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Approximation by generalized Stancu type integral operators involving Sheffer polynomials

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ABSTRACT. In this article, we give a generalization of integral operators which involves Sheffer polynomials introduced by Sucu and Büyükyazici. We obtain approximation properties of our operators with the help of the universal Korovkin's theorem and study convergence properties by using modulus of continuity, the second order modulus of smoothness and Peetre's K-functional. We have also established Voronovskaja type asymptotic formula. Furthermore, we study the convergence of these operators in weighted spaces of functions on the positive semi-axis and estimate the approximation by using weighted modulus of continuity.

1. INTRODUCTION

In 1950, Szász [28] defined the positive linear operators:

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \geq 0$ and for functions $f \in C[0,\infty)$ for which the series is convergent. Motivated by this work, many authors have investigated several interesting properties of these operators. Mazhar and Totik [19] modified the Szász operators given by (1.1) and defined another class of linear positive operators

$$S_n^*(f;x) := ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)dt. \tag{1.2}$$

In 1969, Jakimovski and Leviatan [17] introduced a generalization of Szász operators by means of Appell polynomials. Let $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disk $|z| < R$, $(R > 1)$ and suppose that $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have generating functions of the form

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.3}$$

Under the assumption that $p_k(x) \geq 0$ for $x \in [0,\infty)$, Jakimovski and Leviatan introduced the positive linear operators $P_n(f;x)$ via

$$P_n(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.4}$$

and gave the approximation properties of the operators.

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Ciupa [9] modified the sequence of operators (1.4) as follows:

$$P_n^*(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)dt, \tag{1.5}$$

where Γ denotes gamma function and $\lambda \geq 0$.

Case 1. For $g(1) = 1$, with the help of (1.3) we easily find $p_k(x) = \frac{x^k}{k!}$ and from (1.4), we meet again the Szász operators given by (1.1).

Case 2. For $g(1) = 1$ and $\lambda = 0$ the sequence of operators defined by (1.5) becomes operators S_n^* given by (1.2).

Atakut and Büyükyazici modified the operators (1.5) as follows:

$$P_n^*(f;a_n,b_n;x) := \frac{e^{-a_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-b_n t} t^{\lambda+k} f(t)dt, \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$ are strictly increasing sequences of positive real numbers satisfying $\lim\limits_{n\to\infty} \frac{1}{b_n} = 0$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1 + O\big(\frac{1}{b_n}\big)$ and $p_k(x) = \sum\limits_{r=0}^{k} a_{k-r} b_r x^r,\ k = 0, 1, 2, \ldots$ are Appell type polynomials given by the generating functions (1.3).

Ismail [15] presented another generalization of Szász operators (1.1) and Jakimovski and Leviatan operators (1.4) by using Sheffer polynomials. Let $A(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ and $H(z) = \sum\limits_{k=1}^{\infty} h_k z^k$ $(h_1 \neq 0)$ be analytic functions in the disk $|z| < R$ $(R > 1)$ where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the type

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x) t^k,\ |t| < R. \tag{1.7}$$

Using the following assumptions:

(i) for $x \in [0, \infty)$, $p_k(x) \geq 0$,
(ii) $A(1) \neq 0$ and $H'(1) = 1$.

Ismail investigated the approximation properties of the positive linear operators given by

$$T_n(f;x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \text{for } n \in \mathbb{N}. \tag{1.8}$$

Sucu and Büyükyazici [27] revised the operators T_n via

$$T_n^*(f;x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)dt, \tag{1.9}$$

and gave the approximation properties of these operators.

Case 1. For $H(t) = t$, it can be easily seen that the generating functions (1.7) return to (1.3) and, from this fact, the operators (1.8) and (1.9) reduces to the operators (1.4) and (1.5), respectively.

Case 2. For $H(t) = t$ and $A(t) = 1$, one get the Szász operators from the operators (1.8).

Case 3. For $H(t) = t$, $A(t) = 1$ and $\lambda = 0$ the operator defined by (1.9) becomes operator S_n^* given by (1.2).

