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On weighted Strand's iteration

D. CARP[1], C. POPA[1], T. PRECLIK[2] and U. RÜDE[2]

ABSTRACT. In this paper we present a generalization of Strand's iterative method for numerical approximation of the weighted minimal norm solution of a linear least squares problem. We prove convergence of the extended algorithm, and show that previous iterative algorithms proposed by L. Landweber, J. D. Riley and G. H. Golub are particular cases of it.

1. INTRODUCTION

Let us consider the linear least squares problem: find $x^* \in I\!R^n$ such that

$$\| Ax^* - b \| = \min\{\| Ax - b \|, x \in I\!R^n\} \ \Leftrightarrow \| Ax^* - b \| = \min!, \tag{1.1}$$

where A is an $m \times n$ matrix, $b \in I\!R^m$, $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ will denote the Euclidean scalar product and norm, and P_S will stand for the orthogonal projection onto a vector subspace S; also $\mathcal{N}(A), \mathcal{R}(A)$ denote the null space and range of the matrix A. Let $LSS(A; b)$ be the set of all solutions of (1.1) and x_{LS} its (unique) minimal Euclidean norm one. It is well known that (1.1) can be expressed as a classical linear system through its associated normal equation

$$A^T A x^* = A^T b. \tag{1.2}$$

Let also

$$A = U \Sigma V^T, \ \Sigma = diag(\sigma_1, \dots, \sigma_r, 0, \dots, 0), \ \sigma_1 \geq \dots \geq \sigma_r > 0, \tag{1.3}$$

be a Singular Value Decomposition of A (SVD, for short), where

$$U = col(U^1, \dots, U^m), \ V = col(V^1, \dots, V^n) \tag{1.4}$$

are orthogonal matrices. In the paper [9] O.N. Strand proposed an iterative method for solving (1.2), which in the finite dimensional case of $I\!R^n$ can be written as follows

$$x^0 \in I\!R^n, \ x^k = x^{k-1} + FA^T(b - Ax^{k-1}), \ k \geq 1, \tag{1.5}$$

where F is an $n \times n$ matrix that satisfies

$$F\, V^i = p_i V^i, \ \forall\, i = 1, \dots, r, \tag{1.6}$$

V^i are the right singular vectors from (1.4), and $\lambda_i, \ p_i$ are defined by

$$\lambda_i = \sigma_i^2, \ 0 \ < \ p_i \lambda_i \ < \ 2, \ \forall\, i = 1, \dots, r. \tag{1.7}$$

The convergence of the algorithm (1.5) was analyzed in [9] and is presented in the following result.

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Corresponding author: Constantin Popa; cpopa@univ-ovidius.ro

Theorem 1.1. ([9]) *In the above hypothesis (1.6) - (1.7), $\forall x^0 \in I\!R^n$ the sequence $(x^k)_{k\geq 0}$ generated by (1.5) converges and its limit is given by*

$$\lim_{k\to\infty} x^k = P_{\mathcal{N}(A)}(x^0) + x_{LS} \in LSS(A;b). \tag{1.8}$$

The author observes in [9] that particular matrices F which satisfies (1.6) can be either a polynomial in A^TA, i.e.

$$F = \alpha_0 I + \alpha_1 A^TA + \cdots + \alpha_q (A^TA)^q \tag{1.9}$$

or a rational function on A^TA as e.g.

$$F = (sI + A^TA)^{-1},\ s > 0. \tag{1.10}$$

Remark 1.1. For the choice $F = \omega I$, with I the unit $n \times n$ matrix and $\omega > 0$ (see (1.9)), (1.5) becomes Landweber's iteration from [3], whereas for the choice (1.10) we find the algorithm

$$x^0 \in I\!R^n,\ (sI + A^TA)x^k = sx^{k-1} + A^Tb,\ k \geq 0, \tag{1.11}$$

first considered by Riley in [8] for A square and invertible, and then generalized by Golub in [1], for A arbitrary rectangular.

