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# A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized nonexpansive-type maps

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ABSTRACT. Let $K$ be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space with dual space $E^*$. In this paper, a new iterative algorithm of Halpern-type is constructed and used to approximate a common element of a generalized mixed equilibrium problem and a common fixed points for a countable family of *generalized nonexpansive-type maps*. Application of our theorem, in the case of real Hilbert spaces, complements, extends and improves several important recent results. Finally, we give numerical experiments to illustrate the convergence of our sequence.

## 1. INTRODUCTION

Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^*$. Let $K$ be a nonempty closed and convex subset of $E$ such that $JK$ is closed and convex where $J : E \to E^*$ is the normalized duality map on $E$. Let $\chi : JK \to \mathbb{R}$ be a map, $\Theta : JK \times JK \to \mathbb{R}$ be a bifunction and $B : K \to E^*$ be a nonlinear map. The *generalized mixed equilibrium problem* is to find an element $u \in K$ such that

$$\Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle \geq 0, \ \forall\, z \in K. \tag{1.1}$$

The set of solutions of the generalized mixed equilibrium problem is given by $GMEP(\Theta, B, \chi)$. It is well known that the class of generalized mixed equilibrium problems contains, as special cases, numerous important classes of nonlinear problems such as equilibrium problems, optimization problems, variational inequality problems, and so on (see e.g., Browder *et al.* [3], Onjai-Uea and Kumam [13] and the references contained in them).

Let $E$ be a real normed space with dual space $E^*$. A map $B : E \to 2^{E^*}$ is called *monotone* if for each $u, v \in E$, the following inequality holds: $\langle \eta - \gamma, u - v\rangle \geq 0,\ \eta \in Bu, \gamma \in Bv$. Consider, for example, the following: Let $g : E \to \mathbb{R}$ be a convex functional. The *subdifferential* of $g$, $\partial g : E \to 2^{E^*}$, is defined for each $u \in E$ by $\partial g(u) = \{u^* \in E^* : \langle v - u, u^*\rangle \leq g(v) - g(u),\ \forall\, v \in E\}$. It is easy to see that $\partial g$ is a monotone map on $E$ and that $0 \in \partial g(u)$ if and only if $u$ is a minimizer of $g$. Setting $\partial g = B$, it follows that solving the inclusion $0 \in Bu$, in this case, is equivalent to solving for a minimizer of $g$.

A map $B : E \to E$ is called *accretive* if for each $u, v \in E$, there exists $j(u - v) \in J(u - v)$ such that $\langle Bu - Bv, j(u - v)\rangle \geq 0$. For solving the equation $Bu = 0$, where $B$ is an *accretive operator*, Browder introduced a map, $T : E \to E$ defined by $T := I - B$, where $I$ is the identity map on $E$. He called such a map *pseudocontractive*. It is clear that solutions of $Bu = 0$, in this case, correspond to fixed points of $T$. Consequently, approximating zeros of accretive operators is equivalent to approximating fixed points of pseudocontractive

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maps, assuming existence. This fixed point technique, for obvious reasons, is not applicable in the case where $B : E \to E^*$ is a monotone map.

Motivated by the need to develop a fixed point technique for approximating a solution of the equation $Bu = 0$ when $B$ is monotone, a new notion of *fixed points for maps from $E$ to $E^*$* called *$J$-fixed points* has recently been introduced and studied (see e.g., Zegeye [17], Liu [10], Chidume and Idu [4], Chidume *et al.* [7], Chidume *et al.* [5]). This notion turns out to be very useful and applicable. For example, Chidume and Idu [4] introduced the concept of *$J$-pseudocontractive maps* and proved a strong convergence theorem for approximating $J$-fixed points of a $J$-pseudocontractive map. As an application of their theorem, they proved a strong convergence theorem for approximating a zero of an $m$-accretive operator (Corollary 4.1 of [4]).

It is our purpose in this paper to continue the study of $J$-fixed points and some of their applications. Here, we study a new *Halpern-type* algorithm and prove a strong convergence theorem for obtaining a common element in the solutions of a generalized mixed equilibrium problem and common fixed points for a countable family of generalized-$J$-nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. In the special case of a real Hilbert space, our theorem complements, extends and improves the results of Martinez-Yanes and Xu [11], Nakajo and Takahashi [12], Pen and Yao [14], Qin and Su [15], Tada and Takahashi [16], and a host of other recent results. Finally, we give numerical experiments to illustrate the convergence of our sequence.