In this paper, inspired by the operators (1.6) we introduce Stancu type generalization of the operators (1.9) given by

$$T^*_{n,\alpha,\beta}(f;x) = \frac{e^{-a_n x H(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(a_n x) \frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-b_n t} t^{\lambda+k} f\left(\frac{b_n t+\alpha}{b_n+\beta}\right) dt, \tag{1.10}$$

where $0 \leq \alpha \leq \beta$ are two real parameters and $\{a_n\}$, $\{b_n\}$ are strictly increasing sequences of positive real numbers satisfying $\lim_{n\to\infty} \frac{1}{b_n} = 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = 1+O\left(\frac{1}{b_n}\right)$ and $p_k(x)$ are Sheffer polynomials given by the generating functions (1.7). The case $a_n = b_n = n$ and $\alpha = \beta = 0$ yield the operator T_n^* given by (1.9). For more results on such type of operators, we refer the readers to see [6, 20, 21, 22].

The purpose of this paper is to study the convergence properties of our constructed operators (1.10) by using modulus of continuity, second order modulus of smoothness and Peetre's K-functional. Some graphical examples are also given to in claim of convergence of operators towards the function. We have also established Voronovskaja type asymptotic formula. Furthermore, we study the convergence of these operators in weighted space. One of the most useful examples of positive linear operators is Szász-Mirakyan operators. Recently, many generalizations of these operators have been intensively studied in a different direction (see [1], [2],[3],[5],[26]).

We propose the readers to study the further properties of the operators such as convergence properties via summability method (see, for example [7, 8, 11, 18, 23, 24, 25]).

2. Moments estimation and preliminary results

To obtain the moments of the operators (1.10), we need the following Lemmas:

Lemma 2.1. *By* (1.7), *we obtain that*

$$\begin{aligned}
\sum_{k=0}^{\infty} p_k(a_n x) =& A(1)e^{a_n x H(1)},\\
\sum_{k=0}^{\infty} k p_k(a_n x) =& [a_n x A(1) + A'(1)]e^{a_n x H(1)},\\
\sum_{k=0}^{\infty} k^2 p_k(a_n x) =& [a_n^2 x^2 A(1) + a_n x\{A(1)(1+H''(1)) + 2A'(1)\} + A''(1) + A'(1)]e^{a_n x H(1)},\\
\sum_{k=0}^{\infty} k^3 p_k(a_n x) =& [a_n^3 x^3 A(1) + a_n^2 x^2\{3A(1)(1+H''(1)) + 3A'(1)\} + a_n x\{A(1)(H'''(1) + 3H''(1) + 1)\\
&+ 3A'(1)(2+H''(1)) + 3A''(1)\} + (A'(1) + 3A''(1) + A'''(1))]e^{a_n x H(1)},\\
\sum_{k=0}^{\infty} k^4 p_k(\alpha_n x) =& [a_n^4 x^4 A(1) + a_n^3 x^3\{6A(1)(1+H''(1)) + 4A'(1)\} + a_n^2 x^2\{A(1)(4H'''(1) + 3(H''(1))^2\\
&+ 18H''(1) + 7) + A'(1)(12H''(1) + 18) + 6A''(1)\} + a_n x\{A(1)(H^{(iv)}(1) + 6H'''(1)\\
&+ 7H''(1) + 1) + A'(1)(4H'''(1) + 18H''(1) + 14) + A''(1)(6H''(1) + 18) + 4A'''(1)\}\\
&+ A'(1) + 7A''(1) + 6A'''(1) + A^{(iv)}(1)]e^{a_n x H(1)}.
\end{aligned}$$

Lemma 2.2. *Let* $\left\{T^*_{n,\alpha,\beta}\right\}$ *be the sequence of operators given by* (1.10). *For each* $x \geq 0$, *the following equalities hold:*

(1) $T^*_{n,\alpha,\beta}(1;x)=1$,

(2) $T^*_{n,\alpha,\beta}(t;x)=\frac{1}{b_n+\beta}\left[a_n x+1+\lambda+\alpha+\frac{A'(1)}{A(1)}\right]$,

(3) $T^*_{n,\alpha,\beta}(t^2;x)=\frac{1}{(b_n+\beta)^2}\left[a_n^2x^2+a_nx\left\{2(\lambda+\alpha+2)+H''(1)+2\frac{A'(1)}{A(1)}\right\}+\left\{(\lambda+1)(\lambda+2)+2\alpha(\lambda+1)+\alpha^2+2(\lambda+\alpha+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\right\}\right]$,