But in many practical problems, instead of $LSS(A;b)$ and x_{LS} we are interested in "weighted" minimal norm solutions of (1.1) (or (1.2)), i.e. a vector $x_{LS}^D \in LSS(A;b)$ which satisfies

$$\| D^{\frac{1}{2}} x_{LS}^D \|^2 = \min\{\| D^{\frac{1}{2}} x \|^2, such\ that\ \ \| Ax - b \| = \min!\}, \tag{1.12}$$

whrere D is a given symmetric and positive definite matrix (see e.g. the rigid body dynamics problems in [5]).
In the present paper we propose a generalization of Strand's algorithm (1.5), which will include its original version, together with Landweber, Riley and Golub methods. The paper is organized as follows: in section 2 we present some technical results about the weighted least squares solutions of (1.1). The generalization of algorithm (1.5) together with its convergence proof are presented in section 3. Applications are presented in the last section of the paper.

2. WEIGHTED SOLUTIONS OF THE PROBLEM (1.1)

For the theoretical analysis of the generalization of algorithm (1.5) we we will use the Generalized Singular Value Decomposition (GSVD, for short) of the pair $(A, D^{\frac{1}{2}})$, presented in Theorem 2.2 below, (see e.g. the basic papers [10], [4]).

Theorem 2.2. (GSVD) *If A and D are as before, the following equalities hold*

$$U^TAX = D_A = diag(\alpha_1, \ldots, \alpha_r, 0, \ldots, 0),$$

$$V^TD^{\frac{1}{2}}X = D_B = diag(\beta_1, \ldots, \beta_n), \tag{2.13}$$

with $U : m \times m$ and $V : n \times n$ orthogonal, $X : n \times n$ invertible, $D_A : m \times n$, $D_B : n \times n$, $r = rank(A)$,

$$1 > \alpha_1 \geq \cdots \geq \alpha_r > 0,\ 0 < \beta_1 \leq \cdots \leq \beta_r < \beta_{r+1} = \cdots = \beta_n = 1,$$

$$\alpha_i^2 + \beta_i^2 = 1, i = 1, \ldots, r, \tag{2.14}$$

and the ratios $\frac{\alpha_i}{\beta_i}, i = 1, \ldots, r$ are the nonzero singular values of the matrix $AD^{-\frac{1}{2}}$.

We will now present some results about the least squares solutions $LSS(A;b)$ in terms of the GSVD from Theorem 2.2.

Lemma 2.1. *(i) The set $LSS(A;b)$ is characterized by*

$$LSS(A;b) = \{x^* = Xz, z_i = \frac{w_i}{\alpha_i}, i = 1, \dots, r, \ z_{r+1}, \dots, z_n \in I\!R\} =$$

$$\{\sum_{i=1}^{r} \frac{w_i}{\alpha_i} X^i + \sum_{i=r+1}^{n} z_i X^i, \ z_{r+1}, \dots, z_n \in I\!R\}, \tag{2.15}$$

where X^i denotes the i-th column of the matrix X and $w = U^T b \in I\!R^m$.
(ii) The unique minimal D-norm element from $LSS(A;b)$, denoted by x_{LS}^D (see (1.12)) and the null space of A, denoted by $\mathcal{N}(A)$ are given by

$$x_{LS}^D = \sum_{i=1}^{r} \frac{w_i}{\alpha_i} X^i, \ \mathcal{N}(A) = \{z = \sum_{i=r+1}^{n} z_i X^i, \ z_i \in I\!R\}, \tag{2.16}$$

and

$$\langle x_{LS}^D, z \rangle_D = \langle x_{LS}^D, Dz \rangle = 0, \ \forall z \in \mathcal{N}(A). \tag{2.17}$$

