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Cyclic permutations and crossing numbers of join products of symmetric graph of order six

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ABSTRACT. In the paper, we extend known results concerning crossing numbers for join of graphs of order six. We give the crossing number of the join product $G + D_n$, where the graph G consists of two leaves incident with two opposite vertices of one 4-cycle, and D_n consists on n isolated vertices. The proof is done with the help of software that generates all cyclic permutations for a given number k, and creates a new graph COG for a calculating the distances between all $(k-1)!$ vertices of the graph. Finally, by adding new edges to the graph G, we are able to obtain the crossing number of the join product with the discrete graph D_n for two other graphs. The methods used in the paper are new, and they are based on combinatorial properties of cyclic permutations.

1. INTRODUCTION

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *crossing number* $\mathrm{cr}(G)$ of the graph G is defined as the minimum possible number of edge crossings in a drawing of G in the plane. A drawing with the minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex a crossing. Moreover, no three edges cross in a point. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$.

Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

In the paper, some proofs are based on the Kleitman's result on crossing numbers of the complete bipartite graphs [4]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \le 6.$$

The purpose of this article is to extend the known results in [2], [5], [6], [7], [8], [9], [10], and [11] for another two graphs. The methods used in the paper are new, and they

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are based on combinatorial properties of cyclic permutations. The similar methods were partially used first time in the papers [3], and [10]. In [2], and [11], the properties of cyclic permutations are also verified by the help of software. According to our opinion the methods used in [5], [8], and [9], do not allow to establish the crossing number of the join product $G + D_n$. Let G be the graph consisting of one 4-cycle and of two leaves incident with two opposite vertices of the 4-cycle. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G+D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \le i \le n$, denote the subgraph induced by the six edges incident with the vertex t_i. Thus, $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$G + D_n = G \cup K_{6,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1.1}$$

2. Cyclic permutations and configurations

Let D be a good drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [3]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_it_5, and t_iv_6. We emphasize that a rotation is a cyclic permutation. Hence, for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, every subgraph $T^i \cup T^j$ of the graph $G + D_n$ is isomorphic with the graph $K_{6,2}$. In the paper, we will deal with the minimal necessary number of crossings among edges of a subgraph isomorphic with $K_{6,2}$ in a drawing of $G + D_n$ in which no edge of $K_{6,2}$ crosses G, i.e. with the minimum of necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced by the drawing D of the graph $G + D_n$ depending on the rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$. Two vertices t_i and t_j of $G + D_n$ are *antipodal* in a drawing of $G+D_n$ if the subdrawing of $T^i \cup T^j$ has no crossings. A drawing is *antipodal-free* if it has no antipodal vertices.

As the complete bipartite graph $K_{6,n}$ is a subgraph of $G + D_n$, let us discuss some properties of crossings among edges of its subgraph $K_{6,2}$. Let D be a good drawing of the graph $K_{6,n}$. Woodall [12] proved that in the subdrawing of $T^i \cup T^j \cong K_{6,2}$ induced by D, $\mathrm{cr}_D(T^i, T^j) \ge 6$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$, then $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \le \mathrm{cr}_D(T^i, T^j)$. We will separate the subgraphs T^i, $i = 1, \dots, n$, of the graph $G + D_n$ into three subsets depending on how many considered T^i crosses the edges of G in D. Let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$, for $i \in \{1, 2, \dots, n\}$. Hence, every other subgraph T^i crosses G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given drawing of G, any F^i is exactly represented by $\mathrm{rot}_D(t_i)$. By $D(F^i)$ we will understand subdrawing of F^i induced by D. The list with the short names of $6!/6 = 120$ cyclic permutations of six elements can be generated by the algorithm, see [1]. They are collected in Table 1.