(4) $T^*_{n,\alpha,\beta}(t^3;x)=\frac{1}{(b_n+\beta)^3}\left[a_n^3x^3+a_n^2x^2\left\{3(\lambda+\alpha+3)+3H''(1)+3\frac{A'(1)}{A(1)}\right\}+a_nx\left\{3\lambda^2+12\lambda+12+3\alpha(2\lambda+3)+3\alpha^2+H'''(1)+3H''(1)+3(\lambda+\alpha+2)(1+H''(1)+2\frac{A'(1)}{A(1)})+3(H''(1)+2)\frac{A'(1)}{A(1)}+3\frac{A''(1)}{A(1)}\right\}+\left\{(\lambda+1)(\lambda+2)(\lambda+3)+3\alpha(\lambda+1)(\lambda+2)+\alpha^3+(3\lambda^2+12\lambda+11+3\alpha(2\lambda+3)+3\alpha^2)\frac{A'(1)}{A(1)}+3(\lambda+\alpha+2)\frac{A''(1)+A'(1)}{A(1)}+\frac{A'''(1)+3A''(1)+A'(1)}{A(1)}\right\}\right]$,

(5) $T^*_{n,\alpha,\beta}(t^4;x)=\frac{1}{(b_n+\beta)^4}\left[a_n^4x^4+a_n^3x^3\left\{4(\lambda+\alpha+4)+6H''(1)+4\frac{A'(1)}{A(1)}\right\}+a_n^2x^2\left\{6\lambda^2+30\lambda+42+4\alpha(3\lambda+6)+6\alpha^2+4H'''(1)+3(H''(1))^2+18H''(1)+6(2\lambda+2\alpha+5)(1+H''(1)+\frac{A'(1)}{A(1)})+(12H''(1)+18)\frac{A'(1)}{A(1)}+6\frac{A''(1)}{A(1)}\right\}+a_nx\left\{4\lambda^3+30\lambda^2+70\lambda+51+4\alpha(3\lambda^2+12\lambda+11)+6\alpha^2(2\lambda+3)+4\alpha^3+H^{iv}(1)+6H'''(1)+7H''(1)+(6\lambda^2+30\lambda+35+4\alpha(3\lambda+6)+6\alpha^2)(1+H''(1)+2\frac{A'(1)}{A(1)})+2(2\lambda+2\alpha+5)(H'''(1)+3H''(1)+1+3(H''(1)+2)\frac{A'(1)}{A(1)}+3\frac{A''(1)}{A(1)})+(4H'''(1)+18H''(1)+14)\frac{A'(1)}{A(1)}+(6H''(1)+18)\frac{A''(1)}{A(1)}+4\frac{A'''(1)}{A(1)}\right\}+\left\{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)+4\alpha(\lambda+1)(\lambda+2)(\lambda+3)+6\alpha^2(\lambda+1)(\lambda+2)+4\alpha^3(\lambda+2)+\alpha^4+(4\lambda^3+30\lambda^2+70\lambda+50+4\alpha(3\lambda^2+12\lambda+11)+6\alpha^2(2\lambda+3)+4\alpha^3)\frac{A'(1)}{A(1)}+(6\lambda^2+30\lambda+35+4\alpha(3\lambda+6)+6\alpha^2)\frac{A''(1)+A'(1)}{A(1)}+2(2\lambda+2\alpha+5)\frac{A'''(1)+3A''(1)+A'(1)}{A(1)}+\frac{A^{iv}+6A'''(1)+7A''(1)+A'(1)}{A(1)}\right\}\right]$.

Lemma 2.3. *Let* $\left\{T^*_{n,\alpha,\beta}\right\}$ *be the operators given by* (1.10)*. Then the following equalities hold:*

$$T^*_{n,\alpha,\beta}((t-x);x)=\left(\frac{a_n}{b_n+\beta}-1\right)x+\frac{1}{b_n+\beta}\left(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right), \tag{2.11}$$

$$\begin{aligned}T^*_{n,\alpha,\beta}((t-x)^2;x)&=\left(1-\frac{a_n}{b_n+\beta}\right)^2x^2\\&+\left[\frac{a_n}{(b_n+\beta)^2}\left\{2(\lambda+\alpha+2)+H''(1)+2\frac{A'(1)}{A(1)}\right\}-\frac{2}{b_n+\beta}\left\{\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right\}\right]x\\&+\frac{1}{(b_n+\beta)^2}\left[(\lambda+1)(\lambda+2\alpha+2)+\alpha^2+2(\lambda+\alpha+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\right],\end{aligned} \tag{2.12}$$