Proof. (i) An element x^* from $LSS(A;b)$ is a solution of the normal equation (1.2). According to the GSVD decomposition (2.13) - (2.14) it results

$$A^T A = X^{-T} D_A^T D_A X^{-1}, \ D = (D^{\frac{1}{2}})^T D^{\frac{1}{2}} = X^{-T} D_B^2 X^{-1} \tag{2.18}$$

and we have the equivalencies

$$A^T A x^* = A^T b \ \Leftrightarrow \ X^{-T} D_A^T D_A X^{-1} x^* = X^{-T} D_A^T U^T b \ \Leftrightarrow$$

$$D_A^T D_A X^{-1} x^* = D_A^T U^T b \ \Leftrightarrow \ D_A^T D_A z = D_A^T w, \tag{2.19}$$

where

$$z = X^{-1} x^*, \ w = U^T b. \tag{2.20}$$

From (2.13) and (2.19) we obtain

$$z_i = \frac{w_i}{\alpha_i}, i = 1, \dots, r \text{ and } z_i \in I\!R \text{ arbitrary }, i = r+1, \dots, n. \tag{2.21}$$

Then, (2.15) holds directly from (2.20)-(2.21).
(ii) Let $x^* \in LSS(A;b)$. From (2.13) and (2.15) it holds that

$$\| x^* \|_D^2 = \| D^{\frac{1}{2}} x^* \|^2 = \| V D_B X^{-1} X z \|^2 = \| V D_B z \|^2 = \| D_B z \|^2 =$$

$$\sum_{i=1}^{r} \left(\frac{w_i}{\alpha_i} \right)^2 \beta_i^2 + \sum_{i=r+1}^{n} (z_i)^2 \beta_i^2. \tag{2.22}$$

The minimal value of the sum in (2.22) is obtained for $z_i = 0, i = r+1, \dots, n$ which gives us x_{LS}^D from (2.16). For the second equality, we first observe that from (2.13) we get $A = U D_A X^{-1}$. Thus

$$x \in \mathcal{N}(A) \ \Leftrightarrow \ Ax = 0 \ \Leftrightarrow \ D_A X^{-1} x = 0 \ \Leftrightarrow \ D_A z = 0,$$

for $z = X^{-1} x$. Thus, if $x \in \mathcal{N}(A)$, then $x = Xz$ with $z_i = 0, i = 1, \dots, r$ and $z_i, i = r+1, \dots, n$ arbitrary, i.e. x belongs to the set in the right hand side of the second equality in (2.16). Conversely, if x is an element from that set we have

$$x = \sum_{i=r+1}^{n} z_i X^i = X(0, \dots, 0, z_{r+1}, \dots, z_n)^T,$$

and

$$Ax = UD_A X^{-1} X(0, \ldots, 0, z_{r+1}, \ldots, z_n)^T = UD_A(0, \ldots, 0, z_{r+1}, \ldots, z_n)^T = 0,$$

i.e. $x \in \mathcal{N}(A)$.
The equality (2.17) results by direct computation using the second equality in (2.18) and the characterizations of $\mathcal{N}(A)$ and x_{LS}^D from (2.16). □

Lemma 2.2. *(i) For any vector $e \in I\!R^n$ it holds that*

$$P_{\mathcal{N}(A)}^D(e) = \sum_{i=r+1}^{n} \bar{e}_i X^i, \tag{2.23}$$

where $P_{\mathcal{N}(A)}^D(e)$ is the projection of e onto $\mathcal{N}(A)$ with respect to the energy scalar product induced by the matrix D, and $\bar{e} \in I\!R^n$ is defined by $\bar{e} = X^{-1}e$.
(ii) We have the equality

$$LSS(A; b) = \{P_{\mathcal{N}(A)}^D(x) + x_{LS}^D,\ x \in I\!R^n\}. \tag{2.24}$$