## 2. PRELIMINARES

Let $E$ be a real normed space with dual space $E^*$. Consider a map $\phi : E \times E \to \mathbb{R}$ defined by $\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2,$ for all $x, y \in E$. This map which was introduced by Alber [1] will play a central role in the sequel.

The following lemmas will be needed in the sequel.

**Lemma 2.1.** (Alber, [1]) *Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then, the following are equivalent.*
*(i) $C$ is a sunny generalized nonexpansive retract of $E$,*
*(ii) $C$ is a generalized nonexpansive retract of $E$ and, (iii) $JC$ is closed and convex.*

**Lemma 2.2.** (Alber, [1]) *Let $C$ be a nonempty closed and convex subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$. Then, the following hold: (i) $z = Rx$ iff $\langle y - z, Jz - Jx\rangle \geq 0$, for all $y \in C$ and, (ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$, for all $z \in C$.*

**Lemma 2.3.** (Kamimura and Takahashi, [9]) *Let $E$ be a uniformly convex and uniformly smooth real Banach space and $\{x_n\}$, $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then, $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

**Definition 2.1.** (Chidume and Idu, [4]) A point $x^* \in C$ is called a *$J$-fixed point of $T$* if and only if $Tx^* = Jx^*$. The set of *$J$-fixed points of $T$* will be denoted by $F_J(T)$.

**Lemma 2.4.** (Chidume *et al.*, [6]) *Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^*$. Let $C$ be a closed subset of $E$ such that $JC$ is closed and convex. Let $T$ be a generalized-$J$-nonexpansive map from $C$ to $E^*$ with $F_J(T) \neq \emptyset$. Then, $F_J(T)$ is closed and $JF_J(T)$ is closed and convex.*

**Lemma 2.5.** (Chidume *et al.*, [6]) *Let $E$ be a smooth, strictly convex and reflexive real Banach with dual space $E^*$. Let $C$ be a closed subset of $E$ such that $JC$ is closed and convex. Let $T$ be a* *generalized-J-nonexpansive map from $C$ to $E^*$ such that $F_J(T) \neq \emptyset$. Then, $F_J(T)$ is a sunny generalized-$J$-nonexpansive retract of $E$.*

**Remark 2.1.** (Chidume *et al.*, [6]) From lemma 2.4 we have that $JF_J(T)$ and $JGMEP$ are closed and convex. Since $J$ is one-to-one, we have that $J(F_J(T) \cap GMEP(f, A, \varphi)) = JF_J(T) \cap JGMEP(f, A, \varphi)$. By lemma 2.1, we obtain that $F_J(T) \cap GMEP(f, A, \varphi)$ is a sunny generalized-$J$-nonexpansive retract of $E$.

**Basic Assumptions.** Let $K$ be a nonempty closed subset of a smooth, strictly convex and reflexive real Banach space $E$ with dual space $E^*$ such that $JK$ is closed and convex. Let $\chi : JK \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to E^*$ be continuous and monotone. For solving the generalized mixed equilibrium problems, (1.1), we assume that the bifunctional $\Theta : JK \times JK \to \mathbb{R}$ satisfies the following conditons:
$(B_1)$ $\Theta(u^*, u^*) = 0,\ for\ all\ u^* \in JK$
$(B_2)$ $\Theta$ is monotone, i.e. $\Theta(u^*, v^*) + \Theta(v^*, u^*) \leq 0,\ for\ all\ u^*, v^* \in JK,$
$(B_3)$ $\limsup_{\lambda \downarrow 0} \Theta(u^* + \lambda(z^* - u^*), v^*) \leq \Theta(u^*, v^*),\ for\ all\ u^*, v^*, z^* \in JK,$
$(B_4)$ $\Theta(u^*, \cdot)$ is convex and lower semi-continuous, for all $u^* \in JK$.