$$\begin{aligned}&T^*_{n,\alpha,\beta}((t-x)^4;x)\\&=\left(1-\frac{a_n}{b_n+\beta}\right)^4x^4+\left[\frac{{a_n}^3}{(b_n+\beta)^4}\left\{4(\lambda+\alpha+4)+6H''(1)+4\frac{A'(1)}{A(1)}\right\}-\frac{12{a_n}^2}{(b_n+\beta)^3}\right.\\&((\lambda+\alpha+3)+H''(1)+3\frac{A'(1)}{A(1)}\right\}+\frac{6a_n}{(b_n+\beta)^2}\left\{2(\lambda+\alpha+2)+H''(1)+2\frac{A'(1)}{A(1)}\right\}\\&\left.-\frac{4}{b_n+\beta}\left\{\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right\}\right]x^3+\left[\frac{{a_n}^2}{(b_n+\beta)^4}\left\{6\lambda^2+30\lambda+42+4\alpha(3\lambda+6)+6\alpha^2\right.\right.\\&+4H'''(1)+3(H''(1))^2+18H''(1)+6(2\lambda+2\alpha+5)(1+H''(1)+\frac{A'(1)}{A(1)})\\&\left.+(12H''(1)+18)\frac{A'(1)}{A(1)}+6\frac{A''(1)}{A(1)}\right\}-\frac{4a_n}{(b_n+\beta)^3}\left\{3\lambda^2+12\lambda+12+3\alpha(2\lambda+3)+3\alpha^2\right.\\&\left.+H'''(1)+3H''(1)+3(\lambda+\alpha+2)(1+H''(1)+2\frac{A'(1)}{A(1)})+(H''(1)+2)\frac{A'(1)}{A(1)}+3\frac{A''(1)}{A(1)}\right\}\\&\left.+\frac{6}{(b_n+\beta)^2}\left\{(\lambda+1)(\lambda+2)+2\alpha(\lambda+1)+\alpha^2+2(\lambda+\alpha+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\right\}\right]x^2\end{aligned}$$

$$+\left[\frac{a_n}{(b_n+\beta)^4}\left\{4\lambda^3+30\lambda^2+70\lambda+51+4\alpha(3\lambda^2+12\lambda+11)+6\alpha^2(2\lambda+3)+4\alpha^3+H^{iv}(1)\right.\right.$$
$$+6H'''(1)+7H''(1)+(6\lambda^2+30\lambda+35+4\alpha(3\lambda+6)+6\alpha^2)(1+H''(1)+2\frac{A'(1)}{A(1)})$$
$$+2(2\lambda+2\alpha+5)(H'''(1)+3H''(1)+1+3(H''(1)+2)\frac{A'(1)}{A(1)}+3\frac{A''(1)}{A(1)})+(4H'''(1)+18H''(1)$$
$$\left.+14)\frac{A'(1)}{A(1)}+(6H''(1)+18)\frac{A''(1)}{A(1)}+4\frac{A'''(1)}{A(1)}\right\}-\frac{4}{(b_n+\beta)^3}\left\{(\lambda+1)(\lambda+2)(\lambda+3)\right.$$
$$+3\alpha(\lambda+1)(\lambda+2)+\alpha^3+\left(3\lambda^2+12\lambda+11+3\alpha(2\lambda+3)+3\alpha^2\right)\frac{A'(1)}{A(1)}$$
$$\left.\left.+3(\lambda+\alpha+2)\frac{A''(1)+A'(1)}{A(1)}+\frac{A'''(1)+3A''(1)+A'(1)}{A(1)}\right\}\right]x$$
$$+\frac{1}{(b_n+\beta)^4}\left[(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)+4\alpha(\lambda+1)(\lambda+2)(\lambda+3)+6\alpha^2(\lambda+1)(\lambda+2)\right.$$
$$+4\alpha^3(\lambda+2)+\alpha^4+(4\lambda^3+30\lambda^2+70\lambda+50+4\alpha(3\lambda^2+12\lambda+11)+6\alpha^2(2\lambda+3)+4\alpha^3)\frac{A'(1)}{A(1)}$$
$$+(6\lambda^2+30\lambda+35+4\alpha(3\lambda+6)+6\alpha^2\frac{A''(1)+A'(1)}{A(1)}+2(2\lambda+2\alpha+5)\frac{A'''(1)+3A''(1)+A'(1)}{A(1)}$$
$$\left.+\frac{A^{iv}+6A'''(1)+7A''(1)+A'(1)}{A(1)}\right]. \tag{2.13}$$