Name	Cyclic p.	Name	Cyclic p.	Name	Cyclic p.	Name	Cyclic p.
$P_1 \to$	(1 2 3 4 5 6)	$P_{31} \to$	(1 2 3 6 4 5)	$P_{61} \to$	(1 2 5 6 3 4)	$P_{91} \to$	(1 4 5 6 2 3)
$P_2 \to$	(1 3 2 4 5 6)	$P_{32} \to$	(1 3 2 6 4 5)	$P_{62} \to$	(1 5 2 6 3 4)	$P_{92} \to$	(1 5 4 6 2 3)
$P_3 \to$	(1 2 4 3 5 6)	$P_{33} \to$	(1 2 6 3 4 5)	$P_{63} \to$	(1 2 6 5 3 4)	$P_{93} \to$	(1 4 6 5 2 3)
$P_4 \to$	(1 4 2 3 5 6)	$P_{34} \to$	(1 6 2 3 4 5)	$P_{64} \to$	(1 6 2 5 3 4)	$P_{94} \to$	(1 6 4 5 2 3)
$P_5 \to$	(1 3 4 2 5 6)	$P_{35} \to$	(1 3 6 2 4 5)	$P_{65} \to$	(1 5 6 2 3 4)	$P_{95} \to$	(1 5 6 4 2 3)
$P_6 \to$	(1 4 3 2 5 6)	$P_{36} \to$	(1 6 3 2 4 5)	$P_{66} \to$	(1 6 5 2 3 4)	$P_{96} \to$	(1 6 5 4 2 3)
$P_7 \to$	(1 2 3 5 4 6)	$P_{37} \to$	(1 2 4 6 3 5)	$P_{67} \to$	(1 3 5 6 2 4)	$P_{97} \to$	(1 3 4 5 6 2)
$P_8 \to$	(1 3 2 5 4 6)	$P_{38} \to$	(1 4 2 6 3 5)	$P_{68} \to$	(1 5 3 6 2 4)	$P_{98} \to$	(1 4 3 5 6 2)
$P_9 \to$	(1 2 5 3 4 6)	$P_{39} \to$	(1 2 6 4 3 5)	$P_{69} \to$	(1 3 6 5 2 4)	$P_{99} \to$	(1 3 5 4 6 2)
$P_{10} \to$	(1 5 2 3 4 6)	$P_{40} \to$	(1 6 2 4 3 5)	$P_{70} \to$	(1 6 3 5 2 4)	$P_{100} \to$	(1 5 3 4 6 2)
$P_{11} \to$	(1 3 5 2 4 6)	$P_{41} \to$	(1 4 6 2 3 5)	$P_{71} \to$	(1 5 6 3 2 4)	$P_{101} \to$	(1 4 5 3 6 2)
$P_{12} \to$	(1 5 3 2 4 6)	$P_{42} \to$	(1 6 4 2 3 5)	$P_{72} \to$	(1 6 5 3 2 4)	$P_{102} \to$	(1 5 4 3 6 2)
$P_{13} \to$	(1 2 4 5 3 6)	$P_{43} \to$	(1 3 4 6 2 5)	$P_{73} \to$	(1 2 4 5 6 3)	$P_{103} \to$	(1 3 4 6 5 2)
$P_{14} \to$	(1 4 2 5 3 6)	$P_{44} \to$	(1 4 3 6 2 5)	$P_{74} \to$	(1 4 2 5 6 3)	$P_{104} \to$	(1 4 3 6 5 2)
$P_{15} \to$	(1 2 5 4 3 6)	$P_{45} \to$	(1 3 6 4 2 5)	$P_{75} \to$	(1 2 5 4 6 3)	$P_{105} \to$	(1 3 6 4 5 2)
$P_{16} \to$	(1 5 2 4 3 6)	$P_{46} \to$	(1 6 3 4 2 5)	$P_{76} \to$	(1 5 2 4 6 3)	$P_{106} \to$	(1 6 3 4 5 2)
$P_{17} \to$	(1 4 5 2 3 6)	$P_{47} \to$	(1 4 6 3 2 5)	$P_{77} \to$	(1 4 5 2 6 3)	$P_{107} \to$	(1 4 6 3 5 2)
$P_{18} \to$	(1 5 4 2 3 6)	$P_{48} \to$	(1 6 4 3 2 5)	$P_{78} \to$	(1 5 4 2 6 3)	$P_{108} \to$	(1 6 4 3 5 2)
$P_{19} \to$	(1 3 4 5 2 6)	$P_{49} \to$	(1 2 3 5 6 4)	$P_{79} \to$	(1 2 4 6 5 3)	$P_{109} \to$	(1 3 5 6 4 2)
$P_{20} \to$	(1 4 3 5 2 6)	$P_{50} \to$	(1 3 2 5 6 4)	$P_{80} \to$	(1 4 2 6 5 3)	$P_{110} \to$	(1 5 3 6 4 2)
$P_{21} \to$	(1 3 5 4 2 6)	$P_{51} \to$	(1 2 5 3 6 4)	$P_{81} \to$	(1 2 6 4 5 3)	$P_{111} \to$	(1 3 6 5 4 2)
$P_{22} \to$	(1 5 3 4 2 6)	$P_{52} \to$	(1 5 2 3 6 4)	$P_{82} \to$	(1 6 2 4 5 3)	$P_{112} \to$	(1 6 3 5 4 2)
$P_{23} \to$	(1 4 5 3 2 6)	$P_{53} \to$	(1 3 5 2 6 4)	$P_{83} \to$	(1 4 6 2 5 3)	$P_{113} \to$	(1 5 6 3 4 2)
$P_{24} \to$	(1 5 4 3 2 6)	$P_{54} \to$	(1 5 3 2 6 4)	$P_{84} \to$	(1 6 4 2 5 3)	$P_{114} \to$	(1 6 5 3 4 2)
$P_{25} \to$	(1 2 3 4 6 5)	$P_{55} \to$	(1 2 3 6 5 4)	$P_{85} \to$	(1 2 5 6 4 3)	$P_{115} \to$	(1 4 5 6 3 2)
$P_{26} \to$	(1 3 2 4 6 5)	$P_{56} \to$	(1 3 2 6 5 4)	$P_{86} \to$	(1 5 2 6 4 3)	$P_{116} \to$	(1 5 4 6 3 2)
$P_{27} \to$	(1 2 4 3 6 5)	$P_{57} \to$	(1 2 6 3 5 4)	$P_{87} \to$	(1 2 6 5 4 3)	$P_{117} \to$	(1 4 6 5 3 2)
$P_{28} \to$	(1 4 2 3 6 5)	$P_{58} \to$	(1 6 2 3 5 4)	$P_{88} \to$	(1 6 2 5 4 3)	$P_{118} \to$	(1 6 4 5 3 2)
$P_{29} \to$	(1 3 4 2 6 5)	$P_{59} \to$	(1 3 6 2 5 4)	$P_{89} \to$	(1 5 6 2 4 3)	$P_{119} \to$	(1 5 6 4 3 2)
$P_{30} \to$	(1 4 3 2 6 5)	$P_{60} \to$	(1 6 3 2 5 4)	$P_{90} \to$	(1 6 5 2 4 3)	$P_{120} \to$	(1 6 5 4 3 2)

TABLE 1. Names of Cyclic Permutations of 6-elements set.