## 3. Main results

Let $C$ be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$ with dual space $E^*$. Let $J$ and $J^{-1}$ be the normalized duality maps on $E$ and $E^*$, respectively. Clearly, $J^{-1} = J_*$ exists under this setting.

**Definition 3.2.** (Chidume *et al.* [6]) A map $T : C \to E^*$ is called *generalized-$J$-nonexpansive* if $F_J(T) \neq \emptyset$ and $\phi((J^{-1}oT)x, p) \leq \phi(x, p)$, for all $x \in C$, for all $p \in F_J(T)$.

***NST-Condition***. Let $\{S_n\}$ and $\Upsilon$ be two families of generalized-$J$-nonexpansive maps from $C$ into $E^*$ such that $\cap_{n=1}^{\infty} F_J(S_n) = F_J(\Upsilon) \neq \emptyset$, where $F_J(S_n)$ is the set of $J$-fixed points of $S_n$ and $F_J(\Upsilon)$ is the set of $J$-fixed points of $\Upsilon$. A sequence $\{S_n\}$ from $C$ to $E^*$ is said to satisfy the *NST-condition with* $\Upsilon$ if for each bounded sequence $\{x_n\} \subset C$,
$\lim_{n\to\infty} ||Jx_n - S_n x_n|| = 0 \implies \lim_{n\to\infty} ||Jx_n - Sx_n|| = 0, \forall\, S \in \Upsilon.$

We now prove the following theorem.

**Theorem 3.1.** *Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^*$. Let $K$ be a nonempty closed and convex subset of $E$ such that $JK$ is closed and convex. Let $\chi : JK \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to E^*$ be a continuous and monotone map. Let $\Theta : JK \times JK \to \mathbb{R}$ be a bifunction satisfying conditions $(B_1)-(B_4)$. Let $S_n : K \to E^*, n = 1, 2, \cdots$ be a countable family of generalized-$J$-nonexpansive maps and $\Upsilon$ be a family of closed and generalized-$J$-nonexpansive maps from $K$ to $E^*$ such that $\cap_{n=1}^{\infty} F_J(S_n) \cap GMEP(\Theta, B, \chi) = F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \neq \emptyset$, $\beta_n \in (0, 1)$ with $\lim \beta_n = 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Let $\{x_n\}$ be generated by:*

$$\begin{cases} x_1 = x \in K,\ K_1 = K, \\ z_n = \beta_n x + (1 - \beta_n)(J^{-1}oS_n)x_n, \\ u_n = T_{r_n} z_n, \\ K_{n+1} = \big\{v \in K_n : \phi(u_n, v) \leq \beta_n \phi(x, v) + (1 - \beta_n)\phi(x_n, v)\big\}, \\ x_{n+1} = R_{K_{n+1}} x,\ \forall\, n \geq 1. \end{cases} \tag{3.2}$$

*Assume that $\{S_n\}$ satisfies the NST-condition with $\Upsilon$, then $\{x_n\}$ converges strongly to $R_{F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)} x$, where $R_{F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)}$ is the sunny generalized-$J$-nonexpansive retraction of $E$ onto $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$.*

*Proof.* The proof is divided into 5 steps.

**Step 1:** The sequence $\{x_n\}$ is well defined and $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$.
First, we show that $JK_n$ is closed and convex. Clearly $K_1 = K$ is closed and convex. Assume that $K_n$ is closed and convex for some $n \geq 1$, applying the definition of $K_{n+1}$, it is clear that $K_{n+1} = \{v \in K_n : 2\langle \beta_n x + (1-\beta_n)x_n - u_n, Jv\rangle \leq \beta_n ||x||^2 + (1-\beta_n)||x_n||^2 - ||u_n||^2\}$. Thus, $K_{n+1}$ is closed and convex. Hence $JK_n$ is closed and convex. By lemma 2.1, $K_n$ is a sunny generalized-$J$-nonexpansive retract of $E$. Hence, $\{x_n\}$ is well defined. Next, we show that $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n, \ \forall n \geq 1$. Clearly $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$ is a subset of $K_1$. Assume that $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$ for some $n \geq 1$. Let $q \in F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$. By a result of Zhang [18] and definition of $S_n$, we have that

$$\begin{aligned}
\phi(u_n, q) &= \phi(T_{r_n} z_n, q) \leq \phi(z_n, q) \\
&= ||\beta_n x + (1-\beta_n)(J^{-1}oS_n)x_n||^2 - 2\big\langle \beta_n x + (1-\beta_n)(J^{-1}oS_n)x_n, Jq\big\rangle + ||q||^2 \\
&\leq \beta_n \phi(x, q) + (1-\beta_n)\phi((J^{-1}oS_n)x_n, q) \leq \beta_n\phi(x,q) + (1-\beta_n)\phi(x_n, q).
\end{aligned} \tag{3.3}$$