Let $\delta>0$ and $f\in C[0,\infty)$. The modulus of continuity is denoted by $\omega(f,\delta)$ and is defined as

$$\omega(f,\delta)=\sup_{x,y\in[0,\infty),\ |x-y|\le\delta}|f(x)-f(y)|. \tag{2.14}$$

If λ is any positive real number, then

$$\omega(f,\lambda\delta)\le(1+\lambda)\omega(f,\delta). \tag{2.15}$$

If f is uniformly continuous on $(0,\infty)$, then it is necessary and sufficient that

$$\lim_{\delta\to0}\omega(f,\delta)=0.$$

The second order modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega_2(f,\delta)=\sup_{0<t\le\delta}\|f(.+2t)-2f(.+t)+f(.)\|_{C_B},$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|$.

The Peetre's K-functional [10] of the function $f\in C_B[0,\infty)$ is defined by

$$K(f,\delta):=\inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|_{C_B}+\delta\|g\|_{C_B^2}\right\},$$

where

$$C_B^2[0,\infty):=\left\{g\in C_B[0,\infty):\ g',\ g''\in C_B[0,\infty)\right\},$$

and the norm $\|g\|_{C_B^2}:=\|g\|_{C_B}+\|g'\|_{C_B}+\|g''\|_{C_B}$. It is clear that the following inequality (see [10]) :

$$K(f,\delta)\le M\left\{\omega_2(f,\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B}\right\},$$

is valid, for all $\delta>0$. The constant M is independent of f and δ.

Lemma 2.4. ([14]) *Let $g \in C^2[0,\infty)$ and $(P_n)_{n\geq 0}$ be a sequence of positive linear operators with the property $P_n(1;x) = 1$. Then*

$$|P_n(g;x) - g(x)| \leq \sqrt{P_n\big((s-x)^2;x\big)}\|g'\| + \frac{1}{2}P_n\big((s-x)^2;x\big)\|g''\|. \tag{2.16}$$

Lemma 2.5. ([29]) *Let $f \in C[a,b]$ and $h \in \big(0, \frac{b-a}{2}\big)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied:*

(1) $\|f_h - f\| \leq \frac{3}{4}\omega_2(f,h)$,
(2) $\|f_h''\| \leq \frac{3}{2h^2}\omega_2(f,h)$.

We now consider the weighted spaces of the functions which are defined on the semi-axis $[0,\infty)$ satisfying the inequality $|f(x)| \leq M_f\rho(x)$ where $\rho(x) = x^2+1$ is a weight function and M_f is a constant depending only on f. By $B_\rho[0,\infty)$, we denote the set of functions that satisfy the above inequality and by $C_\rho[0,\infty)$, the subspace of all continuous functions belonging to $B_\rho[0,\infty)$. We also denote by $C_\rho^*[0,\infty)$, the subspace of all continuous functions $f \in C_\rho[0,\infty)$, for which $\lim\limits_{x\to\infty}\frac{f(x)}{\rho(x)} = k_f < \infty$. Obviously, $C_\rho[0,\infty)$ is a normed linear space with the ρ-norm $\|f\|_\rho = \sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}$.

It is well-known that the first and second order modulus of continuity in general do not tend to zero with $\delta \to 0$ on $[0,\infty)$, so we use the following weighted modulus of continuity [16]:

$$\Omega(f,\delta) = \sup_{|t-x|\leq\delta}\ \sup_{t,x\in[0,\infty)} \frac{|f(t)-f(x)|}{[1+(t-x)^2]\rho(x)}. \tag{2.17}$$

We have the following lemma:

Theorem 2.1. ([12, 13]) *Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators which acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ such that*

$$\lim_{n\to\infty}\|L_n(t^k;x) - x^k\|_\rho = 0,\ k \in \{0,1,2\}.$$

Then, for any function $f \in C_\rho^*[0,\infty)$

$$\lim_{n\to\infty}\|L_nf - f\|_\rho = 0.$$

Lemma 2.6. ([16]) *Let $f \in C_\rho^*[0,\infty)$. Then, we have*

(1) $\lim\limits_{\delta\to 0}\Omega(f,\delta) = 0$, *for each $\delta > 0$,*
(2) $\Omega(f,\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f,\delta)$.