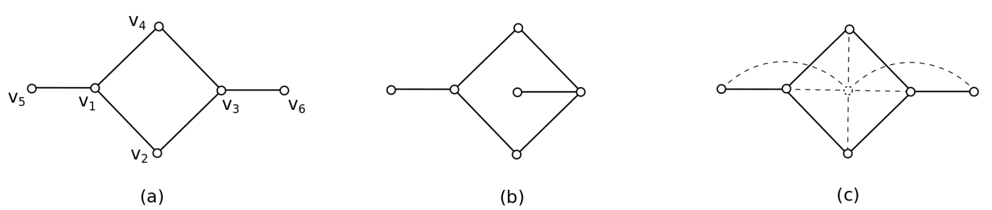

FIGURE 1. Drawings of the graph G with $\mathrm{cr}_D(G) = 0$ and of $G + D_2$.

In the paper, we will dealt with only drawings of the graph G with a possibility of an existence of a subgraph $T^i \in R_D$ because of mentioned arguments in the proof of the main Theorem 3.1. Assume a good drawing D of the graph $G + D_n$ in which the edges of G do not cross each other. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Fig. 1(a).

Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Since there is only one subdrawing of $F^i \setminus \{v_1, v_3\}$ represented by the rotation (2546), we have four possibilities how to obtain the subdrawing of F^i depending on in which region the edges t_iv_1 and t_iv_3 are placed. Thus, there are four different possible configurations of the subgraph F^i denoted as A_1, A_2, A_3, and A_4, i.e. $\mathrm{rot}_D(t_i) = A_j$ for $j = 1, 2, 3, 4$. As for our considerations does not play role which of the regions is unbounded, assume the drawings shown in Fig. 2. In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. Thus, the configurations A_1, A_2, A_3, and A_4 are represented by the cyclic permutations $P_{116} = (154632)$, $P_{44} = (143625)$, $P_{102} = (154362)$, and $P_{47} = (146325)$, respectively. In a fixed drawing of the graph $G + D_n$, some configurations from $\mathcal{M}$ do not must appear. We denote by $\mathcal{M}_D$ the set of all configurations that exist in the drawing D belonging to the set $\mathcal{M} = \{A_1, A_2, A_3, A_4\}$. The unique drawing of the subgraph F^i contains six regions with the vertex t_i on their boundaries, e.g. if F^i has the configuration A_1, then let us denote these six regions by $\omega_{1,2}$, $\omega_{2,3}$, $\omega_{3,6}$, $\omega_{3,6,4}$, $\omega_{1,4,5}$, and $\omega_{1,5}$ depending on which of vertices of G are located on the boundary of the corresponding region. We will shortly write $t_j \in \omega_{x,y}$, if a vertex t_j is placed in the region $\omega_{x,y}$.

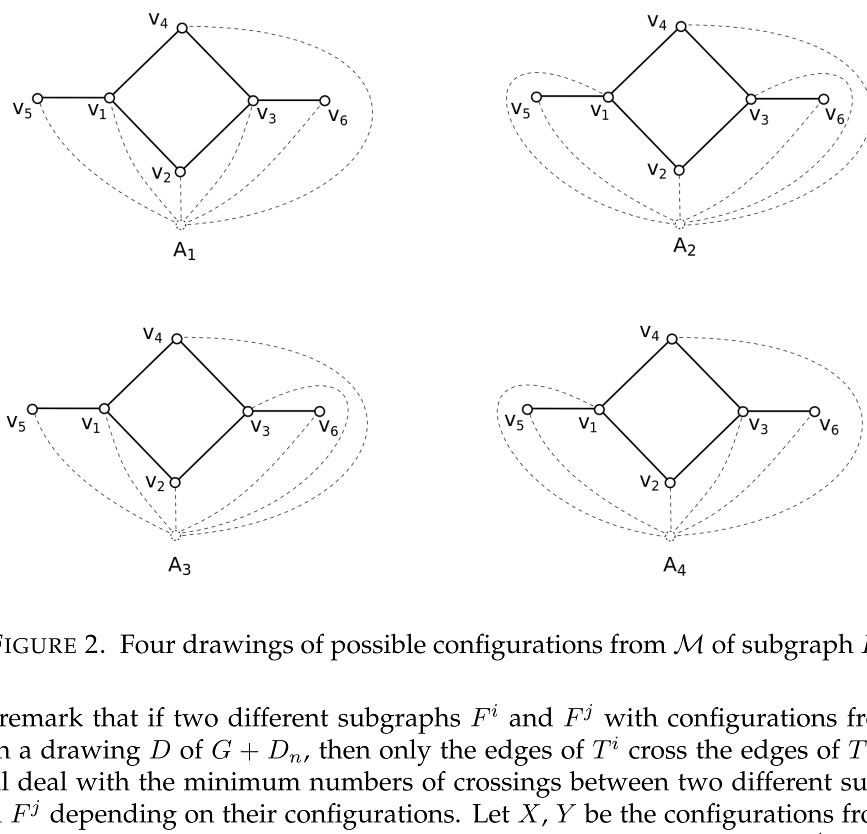

FIGURE 2. Four drawings of possible configurations from $\mathcal{M}$ of subgraph F^i.

We remark that if two different subgraphs F^i and F^j with configurations from $\mathcal{M}_D$ cross in a drawing D of $G + D_n$, then only the edges of T^i cross the edges of T^j. Thus, we will deal with the minimum numbers of crossings between two different subgraphs F^i and F^j depending on their configurations. Let X, Y be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(X, Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X, Y) = \min\{\mathrm{cr}_D(X, Y)\}$ over all good drawings of the graph $G + D_n$ with $X, Y \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(X, Y)$ for all pairs $X, Y \in \mathcal{M}$. In the next statements we are able to use the possibilities of the algorithm of the cyclic permutations of 6-elements set, see [1]. By $\overline{P_i}$ we will understand the inverse cyclic permutation to the permutation P_i, for

$i = 1, \dots, 120$. Woodall [12] defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (i. e. an ordered set with 6 elements). Hence, if $d_{COG}(\text{"rot}_D(t_i)\text{"}, \text{"rot}_D(t_j)\text{"})$ denotes the distance between two vertices correspond to the cyclic permutations $\text{rot}_D(t_i)$ and $\text{rot}_D(t_j)$ in the graph COG, then

$$d_{COG}(\text{"rot}_D(t_i)\text{"}, \text{"}\overline{\text{rot}_D(t_j)}\text{"}) = Q(\text{rot}_D(t_i), \text{rot}_D(t_j)) \leq \text{cr}_D(T^i, T^j) \tag{2.2}$$

for any two different subgraphs T^i and T^j.

