B^C of B^C is maximal monotone. So, we can define the resolvent J_r of ∂B^C for $r > 0$, i.e., $J_r := (I + r\partial B^C)^{-1}$. Letting $x = J_r y$, we find that

$$\begin{aligned} y \in x + r\partial B^C x \quad &\Leftrightarrow \quad y \in rN_C x + x \\ &\Leftrightarrow \quad \langle y - x, v - x \rangle \leq 0, \forall v \in C \\ &\Leftrightarrow \quad P_C y = x, \end{aligned}$$

where P_C is the metric projection from H onto C and $N_C x := \{e \in H : \langle e, v-x \rangle, \forall v \in C\}$.

Corollary 3.4. *Let $A : C \to H$ be an α-inverse strongly monotone mapping. Assume that the solution set of $VI(C,A)$ is not empty. Let $f : C \to C$ be a κ-contraction mapping. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\{e_n\}$ be an error sequence in H and $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C(y_n - r_n A y_n + e_n),$$

where $y_n = (1-\alpha_n)x_n + \alpha_n f(x_n)$, $\forall n \geq 0$. Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(b) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two constants.*
(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to a point $\bar{x}$ which is a solution to $VI(C,A)$, where $\bar{x} = P_{VI(C,A)} f(\bar{x})$.

Proof. Putting $B = \partial B^C$ in Corollary 3.3, we find the desired conclusion immediately. □

Recall that a mapping $T : C \to E$ is said to be α-strictly pseudocontractive if there exits a constant $\alpha \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was first introduced by Browder and Petryshyn [4]. It is known that if T is α-strictly pseudocontractive, then $I - T$ is $\frac{1-\alpha}{2}$-inverse strongly monotone.

Corollary 3.5. *Let $T : C \to C$ be an α-strictly pseudocontractive mapping with a nonempty fixed point set, $f : C \to C$ be a fixed κ-contraction and $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\{e_n\}$ be an error sequence in H and $\{r_n\}$ be a positive real number sequence in $(0, 1 - \alpha)$. Let $x_0 \in C$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P_C((1 - r_n)y_n + r_n Ty_n + e_n),$$

where $y_n = (1 - \alpha_n)x_n + \alpha_n f(x_n)$, $\forall n \geq 0$. Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(b) $0 < a \leq r_n \leq b < 1 - \alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two constants.*
(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in F(T)$, where $\bar{x} = P_{F(T)} f(\bar{x})$.

Proof. Putting $A = I - T$, we find A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We also have $F(T) = VI(C,A)$ and $P_C(y_n - r_n Ay_n + e_n) = P_C((1 - r_n)y_n + r_n Ty_n + e_n)$. So, by using Corollary 3.3, we obtain the desired result. □

For a proper lower semicontinuous convex function $g : H \to (-\infty, \infty]$, the subdifferential mapping ∂g of g is defined by

$$\partial g(x) = \{x^* \in H : \langle y - x, x^* \rangle \leq g(y) - g(x), \forall y \in H\}, \quad \forall x \in H.$$

Rockafellar [15] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ if and only if $g(v) = \min_{x \in H} g(x)$.

Corollary 3.6. *Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial g)^{-1}(0)$ is not empty. Let $f : H \to H$ be a κ-contraction. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\{e_n\}$ be a sequence in H and let $\{r_n\}$ be a positive real number sequence. Let $x_0 \in H$ be an arbitrary initial and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \arg\min_{z \in H} \left\{ g(z) + \frac{\|z - y_n - e_n\|^2}{2r_n} \right\},$$

where $y_n = (1 - \alpha_n)x_n + \alpha_n f(x_n)$, $\forall n \geq 0$. Assume that $\{\alpha_n\}$, $\{e_n\}$ and $\{r_n\}$ satisfy the following conditions:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(b) $0 < a \leq r_n \leq b < \infty$, *where a and b are two constants.*
(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in (\partial f)^{-1}(0)$, where $\bar{x} = P_{(\partial f)^{-1}(0)} f(\bar{x})$.

Proof. Since $g : H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Noting that

$$x_{n+1} = \arg\min_{z \in H} \left\{ g(z) + \frac{\|z - y_n - e_n\|^2}{2r_n} \right\}$$

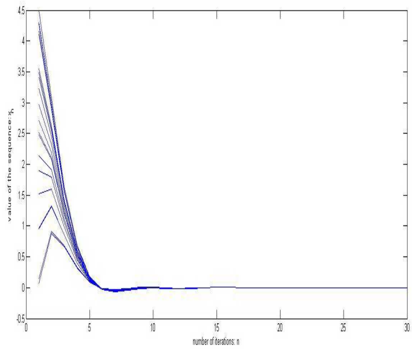
is equivalent to

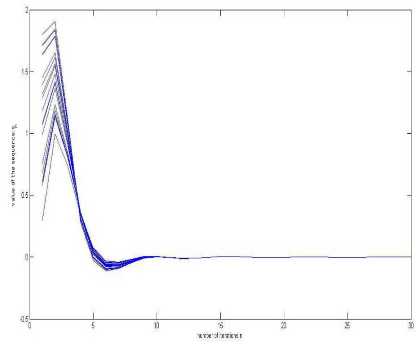
$$0 \in \partial g(x_{n+1}) + \frac{1}{r_n}(x_{n+1} - y_n - e_n).$$

It follows that

$$y_n \in x_{n+1} + r_n \partial g(x_{n+1}) - e_n.$$

Putting $A = 0$, we derive from Corollary 3.3 the desired conclusion immediately. □





In order to illustrate the effectiveness of the algorithm we proposed, we give the following numerical results using software Matlab 7.0. Put $\alpha_n = \frac{1}{n}$ and $e_n = \frac{\sin n}{n^2}$. Let E be the set of real numbers and $C = [0, 5]$. Let $A = 2x$ and let B be the subdifferential of the indicator function of C. Then the zero point of the sum A and B is 0. If we choose $x_0 \in C$ arbitrarily, then for 50 different initial values, we see all the results are convergent in Figure 1.

Let E be the set of real numbers and $C = [0, \pi]$. Let $A = x - \sin x$ and let B be the subdifferential of the indicator function of C. Then the zero point of the sum A and B is 0. If we choose $x_0 \in C$ arbitrarily, then for 20 different initial values, we see all the results are convergent in Figure 2.

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