Proof. (i) By introducing the notation $y = \sum_{i=r+1}^n \bar{e}_i X^i$, we have to prove that

$$\langle e - y, z\rangle = 0,\ \forall z \in \mathcal{N}(A). \tag{2.25}$$

For an arbitrary $z \in \mathcal{N}(A)$, from (2.16) we get $z = \sum_{i=r+1}^n z_i X^i$, for some $z_i \in I\!R, i = r+1, \ldots, n$, thus (by also using the equality $D = X^{-T} D_B^2 X^{-1}$)

$$Dz = X^{-T} D_B^2 \sum_{i=r+1}^{n} z_i X^{-1} X^i = X^{-T} D_B^2 (0, \ldots, 0, 1, \ldots, 1)^T = X^{-T} v \tag{2.26}$$

with $v \in I\!R^n$ given by

$$v = (0, \ldots, 0, \beta_{r+1}^2 z_{r+1}, \ldots, \beta_n^2 z_n)^T. \tag{2.27}$$

From (2.26)-(2.27) we get

$$\langle y, z\rangle_D = \langle X^{-1} y, v\rangle = \langle (0, \ldots, 0, \bar{e}_{r+1}, \ldots, \bar{e}_n)^T, v\rangle = \sum_{i=r+1}^{n} \bar{e}_i \beta_i^2 z_i, \tag{2.28}$$

and by using the definition of the vector $\bar{e}$

$$\langle e, z\rangle_D = \langle X^{-1} e, v\rangle = \langle \bar{e}, v\rangle = \sum_{i=r+1}^{n} \bar{e}_i \beta_i^2 z_i. \tag{2.29}$$

By substracting (2.28) from (2.29) we obtain (2.25).
(ii) If $x^* = P_{\mathcal{N}(A)}^D(x) + x_{LS}^D$ for some $x \in I\!R^n$ then, from (2.23), (2.16) and (2.13) we obtain that x^* is a solution of the normal equation $A^T A x^* = A^T b$, therefore $x^* \in LSS(A; b)$. Conversely, if $x^* \in LSS(A; b)$, from (2.15) and (2.16) (first equality) we get $x^* = x_{LS}^D + \sum_{i=r+1}^n z_i X^i$, with $x^* = Xz$, $z = (\frac{w_i}{\alpha_i}, \ldots, \frac{w_i}{\alpha_i}, z_{r+1}, \ldots, z_n)^T \in I\!R^n$. But $z = X^{-1} x^*$, which according to (2.23) gives us that $\sum_{i=r+1}^n z_i X^i = P_{\mathcal{N}(A)}^D(x^*)$ and the proof is complete. □

Remark 2.2. We have to observe that from (2.16), (2.17) and (2.24) do not result that $x_{LS}^D \in \mathcal{R}(A^T)$, as for the classical Euclidean scalar product case (i.e. $D = I$ in our context); x_{LS}^D is only the unique element from $LSS(A; b)$ with a minimal D-norm among the other elements from $LSS(A; b)$; it also has the D-orthogonality property from (2.17).

3. WEIGHTED STRAND'S ITERATION

According to the GSVD from Theorem 2.2 let

$$\mathcal{V} = sp\{X^1, X^2, \dots, X^r\}, \tag{3.30}$$

and F an $n \times n$ matrix with the property that $I - FA^TA$ is convergent on $\mathcal{V}$, i.e.

$$\lim_{k\to\infty} (I - FA^TA)^k v = 0, \ \forall\, v \in \mathcal{V}. \tag{3.31}$$

Formally, our weighted version of algorithm (1.5) has exactly the same form, but its convergence will be related to the results from section 2.