Let us show mentioned lower-bounds of number of crossing of two configurations. The configurations A_1 and A_2 are represented by the cyclic permutations $P_{116} = (154632)$ and $P_{44} = (143625)$, respectively. Since $\overline{P_{116}} = (123645) = P_{31}$, we have $\text{cr}(A_1, A_2) \geq 4$ using of $d_{COG}(\text{"}P_{31}\text{"}, \text{"}P_{44}\text{"}) = 4$. The same reason gives us $\text{cr}(A_1, A_3) \geq 5$, $\text{cr}(A_1, A_4) \geq 5$, $\text{cr}(A_2, A_3) \geq 5$, $\text{cr}(A_2, A_4) \geq 5$, and $\text{cr}(A_3, A_4) \geq 4$. Moreover, by a discussion of possible subdrawings, we can verify that $\text{cr}(A_3, A_4) \geq 5$. Let F^i and F^j be two different graphs having the configuration A_3, A_4, respectively. We will consider two possibilities for a placement of the vertex t_j into the six regions of F^i with the vertex t_i on their boundaries. Remark that $\text{rot}_D(t_j) = A_4$ is represented by the cyclic permutation $P_{47} = (146325)$, and let us denote $\omega^\star = \omega_{1,2} \cup \omega_{2,3,6} \cup \omega_{3,6}$ and $\omega^{\star\star} = \omega_{3,4} \cup \omega_{1,4,5} \cup \omega_{1,5}$. If $t_j \in \omega^\star$, then the subdrawing $D(F^j)$ induced by the edges incident with the vertices v_1, v_4, and v_5 crosses the edges of T^i on the boundary $\omega^\star$ at least twice, once, and once, respectively. Since $T^j \in R_D$ and $\text{rot}_D(t_j) = A_4$, the edges of T^j must cross the edges of the region $\omega_{2,3,6}$ at least once. If $t_j \in \omega^{\star\star}$, then the subdrawing $D(F^j)$ induced by the edges incident with the vertices v_3, v_2, and v_6 crosses the edges of T^i on the boundary $\omega^{\star\star}$ also at least twice, once, and once, respectively. Similarly, the edges of T^j must cross the edges of the region $\omega_{1,4,5}$ also at least once. Thus, the edges of F^j cross the edges of F^i together at least five times. Clearly, also $\text{cr}(A_i, A_i) \geq 6$ for any $i = 1, 2, 3, 4$. Thus, all lower-bounds of number of crossing of configurations from $\mathcal{M}$ are summarized in Table 2.

$-$	A_1	A_2	A_3	A_4
A_1	6	4	5	5
A_2	4	6	5	5
A_3	5	5	6	5
A_4	5	5	5	6

TABLE 2. The necessary number of crossings between T^i and T^j for the configurations of F^i and F^j from $\mathcal{M}$.

3. THE CROSSING NUMBER OF $G + D_n$

In the proof of Theorem 3.1, the following lemmas related to some restricted drawings of the graph $G + D_n$ are needed. Let us note that if the edges of G do not cross each other, in D, then $\text{cr}_D(T^i \cup T^j) \geq 4$ for any two different subgraphs $T^i, T^j \in R_D$ by Table 2.

Lemma 3.1. *Let D be a good and antipodal-free drawing of $G + D_n$, $n > 2$. Let $2|R_D| + |S_D| > 2n - 2\left\lfloor \frac{n}{2} \right\rfloor$ and let $T^i, T^j \in R_D$ be two different subgraphs with $\text{cr}_D(T^i \cup T^j) \geq 4$. If both conditions*

$$\text{cr}_D(G \cup T^i \cup T^j, T^l) \geq 10 \qquad \textit{for any } T^l \in R_D \setminus \{T^i, T^j\}, \tag{3.3}$$

(3.4) $\quad \mathrm{cr}_D(G \cup T^i \cup T^j, T^l) \geq 7 \qquad$ *for any* $T^l \in S_D$

hold, then there are at least $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *crossings in* D.

*P*roof. We denote by $r = |R_D|$ and $s = |S_D|$. By the antipodal-free drawing D, any subgraph $T^l \notin R_D \cup S_D$ satisfies the condition $\mathrm{cr}_D(G \cup T^i \cup T^j, T^l) \geq 4$, and the number of T^l that cross the graph G at least twice is equal to $n - r - s$. By fixing of the graph $G \cup T^i \cup T^j$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G \cup T^i \cup T^j) + \mathrm{cr}_D(G \cup T^i \cup T^j)$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 10(r-2) + 7s + 4(n-r-s) + 4 = 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 3(2r+s)$$

$$+4n - 16 \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 3\Big(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\Big) + 4n - 16 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

□

Remark that if D is a good and antipodal-free drawing of $G + D_n$, and $T^i \in R_D$ such that F^i has configuration $A_j \in \mathcal{M}_D$, then $\mathrm{cr}_D(G \cup T^i, T^l) \geq 3$ for any subgraph T^l, $l \neq i$, see Fig. 2. Moreover, there are possibilities of an existence of a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$ only for the cases of the configurations A_1, and A_2 of F^i.