From this inequality, for $t,x \in [0,\infty)$, we get

$$|f(t)-f(x)| \leq 2\Big(1+\frac{1}{\delta}|t-x|\Big)(1+\delta^2)(1+x^2)(1+(t-x)^2)\Omega(f,\delta). \tag{2.18}$$

3. Approximation results

Theorem 3.2. *Let $T^*_{n,\alpha,\beta}$ be the operators given by* (1.10)*. Then, for any function $f \in C[0,\infty) \cap E$,*

$$\lim_{n\to\infty}T^*_{n,\alpha,\beta}(f;x) = f(x),$$

uniformly on each compact subset of $[0,\infty)$*, where*

$$E := \{f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\}.$$

Proof. According to Lemma 2.2 (1)-(3), we have

$$\lim_{n\to\infty} T^*_{n,\alpha,\beta}(t^i;x) = x^i,\ i\in\{0,1,2\}.$$

If we apply the Korovkin theorem [4], we obtain the desired result. □

For $A(t)=\exp(t), H(t)=t$ and $A(t)=t, H(t)=t$, comparison of the convergence of $T^*_n(f;x)$(blue) and $T^*_{n,0,0}(f;x)$(red) towards the function $f(x)=x^2\exp(-3x)$(black $--$) is illustrated in Fig.1 and Fig.2, respectively, where $a_n=n+\sqrt{n+1},\ b_n=n+3$ and $n=9$.

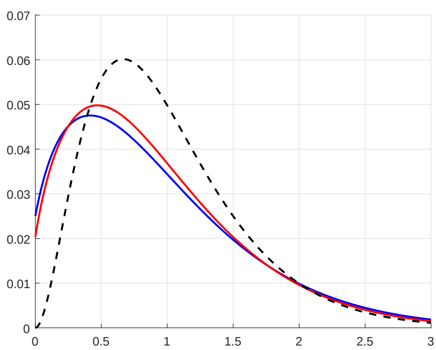

FIGURE 1

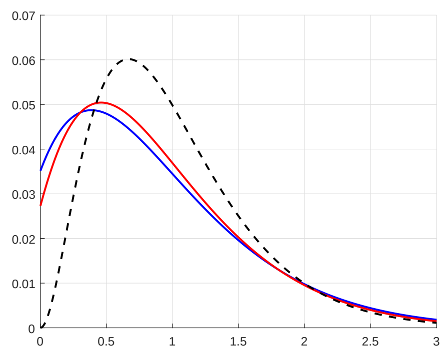

FIGURE 2

Theorem 3.3. *If $f\in C[0,\infty)\cap E$, then we have*

$$|T^*_{n,\alpha,\beta}(f,x)-f(x)| \le \Big\{1+\vartheta_{n,\alpha,\beta}(x)\Big\}\omega\Big(f,\frac{1}{\sqrt{b_n+\beta}}\Big),$$

where

$$\begin{aligned}\vartheta_{n,\alpha,\beta}(x) &= \Bigg[\frac{1}{b_n+\beta}\Big\{(\lambda+1)(\lambda+2)+\alpha^2+2\alpha(\lambda+1)+2(\lambda+\alpha+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\Big\}\\ &\quad+\Big\{\frac{a_n}{b_n+\beta}\Big(2(\lambda+\alpha+2)+H''(1)+2\frac{A'(1)}{A(1)}\Big)\\ &\quad-2\Big(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\Big)\Big\}x+\frac{(b_n+\beta-a_n)^2}{b_n+\beta}x^2\Bigg]^{\frac{1}{2}}.\end{aligned}$$

Proof. From the definition of $T^*_{n,\alpha,\beta}$ and the well-known properties of the modulus of continuity, we have

$$\begin{aligned}|T^*_{n,\alpha,\beta}(f;x)-f(x)| \le& \frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k}\left|f\left(\frac{b_nt+\alpha}{b_n+\beta}\right)-f(x)\right|dt\\ \le&\Big\{1+\frac{1}{\delta}\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k}\left|\frac{b_nt+\alpha}{b_n+\beta}-x\right|dt\Big\}\\ &\times\omega(f,\delta).\end{aligned}$$