Theorem 3.3. *If the matrix F satisfies (3.31) then, $\forall\, x^0 \in I\!R^n$ the sequence $(x^k)_{k\geq 0}$ generated by (1.5) converges and*

$$\lim_{k\to\infty} x^k = P^D_{\mathcal{N}(A)}(x^0) + x^D_{LS} \in LSS(A;b). \tag{3.32}$$

Proof. We will adapt the main steps from the proof of Theorem 5 in [9]. The weighted minimal norm solution x^D_{LS} is an element of $LSS(A;b)$ thus (see e.g. [7])

$$Ax^D_{LS} = P_{\mathcal{R}(A)}(b). \tag{3.33}$$

By following the computations in [9], page 807 and the notational convention there we obtain for the sequence $(x^k)_{k\geq 0}$ generated by (1.5)

$$x^k = x^0 + \left\{\frac{I - (I - FA^TA)^k}{FA^TA}\right\} FA^T(b - Ax^0). \tag{3.34}$$

Then, by successively using (3.34), (3.33) and the above mentioned notational convention we get

$$\begin{aligned} x^k = x^0 + \left\{\frac{I - (I - FA^TA)^k}{FA^TA}\right\} FA^T \left(A(x^D_{LS} - x^0) + P_{\mathcal{N}(A^T)}(b)\right) = \\ x^0 + \left\{\frac{I - (I - FA^TA)^k}{FA^TA}\right\} FA^TA \left(x^D_{LS} - x^0\right) = \\ x^0 + \left(I - (I - FA^TA)^k\right)\left(x^D_{LS} - x^0\right) = \\ x^D_{LS} - (I - FA^TA)^k(x^D_{LS} - x^0). \end{aligned} \tag{3.35}$$

Moreover, from (2.16) and (2.23) we have that

$$x^D_{LS} \in sp\{X^1, \dots, X^r\} = \mathcal{V}, \ P^D_{\mathcal{N}(A)}(z) \in sp\{X^{r+1}, \dots, X^n\}, \tag{3.36}$$

for any vector $z \in I\!R^n$, and because $x^D_{LS}, x^0 \in I\!R^n$ and $\{X^1, \dots, X^n\}$ is a basis in $I\!R^n$ it holds that

$$x^D_{LS} - x^0 = \sum_{i=1}^n \gamma_i X^i, \ x^0 = \sum_{i=1}^n z_i X^i, \tag{3.37}$$

for some scalars $\gamma_i, z_i \in I\!R$. From (3.36) and (3.37) we obtain that

$$x^D_{LS} - x^0 = \sum_{i=1}^r \gamma_i X^i + \sum_{i=r+1}^n z_i X^i. \tag{3.38}$$

Moreover, because X is invertible x^0 can be expressed as

$$x^0 = Xz, \tag{3.39}$$

which according to (2.23) gives us

$$P^D_{\mathcal{N}(A)}(x^0) = \sum_{i=r+1}^{n} z_i X^i. \tag{3.40}$$

Now, from (3.35), (3.38) and (3.40) we obtain

$$x^k = x^D_{LS} - \left(I - FA^TA\right)^k \left(\sum_{i=1}^{r} \gamma_i X^i\right) + \left(I - FA^TA\right)^k P^D_{\mathcal{N}(A)}(x^0). \tag{3.41}$$

Because $P^D_{\mathcal{N}(A)}(x^0) \in \mathcal{N}(A)$ we get that

$$\left(I - FA^TA\right)^k P^D_{\mathcal{N}(A)}(x^0) = P^D_{\mathcal{N}(A)}(x^0),$$

which together with (3.41), (3.31) and because X is invertible give us (3.32) and completes the proof. □

Remark 3.3. Following the proofs of theorem 3.3 and the one in Strand's paper [9], it can be easily shown that Strand's result in Theorem 1.1 rests true if we replace assumptions (1.7)-(1.6) by

$$\lim_{k\to\infty} (I - FA^TA)^k v = 0,\ \forall v \in sp\{V^1, \ldots, V^r\}, \tag{3.42}$$

which is similar to our asumption (3.31) and is clearly weaker that (1.7)-(1.6).

4. Applications

We will start this section by first identifying, as Strand did in his original paper, two important classes of functions F which satisfy the assumptions (3.30)-(3.31).