Lemma 3.2. *Let* D *be a good and antipodal-free drawing of* $G + D_n$, $n > 2$. *Let* $T^i \in R_D$ *be a subgraph such that* F^i *has configuration* $A_j \in \mathcal{M}_D$, $j \in \{1, 2\}$. *If there is a subgraph* $T^k \in S_D$ *with* $\mathrm{cr}_D(T^i, T^k) = 2$, *then*

a) $\mathrm{cr}_D(T^i \cup T^k, T^l) \geq 4$ *for any subgraph* T^l, $l \neq i, k$;
b) $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 7$ *for any subgraph* $T^l \in S_D$, *with* $\mathrm{cr}_D(T^i, T^l) = 3$;
c) $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 6$ *for any subgraph* $T^l \in S_D$, *with* $\mathrm{cr}_D(T^i, T^l) = 4$.

*P*roof. Let us assume the configuration A_1 of F^i, and remark that it is represented by the cyclic permutation $P_{116} = (154632)$.

a) The unique drawing of the subgraph F^i contains six regions with the vertex t_i on their boundaries, see Fig. 2. If there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the vertex t_k must be placed in the quadrangular region with three vertices of G on its boundary, i.e. $t_k \in \omega_{3,6,4}$ or $t_k \in \omega_{1,4,5}$. Thus, the subgraph F^k has a configuration which can be represented only by two possible cyclic permutations $P_{27} = (124365)$ or $P_{28} = (142365)$, see Fig. 3. For example, with the help of the algorithm [1], the reader can easy to verify there is no cyclic permutation P_m different from P_{116}, P_{27}, and P_{28} with $d_{COG}(\text{"}P_{116}\text{"}, \text{"}P_m\text{"}) + d_{COG}(\text{"}P_{27}\text{"}, \text{"}P_m\text{"}) < 4$, or $d_{COG}(\text{"}P_{116}\text{"}, \text{"}P_m\text{"}) + d_{COG}(\text{"}P_{28}\text{"}, \text{"}P_m\text{"}) < 4$. Thus, Woodall's result implies that there is no subgraph T^l with $\mathrm{cr}_D(T^i \cup T^k, T^l) < 4$ for any $l \neq i, k$.

b) Let $T^l \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^l) = 3$. The reader can easy to verify by a discussion that if $t_l \in \omega_{1,2} \cup \omega_{1,5} \cup \omega_{2,3} \cup \omega_{3,6}$, then $\mathrm{cr}_D(T^i, T^l) \geq 4$. Let the vertex t_l be placed in one of the region $\omega_{3,6,4}$, $\omega_{1,4,5}$. Then the subdrawing $D(F^l)$ induced by the edges incident with the vertices v_4, v_5, and v_6 crosses the edges of T^i once. Since $T^l \in S_D$, the subdrawing $D(F^l)$ induced by the edges incident with the vertices v_1, and v_2 crosses the edge of G at most once. Thus, the subgraph F^l has a configuration represented by cyclic permutations containing the cyclic sub-permutation (546), but not containing the cyclic sub-permutations (4126) and (4216). Hence, let us define

$$\mathrm{Perm} = \{P_l : (546) \subset P_l \wedge (4126) \not\subset P_l \wedge (4216) \not\subset P_l\},$$

where $l \in \{1, \dots, 120\}$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^k, T^i) = 2$, then $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 1 + 3 + 3 = 7$ for any $T^l \in S_D$ with $\mathrm{cr}_D(T^k, T^l) \geq 3$.

Assume that there is a subgraph $T^l \in S_D$, $l \neq k$ with $\mathrm{cr}_D(T^k, T^l) < 3$. Since $\overline{P_{27}} = P_{113}$, $\overline{P_{28}} = P_{71}$, and $\overline{P_{116}} = P_{31}$, let us define the following sets of the cyclic permutations $\mathrm{Perm}_{27} = \{P_l : d_{COG}("P_l", "P_{113}") < 3 \wedge d_{COG}("P_l", "P_{31}") \leq 3\}$, and $\mathrm{Perm}_{28} = \{P_l : d_{COG}("P_l", "P_{71}") < 3 \wedge d_{COG}("P_l", "P_{31}") \leq 3\}$. Hence, the subgraph F^l has a configuration represented by a $P_l \in \mathrm{Perm}_{27} \cup \mathrm{Perm}_{28}$, see [12]. By the help of the algorithm [1], we can verify that

$$\mathrm{Perm}_{27} = \{P_{36}, P_{45}, P_{46}, P_{51}, P_{61}, P_{62}, P_{63}, P_{65}, P_{68}, P_{106}, P_{109}, P_{110}, P_{112}\},$$

$$\mathrm{Perm}_{28} = \{P_2, P_{12}, P_{34}, P_{35}, P_{36}, P_{46}, P_{54}, P_{61}, P_{62}, P_{65}, P_{66}, P_{68}, P_{110}\}.$$

Since the sets Perm and $\mathrm{Perm}_{27} \cup \mathrm{Perm}_{28}$ are disjoint, then there is no subgraph $T^l \in S_D$, $l \neq k$ with $\mathrm{cr}_D(T^k, T^l) < 3$.

c) If there is a subgraph $T^l \in S_D$ with $\mathrm{cr}_D(T^i, T^l) = 4$, then $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) \geq 1 + 4 + 1 = 6$ by the antipodal-free drawing D.

Due to symmetry of the configurations A_1 and A_2, let us define the function

$$\Pi : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}, \text{ with } 1 \leftrightarrow 3,\ 2 \leftrightarrow 4, \text{ and } 5 \leftrightarrow 6.$$

Thus, the configuration A_2 is obtained from A_1 using the transformation Π, and this completes the proof of Lemma. □

Remark that the lower bound 6 in case c) of Lemma 3.2 can not be higher, see Fig. 3.