By using Cauchy-Schwartz inequality,

$$\int_0^\infty e^{-b_nt}t^{\lambda+k}\left|\frac{b_nt+\alpha}{b_n+\beta}-x\right|dt \le \left(\int_0^\infty e^{-b_nt}t^{\lambda+k}dt\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-b_nt}t^{\lambda+k}\left(\frac{b_nt+\alpha}{b_n+\beta}-x\right)^2dt\right)^{\frac{1}{2}}$$
$$\le \frac{\Gamma(\lambda+k+1)}{b_n^{\lambda+k+1}}\left\{\frac{1}{(b_n+\beta)^2}\Big((\lambda+k+1)(\lambda+k+2)+\alpha^2 + 2\alpha(\lambda+k+1)\Big)-\frac{2x}{b_n+\beta}(\lambda+\alpha+k+1)+x^2\right\}^{\frac{1}{2}}.$$

So, we have

$$|T^*_{n,\alpha,\beta}(f;x)-f(x)| \le \left\{1+\frac{1}{\delta}\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^\infty p_k(a_nx)\left[\frac{1}{(b_n+\beta)^2}\Big((\lambda+k+1)(\lambda+k+2) +\alpha^2+2\alpha(\lambda+k+1)\Big)-\frac{2x}{b_n+\beta}(\lambda+\alpha+k+1)+x^2\right]^{\frac{1}{2}}\right\}\omega(f,\delta).$$

Again, applying Cauchy-Schwartz inequality, we have

$$\begin{aligned}|T^*_{n,\alpha,\beta}(f;x)-f(x)| &\le \left\{1+\frac{1}{\delta}\left[\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^\infty p_k(a_nx)\left\{\frac{1}{(b_n+\beta)^2}\Big((\lambda+k+1)(\lambda+k+2) +\alpha^2+2\alpha(\lambda+k+1)\Big)-\frac{2x}{b_n+\beta}(\lambda+\alpha+k+1)+x^2\right\}\right]^{\frac{1}{2}}\right\}\omega(f,\delta)\\ &\le \left\{1+\frac{1}{\delta}\frac{1}{\sqrt{b_n+\beta}}\left[\frac{1}{b_n+\beta}\left\{(\lambda+1)(\lambda+2)+\alpha^2+2\alpha(\lambda+1)+2(\lambda+\alpha+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\right\}+\left\{\frac{a_n}{b_n+\beta}\left(2(\lambda+\alpha+2)+H''(1)+2\frac{A'(1)}{A(1)}\right)\right.\right.\right.\\ &\left.\left.\left.-2\left(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right)\right\}x+\frac{(b_n+\beta-a_n)^2}{b_n+\beta}x^2\right]^{\frac{1}{2}}\right\}\omega(f,\delta).\end{aligned}$$

Choosing $\delta = \dfrac{1}{\sqrt{b_n+\beta}}$ in the above inequality, we obtain the desired result. □

Theorem 3.4. *Let f be defined on* $[0,\infty)$ *and* $f\in C[0,a]$, *then the rate of convergence of the operators* $T^*_{n,\alpha,\beta}$ *is given by*

$$|T^*_{n,\alpha,\beta}(f;x)-f(x)| \le \frac{2}{a}\|f\|h^2+\frac{3}{4}(a+2+h^2)\omega_2(f,h),$$

where

$$h := h_n(x) = \sqrt[4]{T^*_{n,\alpha,\beta}((t-x)^2;x)},$$

and the second order modulus of continuity is given by $\omega_2(f,\delta)$ *with the norm* $\|f\| = \max_{x\in[a,b]}|f(x)|$.

Proof. Let f_h be the second-order Steklov function attached to the function f. By virtue of the identity $T^*_{n,\alpha,\beta}(1;x)=1$, we have

$$\begin{aligned}|T^*_{n,\alpha,\beta}(f;x)-f(x)| &\le |T^*_{n,\alpha,\beta}(f-f_h;x)|+|T^*_{n,\alpha,\beta}(f_h;x)-f_h(x)|+|f_h(x)-f(x)|\\ &\le 2\|f_h-f\|+|T^*_{n,\alpha,\beta}(f_h;x)-f_h(x)|. \qquad (3.19)\end{aligned}$$

Taking into account the fact that $f_h \in C^2[0,a]$, it follows from Lemma 2.4 that

$$|T^*_{n,\alpha,\beta}(f_h;x) - f_h(x)| \leq \|f'_h\|\sqrt{T^*_{n,\alpha,\beta}((t-x)^2;x)} + \frac{1}{2}\|f''_h\|T^*_{n,\alpha,\beta}((t-x)^2;x). \tag{3.20}$$