Proposition 4.1. *If the matrix F is of the form*

$$F = \omega D^{-1} \tag{4.43}$$

or a rational function as

$$F = (sD + A^TA)^{-1}, \tag{4.44}$$

then it satisfies the assumptions (3.30)-(3.31).

Proof. According to equalities (2.13) we have $D^{-1} = XD_B^{-2}X^T$ and

$$(sD + A^TA)^{-1} = \left(X^{-T}(sD_B^2 + D_A^TD_A)X^{-1}\right)^{-1} =$$

$$X diag(\frac{s\beta_1^2}{\alpha_1^2 + s\beta_1^2}, \ldots, \frac{s\beta_r^2}{\alpha_r^2 + s\beta_r^2}, 1, \ldots, 1)X^T. \tag{4.45}$$

Let now $i \in \{1, \ldots, r\}$ be arbitrary fixed. From (2.13) and (4.45) we get for the matrix $I - FA^TA$ that (e_i is the i-th vector of the cannonical basis in $I\!R^n$)

$$(I - FA^TA)^k = \left[X\left(I - \left(D_A^TD_A + sD_B^2\right)^{-1} D_A^TD_A\right)X^{-1}\right]^k =$$

$$X\left(I - \left(D_A^TD_A + sD_B^2\right)^{-1} D_A^TD_A\right)^k X^{-1} = X diag(\frac{(s\beta_1^2)^k}{(s\beta_1^2 + \alpha_1^2)^k}, \ldots, \frac{(s\beta_r^2)^k}{(s\beta_r^2 + \alpha_r^2)^k}, 0, \ldots, 0)X^{-1}.$$

Therefore, for $1 \le i \le r$

$$(I - FA^TA)^k X^i = X diag(\frac{(s\beta_1^2)^k}{(s\beta_1^2 + \alpha_1^2)^k}, \ldots, \frac{(s\beta_r^2)^k}{(s\beta_r^2 + \alpha_r^2)^k}, 0, \ldots, 0)e_i =$$

$$X\frac{(s\beta_i^2)^k}{(s\beta_i^2+\alpha_i^2)^k}e_i = \frac{(s\beta_i^2)^k}{(s\beta_i^2+\alpha_i^2)^k}Xe_i = \frac{(s\beta_i^2)^k}{(s\beta_i^2+\alpha_i^2)^k}X^i \longrightarrow 0 \text{ for } k\to\infty, \tag{4.46}$$

i.e the equality (3.31).
If F is given by eqfinal, from 14 we get

$$(I-FA^TA)^k = (X(I-\omega D_B^{-2}D_A^TD_A)X^{-1})^k = X(I-\omega D_B^{-2}D_A^TD_A)^kX^{-1} =$$
$$Xdiag((1-\omega\frac{\alpha_1^2}{\beta_1^2})^k,\ldots,(1-\omega\frac{\alpha_r^2}{\beta_r^2})^k,1,\ldots,1)X^{-1}.$$

Therefore, for $i\in\{1,\ldots,r\}$ we obtain

$$(I-FA^TA)^kX^i = Xdiag((1-\omega\frac{\alpha_1^2}{\beta_1^2})^k,\ldots,(1-\omega\frac{\alpha_r^2}{\beta_r^2})^k,1,\ldots,1)X^{-1}X^i =$$

$$Xdiag((1-\omega\frac{\alpha_1^2}{\beta_1^2})^k,\ldots,(1-\omega\frac{\alpha_r^2}{\beta_r^2})^k,1,\ldots,1)e_i = (1-\omega\frac{\alpha_i^2}{\beta_i^2})^kX^i. \tag{4.47}$$

In order to get the assumption (3.30) - (3.31), according to (4.47) we impose the equivalent condition

$$-1 < 1-\omega\frac{\alpha_i^2}{\beta_i^2} < 1, \forall i=1,\ldots,r \;\Leftrightarrow\; \omega\in(0,\frac{2}{\max_{1\le i\le r}\frac{\alpha_i^2}{\beta_i^2}}) \tag{4.48}$$