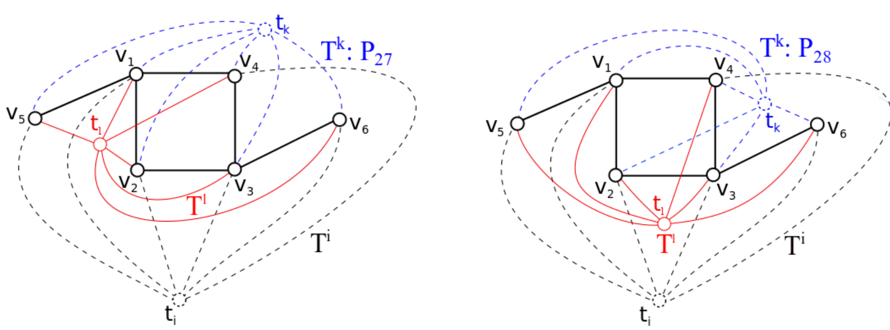

FIGURE 3. Two drawings of $G \cup T^i \cup T^k \cup T^l$ with $\mathrm{cr}_D(G \cup T^i \cup T^k, T^l) = 6$ for $T^i \in R_D$ with the configuration A_1 of $F^i = G \cup T^i$, and $T^k, T^l \in S_D$.

Lemma 3.3. *Let D be a good and antipodal-free drawing of $G + D_n$, $n > 2$, and let $\mathcal{M}_D$ be non-empty set with $\{A_1, A_2\} \subseteq \mathcal{M}_D$. If T^i, $T^j \in R_D$ are different subgraphs such that F^i, F^j have configurations A_1, A_2, respectively, then*

$$\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 7 \qquad \text{for any } T^k \in S_D.$$

Proof. Let us assume the configurations A_1 of F^i, and A_2 of F^j. If there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then the subgraph F^k can be represented by one of the cyclic permutations $P_{27} = (124365)$, and $P_{28} = (142365)$. Let us note that the configuration A_2 is represented by P_{44}. Using $\overline{P_{27}} = (156342) = P_{113}$, $\overline{P_{28}} = (156324) = P_{71}$, and $d_{COG}("P_{71}", "P_{44}") = d_{COG}("P_{113}", "P_{44}") = 4$ we obtain $\mathrm{cr}_D(T^j, T^k) \geq 4$. Hence, $\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 1 + 2 + 4 = 7$. Due to symmetry, we can apply the same

Let τ, σ be two cyclic permutations. We will say that τ is a cyclic sub-permutation of σ, if each cycle of τ is a sub-cycle of some cycle of σ in the obvious sense of preserving cyclic order. We denote this by $\tau \subset \sigma$.

idea for the case, if there is a $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 2$. If $\mathrm{cr}_D(T^i, T^k) \geq 3$, and $\mathrm{cr}_D(T^j, T^k) \geq 3$ for any $T^k \in S_D$, then $\mathrm{cr}_D(G \cup T^i \cup T^j, T^k) \geq 1+3+3 = 7$ trivially holds for any $T^k \in S_D$, and this completes the proof. □

The exact values of the crossing numbers of small graphs can be also computed using the algorithm located on the website `http://crossings.uos.de/`. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs. So, we obtain the following result.

Lemma 3.4. $\mathrm{cr}(G + D_2) = 2$.

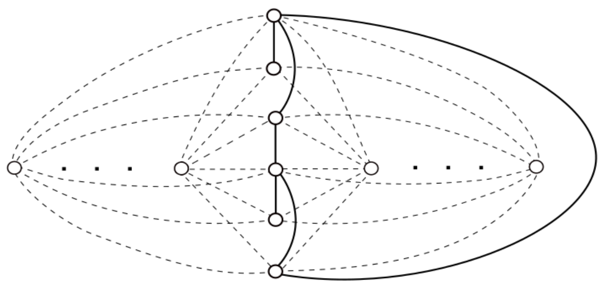

FIGURE 4. A good drawing of $G + D_n$.

Now we are able to prove the main results of the paper.

Theorem 3.1. $\mathrm{cr}(G + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

Proof. In Fig. 4 there is a drawing of $G + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$. We prove the reverse inequality by induction on n. The graph $G + D_1$ is planar, hence $\mathrm{cr}(G + D_1) = 0$. By Lemma 3.4 the result is true for $n = 2$. Suppose now that, for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor, \tag{3.5}$$

and let

$$\mathrm{cr}(G + D_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor \qquad \text{for any integer } m < n. \tag{3.6}$$

Let us first show that the considered drawing D must be antipodal-free. As a contradiction we can suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. One can easy to verify that $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 2$. The known fact that $\mathrm{cr}(K_{6,3}) = 6$ implies that any T^k, $k = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D\,(G + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n)$$

$$+\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 6(n-2) + 2 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

This contradiction confirms that D is antipodal-free. Our assumption on D together with $\mathrm{cr}(K_{6,n}) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{6,n}) < 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, if we denote $r = |R_D|$ and $s = |S_D|$, then

$$\mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Hence, $r \geq 1$, $2r+s > 2n - 2\left\lfloor \frac{n}{2} \right\rfloor$, and $r > n-r-s$. For $T^i \in R_D$, we will discuss the existence of possible configurations of F^i in the drawing D in the following cases:

Case 1: $\mathrm{cr}_D(G) = 0$.