This implies that $q \in K_{n+1}$. Hence, $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n, \ \forall\, n \geq 1$.

**Step 2:** $\lim\limits_{n\to\infty} x_n = x^*$, $\lim\limits_{n\to\infty} u_n = x^*$ and $\lim\limits_{n\to\infty} z_n = x^*$.
First, we show that $\{x_n\}$ is bounded. From the definition of $\{x_n\}$ and lemma 2.2, $(ii)$, we have that $\phi(x, x_n) = \phi(x, R_{K_n}x) \leq \phi(x, q) - \phi(R_{K_n}x, q) \leq \phi(x, q)$, for every $q$ in $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$. This implies that $\{\phi(x, x_n)\}$ is bounded. It follows from the definition of $\phi$ that $\{x_n\}$ is bounded. Since $x_{n+1} = R_{K_{n+1}}x \in K_{n+1} \subset K_n$ and $x_n = R_{K_n}x$, we have that $\phi(x, x_n) \leq \phi(x, x_{n+1})$ and this implies that $\{\phi(x, x_n)\}$ is nondecreasing. Hence, $\lim\limits_{n\to\infty} \phi(x, x_n)$ exists. Also, for $m > n$, from lemma 2.2 and $x_n = R_{K_n}x$, we have that

$$\begin{aligned}
\phi(x_n, x_m) = \phi(R_{K_n}x, R_{K_m}x) &\leq \phi(x, R_{K_m}x) - \phi(x, R_{K_n}x) \\
&= \phi(x, x_m) - \phi(x, x_n) \to 0 \ (as\ n \to \infty).
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} \phi(x_n, x_m) = 0$. It follows from a Lemma 2.3 that $\lim\limits_{n\to\infty} ||x_n - x_m|| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $K$. Thus, there exists $x^* \in K$ such that $\lim\limits_{n\to\infty} x_n = x^*$. From inequality (3.3) and using the fact that $\lim\limits_{n\to\infty} \beta_n = 0$ by assumption, it follows that $\phi(u_n, x_m) \leq \beta_n \phi(x, x_m) + (1-\beta_n)\phi(x_n, x_m) \to 0 \ (as\ n \to \infty)$. By Lemma 2.3, we have that

$$\lim_{n\to\infty} ||u_n - x_m|| = 0. \text{ Hence, } \lim_{n\to\infty} ||u_n - x_n|| = 0. \text{ This implies that } \lim_{n\to\infty} u_n = x^*. \tag{3.4}$$

From inequality (3.3), a result of Zhang [18] and equation (3.4), we get that

$$\begin{aligned}
\phi(z_n, u_n) &= \phi(z_n, T_{r_n} z_n) \leq \phi(z_n, q) - \phi(u_n, q) \\
&\leq \beta_n \phi(x, q) + (1-\beta_n)\phi(x_n, q) - \phi(u_n, q) \\
&\leq \beta_n \phi(x, q) + \phi(x_n, q) - \phi(u_n, q) \to 0.
\end{aligned}$$

By Lemma 2.3, it follows that $\lim\limits_{n\to\infty} ||u_n - z_n|| = 0$. Thus, $\lim\limits_{n\to\infty} z_n = x^*$. Using this and equation (3.4), we conclude that $\lim\limits_{n\to\infty} x_n = x^*$, $\lim\limits_{n\to\infty} u_n = x^*$ and $\lim\limits_{n\to\infty} z_n = x^*$.