□

Remark 4.4. The case (4.43) corresponds to Landweber iteration, whereas (4.44) to Riley-Golub one (see Remark 1.1), but for computing x_{LS}^D instead of x_{LS}. Unfortunately in this weighted case a polynomial expression of F of the type

$$F = a_0I + a_1D^{-1}A^TA + \cdots + a_q(D^{-1}A^TA)^q$$

does not any more satisfy the assumption (3.30) - (3.31) as for the classical (non-weighted) Strand method (see (1.9)). This is due to the fact that in the GSVD decomposition (2.13) the invertible matrix X is not also orthogonal (i.e. $X^{-1}\neq X^T$).

We performed numerical experiments with both constructions (4.43) and (4.44) for the matrix F. In case (4.44) we used four matrices comming from rigid body dynamics problems (see e.g. [2] and [5]), with dimensions and ranks indicated in the first two columns of Table 1. We performed in each case 100 iterations and the optimal value of the parameter s was chosen experimentally. We indicated in each case the 1-norms of the corresponding errors (i.e. $\parallel z \parallel_1 = \max_{1\le i\le n}|z_i|, z\in I\!R^n$). The (good) results are presented in Table 1. The exact value of the weighted solution x_{LS}^D was computed using the $pinv$ Matlab function with the formula

$$x_{LS}^D = \begin{bmatrix} A \\ \sqrt{\sigma}D^{\frac{1}{2}} \end{bmatrix}^+ \begin{bmatrix} b \\ 0 \end{bmatrix},$$

for $\sigma = 10^{-15}$ (see for details [5]).
In case (4.43) we used three matrices coming from Electromagnetic geotomography (see e.g. [6]), with dimensions and ranks indicated in the first two columns of Table 2. The optimal value of the parameter ω was chosen experimentally. We indicated in each case the number of iterations used and the 1-norms of the corresponding errors. As expected, the behavior of the choice (4.43) for F is much less efficient than (4.44).

Dimension	Rank	$\parallel x_{LS} - x_{LS}^D \parallel_1$	$\parallel x^{100} - x^{99} \parallel_1$	$\parallel x^{100} - x_{LS}^D \parallel_1$	Opt. s
1155×1240	1000	393	10^{-11}	10^{-6}	10^{-1}
1013×570	570	393	10^{-11}	10^{-6}	10^{-1}
1200×6240	1200	2256	10^{-13}	10^{-5}	10^{-5}
282×498	282	583	10^{-11}	10^{-4}	10^{-5}

TABLE 1. Experiments with $F = sD + A^T A$.

Dimension	Rank	Opt. ω	Niter	$\parallel x_{LS} - x_{LS}^D \parallel_1$	$\parallel x^{100} - x^{99} \parallel_1$	$\parallel x^{100} - x_{LS}^D \parallel_1$
36×144	35	0.01	10^5	43	10^{-10}	10^{-6}
144×144	120	0.01	$5 \cdot 10^5$	18	10^{-6}	4
144×144	120	0.01	10^6	18	10^{-6}	2
144×144	120	0.01	$2 \cdot 10^6$	18	10^{-7}	1
576×144	133	0.006	10^5	35	10^{-5}	6
576×144	133	0.006	10^6	35	10^{-7}	0.3

TABLE 2. Experiments with $F = \omega D^{-1}$.

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[1] OVIDIUS UNIVERSITY OF CONSTANTA
FACULTY OF MATHEMATICS AND INFORMATICS
BD. MAMAIA 124, 900527, CONSTANTA, ROMANIA
E-mail address: doina.carp@gmail.com
E-mail address: cpopa@univ-ovidius.ro

[2] FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
INFORMATIK 10 - SYSTEM SIMULATION
CAUERSTRASSE 11, 91058, ERLANGEN, GERMANY
E-mail address: ulrich.ruede@fau.de
E-mail address: tobias.preclik@fau.de