Since $r \geq 1$, i.e. there is a subgraph $T^i \in R_D$, we can choose the vertex notations of the graph in such a way as shown in Fig. 1(a). Thus, we will deal with the set of configurations belonging to $\mathcal{M}_D$.

a) $A_j \in \mathcal{M}_D$ for some $j \in \{3,4\}$.
Without lost of generality, we can assume that $T^n \in R_D$ with F^n having configuration A_j, $j \in \{3,4\}$. The subdrawing of F^n induced by D can be obtained from the drawings in Fig. 2. Thus, we can easy to verify that there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) \leq 2$. Moreover, $\mathrm{cr}_D(T^n, T^i) \geq 5$ for any $T^i \in R_D$ by Table 2. Hence, by fixing of the graph $G \cup T^n$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)$$

$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 4s + 3(n-r-s) = 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + (2r+s)$$

$$+3n-5 \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + \left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + 3n - 5 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

In addition, let us consider that $A_3 \notin \mathcal{M}_D$ and $A_4 \notin \mathcal{M}_D$.

b) $\{A_1, A_2\} \subseteq \mathcal{M}_D$.
Without lost of generality, let us fix two T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations A_1, A_2, respectively. Then condition (3.3) is true by summing the values in all columns in the first two rows of Table 2, and condition (3.4) holds by Lemma 3.3. Thus, all assumptions of Lemma 3.1 are fulfilled.

c) $\mathcal{M}_D = \{A_j\}$ for only one $j \in \{1,2\}$.
Without lost of generality, we can assume the configuration A_1 of F^n. Let us denote $S_D(T^n) = \{T^i \in S_D : \mathrm{cr}_D(F^n, T^i) = 3\}$, and $S'_D(T^n) = \{T^i \in S_D : \mathrm{cr}_D(F^n, T^i) = 4\}$. We denote by $s_1 = |S_D(T^n)|$ and $s_2 = |S'_D(T^n)|$. Remark that $S_D(T^n)$ and $S'_D(T^n)$ are disjoint subsets of S_D, and $s_1 + s_2 \leq s$, i.e. $s - s_1 - s_2 \geq 0$. Hence, we will discuss two possibilities:

1) If $s_1 \leq s - s_1 - s_2$, then by fixing of the graph $G \cup T^n$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)$$

$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(r-1) + 3s_1 + 4s_2 + 5(s - s_1 - s_2) + 3(n-r-s)$$

$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(r-1) + 4s + 3(n-r-s) = 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$+r + (2r+s) + 3n - 6 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

2) Let us assume that $s_1 > s - s_1 - s_2$, i.e. $s_1 - 1 \geq s - s_1 - s_2$. Let T^k be a subgraph from non-empty set $S_D(T^n)$. As we assume $\mathcal{M}_D = \{A_1\}$, then we have $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 6 + 2 = 8$ for any $T^i \in R_D$, $i \neq n$. In the proof of Lemma 3.2, it was showed that the subgraph F^k can have only configuration represented by one of the cyclic permutations $P_{27} = (124365)$, and $P_{28} = (142365)$. Using $\overline{P_{28}} = P_{71}$, and $d_{COG}(\text{"}P_{27}\text{"}, \text{"}P_{71}\text{"}) = 5$ we obtain

$\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 1 + 2 + 5 = 8$ for any $T^i \in S_D(T^n)$, $i \neq k$. Again by Lemma 3.2, we can verify that $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 7$ for any $T^i \in S'_D(T^n)$, and $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 6$ for any $T^i \in S_D$ with $\mathrm{cr}_D(F^n, T^i) \geq 5$. Moreover, $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 2 + 4 = 6$ for any $T^i \notin R_D \cup S_D$. Thus, by fixing of the graph $G \cup T^n \cup T^k$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G \cup T^n \cup T^k) + \mathrm{cr}_D(G \cup T^n \cup T^k)$$
$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-1) + 8(s_1-1) + 7s_2 + 6(s-s_1-s_2) + 6(n-r-s) + 3$$
$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(r-1) + 7(s-1) + 6(n-r-s) + 3 = 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$
$$+(2r+s) + 6n - 12 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Due to symmetry, the same arguments are used for the case $\mathcal{M}_D = \{A_2\}$.

Case 2: $\mathrm{cr}_D(G) = 1$.

There are six drawings of G with one crossing among its edges, see Fig. 5. Since $R_D \neq \emptyset$, we will dealt with only two drawings of G, because there is a possibility of an existence of a subgraph $T^i \in R_D$ only for the drawing of G as in Fig. 5(a), (b).

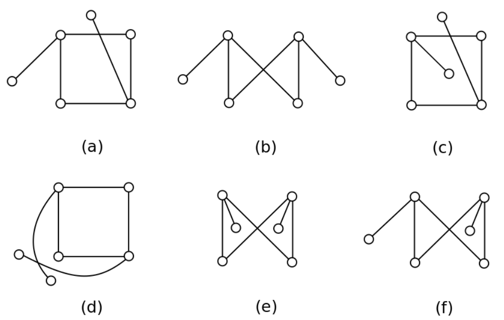

FIGURE 5. Six possible drawings of the graph G with $\mathrm{cr}_D(G) = 1$.

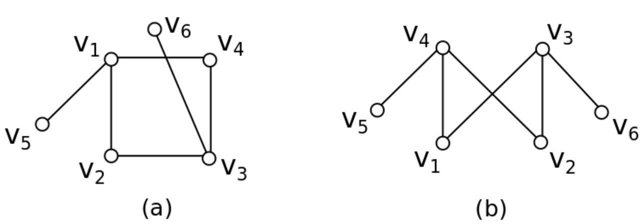

FIGURE 6. The vertex notations of the graph G with $\mathrm{cr}_D(G) = 1$.

a) $\mathrm{cr}_D(G) = 1$ as in Fig. 5(a).
Without loss of generality, we can choose the vertex notations of the graph G in such a way as shown in Fig. 6(a). Thus, we can list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Let us start with the subdrawing of $D(F^i)$ induced by the edges incident with the vertices v_1, and v_3. These two edges together with the edges of G divide the plane into a few regions, but the vertices v_2, and v_4, v_6 must be placed in two different of them.