**Step 3:** $\lim\limits_{n\to\infty} ||Jx_n - Sx_n|| = 0, \ \forall\, S \in \Upsilon$.
From equation (3.1), we obtain that

$$(1-\beta_n)||x_n - (J^{-1}oS_n)x_n|| \leq ||x_n - z_n|| + \beta_n ||x_1 - x_n||. \tag{3.5}$$

First, we observe that $\{(J^{-1}oS_n)x_n\}$ is bounded in $E$. Using step 2 and the fact that $\lim\limits_{n\to\infty}\beta_n = 0$ by assumption in inequality (3.5), we obtain that $\lim\limits_{n\to\infty}||x_n-(J^{-1}oS_n)x_n|| = 0$. By uniform continuity of $J$ on bounded subset of $E$, we get that $\lim\limits_{n\to\infty}||Jx_n - S_nx_n|| = 0$. Since $\{S_n\}$ satisfies the NST-condition with $\Upsilon$, we conclude that $\lim\limits_{n\to\infty}||Jx_n - Sx_n|| = 0$, $\forall\, S \in \Upsilon$.

**Step 4:** $x^* \in F_J(\Upsilon)\cap GMEP(\Theta, B, \chi)$.
From step 3, we have that $\lim\limits_{n\to\infty}||Jx_n - Sx_n|| = 0,\ \forall\, S \in \Upsilon$. We also proved that $x_n \to x^* \in K$. Since $S$ is closed, we conclude that $x^* \in F_J(\Upsilon)$. Furthermore, from step 2 and the property of $J$ on $E$, we get that $\lim\limits_{n\to\infty}||Jz_n-Ju_n|| = 0$. Since $\{r_n\} \subset [a,\infty)$ by assumption, we obtain that $\lim\limits_{n\to\infty}\frac{||Jz_n-Ju_n||}{r_n} = 0$. Since $u_n = T_{r_n}z_n$ in equation (3.2) and by a result of Zhang [18], we have that

$$F(Ju_n, Jz) + \frac{1}{r_n}\Big\langle z - u_n, Ju_n - Jz_n\Big\rangle \geq 0,\ \forall\, z \in K. \tag{3.6}$$

By $B_2$, we have that $\frac{1}{r_n}\Big\langle z-u_n, Ju_n-Jz_n\Big\rangle \geq F(Jz, Ju_n)$. Since $z \mapsto F(Ju, Jz)$ is convex and lower semi-continuous, we obtain from the above inequality that $0 \geq F(Jz, Jx^*)$, $\forall\, z \in K$. For $\lambda \in (0,1]$ and $z \in K$, letting $z^*_\lambda = \lambda Jz + (1-\lambda)Jx^*$, then $z^*_\lambda \in JK$ since $JK$ is closed and convex. Hence, $0 \geq F(z^*_\lambda, Jx^*),\ \forall\, z \in K$. By $B_1$, we have that

$$0 = F(z^*_\lambda, z^*_\lambda) \quad\leq\quad \lambda F(z^*_\lambda, Jz) + (1-\lambda)F(z^*_\lambda, Jx^*) \leq F(Jx^* + \lambda(Jz - Jx^*), Jz).$$

Letting $\lambda \downarrow 0$, by $B_3$, we obtain that $F(Jx^*, Jz) \geq 0$. Hence, $x^* \in GMEP(\Theta, B, \chi)$. Using this and the fact that $x^* \in F_J(\Upsilon)$, we conclude $x^* \in F_J(\Gamma)\cap GMEP(\Theta, B, \chi)$.

**Step 5:** $\lim\limits_{n\to\infty} x_n = R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x$. From Lemma 2.2, we obtain that

$$\phi(x, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x) \leq \phi(x, x^*). \tag{3.7}$$

Also, for $x^* \in F_J(\Upsilon)\cap GMEP(\Theta, B, \chi) \subset K_{n+1},\ x_{n+1} = R_{K_{n+1}}x$, and by Lemma 2.2, we have that $\phi(x, x_{n+1}) \leq \phi(x, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x)$. Since $\lim\limits_{n\to\infty} x_n = x^*$, we get that $\phi(x, x^*) \leq \phi(x, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x)$. Using this and inequality (3.7), we get that $\phi(x, x^*) = \phi(x, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x)$. By uniqueness of $R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x$, we conclude that $x^* = R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}x$. The proof is complete. □

## 4. An example

Let $E = l_p,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$, $K = \overline{B_{l_p}}(0,1) = \{u \in l_p : ||u||_{l_p} \leq 1\}$.
Consider the following maps:
$\chi : JK \to \mathbb{R}$ defined by $\chi(u^*) = ||u^*||,\ \forall\, u^* \in JK$;
$\Theta : JK \times JK \to \mathbb{R}$ defined by $\Theta(u^*, v^*) = \langle J^{-1}u^*, v^* - u^*\rangle,\ \forall\, v^* \in JK$;
$B : K \to l_q$ defined by $Bu = J(u_1, u_2, u_3, \cdots),\ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$;
$S : K \to l_q$ defined by $Su = J(0, u_1, u_2, u_3, \cdots),\ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$;
$S_n : K \to l_q$ defined by $S_nu = J(\alpha_n u + (1-\alpha_n)J^{-1}oSu),\ \forall\, n \geq 1,\ u \in K$ and $\alpha_n \in (0,1)$, $\liminf \alpha_n(1-\alpha_n) > 0$.

Let $\beta_n := \frac{1}{n+1},\ \forall\, n \geq 1,\ \{r_n\} \subset [1,\infty),\ \forall\, n \geq 1$ and $\Upsilon = S$. Then,

$(a)$ $E,\ K,\ JK,\ \chi,\ \Theta$ and $B$ satisfy all the conditions of Theorem 3.1. In particular, $\Theta$ satisfies conditions $(B_1)$ to $(B_4)$ as follows: conditions $(B_1)$ and $(B_4)$ follow easily from direct computation; $(B_2)$ follows from the monotonicity of the normalized duality map $J^{-1}$, and condition $(B_3)$ follows from the continuity of $J^{-1}$. Furthermore, $0 \in GMEP(\Theta, B, \chi)$.

$(b)$ $S_n$ is a generalized-$J$-nonexpansive map and satisfies the NST-condition with $\Upsilon$. $F_J(S) = F_J(S_n) = F_J(\Upsilon) = \{0\},\ \forall\, n \geq 1$. Moreover, $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) = \{0\}$.

Hence, by Theorem 3.1, the sequence $\{x_n\}$ generated by equation (3.2) converges strongly to an element of $F(\Upsilon) \cap GMEP(\Theta, B, \chi)$. This completes the example.

**Remark 4.2.** Theorem 3.1 is applicable in $L_p,\ l_p$ and $W_p^m(\Omega)$ spaces, $1 < p < \infty$, where $W_p^m(\Omega)$ denote the usual Sobolev space, since these spaces are uniformly convex and uniformly smooth. For the analytical representations of $J$ and $J^{-1}$ in these spaces where $p^{-1} + q^{-1} = 1$, the reader is referred to Theorem 3.3, of Alber and Ryazantseva [2]; page 36.

In the case that $E$ is a real Hilbert space, we have the following corollary.

**Corollary 4.1.** *Let $H$ be a real Hilbert space. Let $K$ be a nonempty closed and convex subset of $H$. Let $\chi : K \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to H$ be a continuous and monotone map. Let $\Theta : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions $(B_1) - (B_4)$. Let $S_n : K \to H,\ n = 1, 2, \cdots$ be a countable family of generalized nonexpansive maps and $\Upsilon$ be a family of closed and generalized nonexpansive maps from $K$ to $H$ such that $\cap_{n=1}^{\infty} F(S_n) \cap GMEP(\Theta, B, \chi) = F(\Upsilon) \cap GMEP(\Theta, B, \chi) \neq \emptyset$, $\beta_n \in (0,1)$ with $\lim_{n\to\infty} \beta_n = 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Let $\{x_n\}$ generated by:*

$$
\begin{cases}
x_1 = x \in K,\ K_1 = K,\\
z_n = \beta_n x + (1-\beta_n) S_n x_n,\\
u_n = T_{r_n} z_n,\\
K_{n+1} = \big\{v \in K_n : ||u_n - v||^2 \leq \beta_n ||x - v||^2 + (1-\beta_n)||x_n - v||^2\big\},\\
x_{n+1} = P_{K_{n+1}} x, \quad \forall\, n \geq 1.
\end{cases}
\tag{4.8}
$$