Since there are two possibilities in which region the vertex v_5 can be placed, we obtain two different possible configurations of F^i denoted as B_1, and B_2. They are are represented by the cyclic permutations $P_{48} = (164325)$ and $P_{119} = (156432)$, respectively.

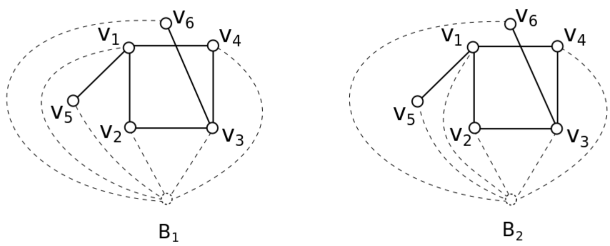

FIGURE 7. Two drawings of possible configurations of subgraph F^i.

The condition $\overline{P_{119}} = (123465) = P_{25}$ together with $d_{COG}("P_{48}", "P_{25}") = 5$ imply $\mathrm{cr}(B_1, B_2) \geq 5$. Moreover, if we assume that $T^n \in R_D$, then there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) \leq 2$. Hence, by fixing of the graph F^n we can use the same inequalities as in **C**ase 1a).

b) $\mathrm{cr}_D(G) = 1$ as in Fig. 5(b).
Without loss of generality, we can choose the vertex notations of the graph G in such a way as shown in Fig. 6(b). Again, our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Since there is only one subdrawing of $F^i \setminus \{v_3, v_4\}$ represented by the rotation (1562), we have four drawings of F^i depending on in which region the vertices v_3, and v_4 are placed. We denote by $\mathcal{N}_D$ the set of all configurations that exist in the drawing D belonging to the set $\mathcal{N} = \{C_1, C_2, C_3, C_4\}$, see Fig. 8. Moreover, all lower-bounds of number of crossing of configurations from $\mathcal{N}$ are same like in Table 2, of course with the corresponding indexes.

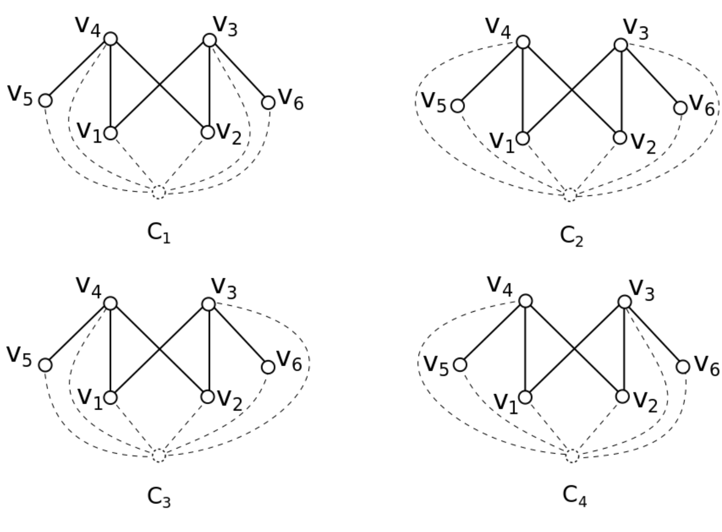

FIGURE 8. Four drawings of possible configurations from $\mathcal{N}$ of subgraph F^i.

It is easy to see, if a $T^i \in R_D$ with the configuration C_2 of F^i, then $\mathrm{cr}_D(G \cup T^i, T^j) \geq 4$ for any T^j, $j \neq i$. Thus, we will consider two cases:

1) $C_2 \in \mathcal{N}_D$.
Without lost of generality, we can assume the configuration C_2 of F^n. Then, by fixing of the graph $G \cup T^n$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

2) $C_2 \notin \mathcal{N}_D$.
Without lost of generality, let us assume $T^n \in R_D$ such that the subgraph F^n has the configuration $C_j \in \mathcal{N}_D$, $j \neq 2$. By the drawings of the subgraph F^n in Fig. 8, there is no $T^i \in S_D$ with $\mathrm{cr}_D(G \cup T^n, T^i) \leq 3$. Using the lower-bounds of number of crossings of configurations in Table 2, if we fix the graph $G \cup T^n$, then

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 4s + 3(n-r-s) + 1 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Case 3: $\mathrm{cr}_D(G) \geq 2$.
The reader can easy to verify over all possible drawings D with the non-empty set R_D that if a $T^i \in R_D$, then $\mathrm{cr}_D(G \cup T^i, T^j) \geq 4$ for any subgraph T^j, $j \neq i$. Thus, by fixing of the graph $G \cup T^i$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^i) + \mathrm{cr}_D(G \cup T^i)$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 2 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, it was shown that there is no good drawing D of the graph $G + D_n$ with less than $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. This completes the proof of the main theorem. □

4. Corollaries

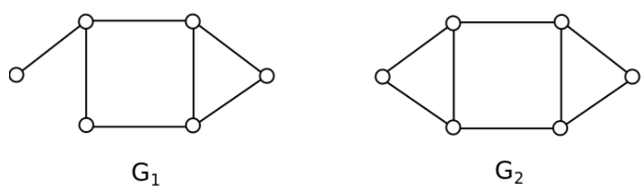

FIGURE 9. Two graphs G_1 and G_2 by adding new edges to the graph G.

In Fig. 4 we are able to add some edges to the graph G without additional crossings. So the drawing of the graphs $G_1 + D_n$ and $G_2 + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings is obtained. Thus, the next results are obvious.

Corollary 4.1. $\mathrm{cr}(G_i + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$, *where* $i = 1, 2$.

Remark that the crossing numbers of the graph $G_2 + D_n$ was obtained in [5] without using the vertex rotation.

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