*Assume that $\{S_n\}$ satisfies the NST-condition with $\Upsilon$, then, $\{x_n\}$ converges strongly to $P_{F(\Upsilon)\cap GMEP(\Theta,B,\chi)} x$.*

*Proof.* In a Hilbert space, $J$ is the identity map and $\phi(y, z) = ||y - z||^2,\ \forall\, y, z \in H$. The result follows from Theorem 3.1. □

**Remark 4.3.** Theorem 3.1 extends and improves the theorem of Martinez-Yanes and Xu [11], Nakajo and Takahashi [12], in the sense that these theorems are special cases of Theorem 3.1 in which $E$ is a real Hilbert space. Furthermore, in the theorem of Martinez-Yanes and Xu [11], $T$ is a single self-map on $C \subset E$ while in Theorem 3.1, $\{S_n\}$ is a family of maps from a subset $C \subset E$ *to the dual space* $E^*$. Finally, in theorem 3.1, generalized mixed equilibrium problem is also studied which is not the case in either the theorem of Martinez-Yanes and Xu [11] or that of Nakajo and Takahashi [12].

**Remark 4.4.** Corollary 4.1 improves significantly the result in Pen and Yao [14], Qin and Su [15], Tada and Takahashi [16] in the following sense:

(1) In Corollary 4.1, the set of generalized mixed equilibrium problem is studied which is not considered in Pen and Yao [14], Qin and Su [15], Tada and Takahashi [16].

(2) Corollary 4.1 extends the result in Pen and Yao [14], Qin and Su [15], Tada and Takahashi [16] from a nonexpansive self-map to a countable family of generalized nonexpansive non self-maps.

(3) The iteration process of Corollary 4.1 is more efficient than that considered in Pen and Yao [14] which requires more arithmetic at each stage to implement because of the extra $y_n$ and $z_n$ terms involved in the iteration process.

(4) Finally, the sequence of *Halpern-type* algorithm considered in theorem 4.1 requires less computation time at each step of the iteration process than the sequence of *Mann-type* algorithm studied in Pen and Yao [14], Qin and Su [15], Tada and Takahashi [16], thereby reducing computational cost.

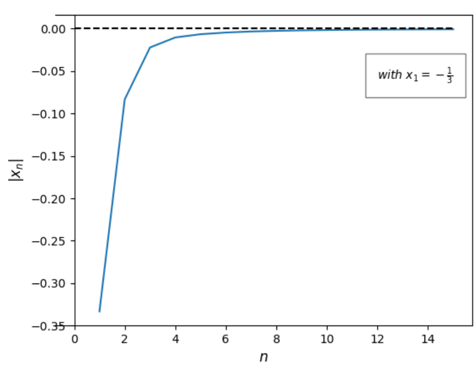
## 5. NUMERICAL EXPERIMENTS

Here, we present numerical examples to illustrate the convergence of our sequence $\{x_n\}$ in Theorem 3.1.

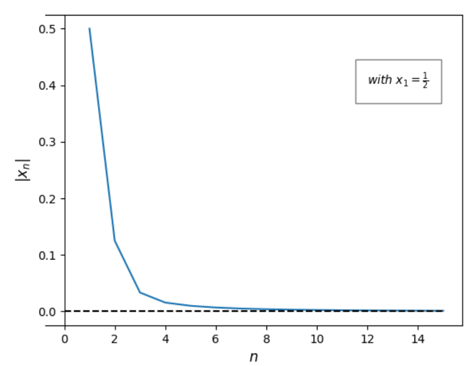
**Example 5.1.** Let $E = \mathbb{R}$, $K = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$. Clearly, $x \in \mathbb{R}$,

$$P_K x \begin{cases} \alpha, & if \quad x < \alpha, \\ x, & if \quad x \in [\alpha, \beta], \\ \beta, & if \quad x > \beta. \end{cases} \tag{5.9}$$

Now, set $K = [-1, 3]$ and $Sx = sin(x)$ in Theorem 3.1. Clearly, $S$ is generalized nonexpansive with $0$ as its unique fixed point. With $x_1 = \frac{-1}{3}$ and $x_1 = \frac{1}{2}$ in $K$ respectively, by Theorem 3.1, the sequence generated by algorithm (3.2) converges strongly to zero. The numerical results are sketched in figure $(1)$ with initial point $x_1 = \frac{-1}{3}$ and figure $(2)$ with initial point $x_1 = \frac{1}{2}$, respectively, where the $y$-$axis$ represents the value of $|x_n - 0|$ while the $x$-$axis$ represents the number of iterations $(n)$.




Numerical example with $x_1 = \dfrac{-1}{3}$




Numerical example with $x_1 = \dfrac{1}{2}$

All computations and graphs were implemented in python 3.6 using some abstractions developed at *AUST* and other open source python library such as numpy and matplotlib on Zinox with intel core $i7$ processor.

## REFERENCES


[1] Alber, Y., *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Appl. of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), 15–50

[2] Alber, Y. and Ryazantseva, I., *Nonlinear Ill Posed problems of Monotone Type*, Springer, London, UK, 2006

[3] Browder, F. E., *Existence and approximation of solutions of nonlinear variational inequalities*, Proc. Natl. Acad. Sci. USA **56** (1966), No. 4, 1080–1086

[4] Chidume, C. E. and Idu, K. O., *Approximation of zeros of bounded maximal monotone maps, solutions of Hammerstein integral equations and convex minimization problems*, FPTA, **97** (2016), DOI 10.1186/s13663-016-0582-8

[5] Chidume, C. E., Otubo, E. E., Ezea, C. G. and Uba, M. O., *A new monotone hybrid algorithm for a convex feasibility problems for an finite family of nonexpansive-type maps with applications*, AFPT, Vol. 7 (2017), No. 3, 413–431

[6] Chidume, C. E., Nnakwe, M. O. and Otubo, E. E., *A new iterative algorithm for a generalized mixed equilibrium problem and a countable family of nonexpansive-type maps, with applications*, Submitted to FPT, 2017

[7] Chidume, C. E., Uba, M. O., Uzochukwu, M. I, Otubo, E. E. and Idu, K. O., *A strong convergence theorem for zeros of maximal monotone maps with applications to convex minimization and variational inequality problems*, Proc. Edinburgh Math. Soc., Accepted January, 2018

[8] Ibaraki, T. and Takahashi, W., *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory, **149** (2007), 1–14

[9] Kamimura, S. and Takahashi, W., *Strong convergence of a proximal-type algorithm in a Banach space*, SIAMJ. Optim., **13** (2002), No. 3, 938–945

[10] Liu, B., *Fixed point of strong duality pseudocontractive mappings and applications*, APA Vol. 2012, Article ID 623625, 7 pages, Doi:10.1155/2012/623625

[11] Martinez-Yanes, C. and Xu, H. K., *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64** (2006), 2400–2411

[12] Nakajo, K. and Takahashi, W., *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372–379.

[13] Onjai-uea, N. and Kumam, P., *Algorithms of Common Solutions to Generalized Mixed Equilibrium Problem and a System of Quasi variational Inclusions for Two Difference Nonlinear Operators in Banach Spaces*, Hindawi Publishing Corporation FPTA Vol. 2011, Article ID 601910, 23 pages Doi:10.1155/2011/601910

[14] Peng, J. and Yao, J., *A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems*, Taiwanese J. Math., **12** (2008), No. 6, 1401–1432

[15] Qin, X. and Su, Y., *Strong convergence of monotone hybrid method for fixed point iteration process*, J. Syst. Sci. and Complexity, **21** (2008), 474–482

[16] Tada, A. and Takahashi, W., *Weak and Strong Convergence Theorems for a Nonexpansive Mapping and an Equilibrium Problem*, J. Optim. Theory Appl., **133** (2007), 359–370

[17] Zegeye, H., *Strong convergence theorems for maximal monotone mappings in Banach spaces*, J. Math. Anal. Appl., **343** (2008) 663–671

[18] Zhang, S., *Generalized mixed equilibrium problems in Banach spaces* , Appl. Math. Mech. -Engl. Ed. 30(9) (2009), 1105-1112 DOI: 10.1007/s10483-009-0904-6



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