Combining the Landau inequality and Lemma 2.5, we can write

$$\begin{aligned} \|f'_h\| &\leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f''_h\| \\ &\leq \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f,h). \end{aligned} \tag{3.21}$$

From the last inequality, (3.20) becomes, on taking $h = \sqrt[4]{T^*_{n,\alpha,\beta}((t-x)^2;x)}$,

$$|T^*_{n,\alpha,\beta}(f_h;x) - f_h(x)| \leq \frac{2}{a}\|f\|h^2 + \frac{3a}{4}\omega_2(f,h) + \frac{3}{4}h^2\omega_2(f,h). \tag{3.22}$$

Substituting (3.22) in (3.19), Lemma 2.5 hence gives the proof of the theorem. □

Theorem 3.5. *Let $f \in C_B^2[0,\infty)$. Then*

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq \frac{1}{b_n+\beta}\gamma(x)\|f\|_{C_B^2},$$

where

$$\gamma(x) = \left(1 + \frac{H''(1)}{2}\right)x + \frac{1}{2}\left((\lambda+1)(\lambda+2+2\alpha) + 2(\lambda+\alpha+2)\frac{A'(1)}{A(1)} + \frac{A''(1)}{A(1)} + \alpha^2\right).$$

Proof. Using the Taylor expansion of the function $f \in C_B^2$, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(\xi)(t-x)^2,$$

where $\xi \in (x,t)$. Due to linearity property of the operators $T^*_{n,\alpha,\beta}$, we can write

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq \|f'\|_{C_B}T^*_{n,\alpha,\beta}(t-x;x) + \frac{1}{2}\|f''\|_{C_B^2}T^*_{n,\alpha,\beta}\big((t-x)^2;x\big).$$

Using Lemma 2.3, we have

$$\begin{aligned} |T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq\ & \|f'\|_{C_B}\left\{\left(\frac{a_n}{b_n+\beta} - 1\right)x + \frac{1}{b_n+\beta}\left(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right)\right\} \\ & + \frac{1}{2}\|f''\|_{C_B}\left[\left(1 - \frac{a_n}{b_n+\beta}\right)^2 x^2 + \left\{\frac{a_n}{(b_n+\beta)^2}\left((\lambda+\alpha+2) + H''(1)\right.\right.\right. \\ & \left. + 2\frac{A'(1)}{A(1)}\right) - \frac{2}{b_n+\beta}\left(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right)\Big\}x + \frac{1}{(b_n+\beta)^2}\Big\{(\lambda+1) \\ & \left.\left.(\lambda+2\alpha+2) + \alpha^2 + 2(\lambda+\alpha+2)\frac{A'(1)}{A(1)} + \frac{A''(1)}{A(1)}\right\}\right]. \end{aligned}$$

For sufficiently large n, we obtain

$$\begin{aligned} |T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq\ & \frac{1}{b_n+\beta}\left\{\left(1 + \frac{H''(1)}{2}\right)x + \frac{1}{2}\Big((\lambda+1)(\lambda+2+2\alpha) + 2(\lambda+\alpha+2)\right. \\ & \left.\frac{A'(1)}{A(1)} + \frac{A''(1)}{A(1)} + \alpha^2\Big)\right\}\|f\|_{C_B^2}. \end{aligned}$$

□

Theorem 3.6. *Let $f \in C_B[0,\infty)$. Then*

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq 2M\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\},$$

where $\delta = \dfrac{\gamma(x)}{2(b_n+\beta)}$ and $M > 0$ is a constant independent of the function f and δ. Note that $\gamma(x)$ is defined as in Theorem 3.5.

Proof. Let $g \in C_B^2[0,\infty)$. Theorem 3.5 allows us to write

$$\begin{aligned} |T^*_{n,\alpha,\beta}(f;x) - f(x)| &\leq |T^*_{n,\alpha,\beta}(f-g;x)| + |T^*_{n,\alpha,\beta}(g;x) - g(x)| + |g(x) - f(x)| \\ &\leq 2\|f-g\|_{C_B} + \frac{1}{b_n+\beta}\gamma(x)\|g\|_{C_B^2} \\ &= 2\Big\{\|f-g\|_{C_B} + \frac{1}{2(b_n+\beta)}\gamma(x)\|g\|_{C_B^2}\Big\}. \end{aligned} \tag{3.23}$$

The left-hand side of inequality (3.23) does not depend on the function $g \in C_B^2[0,\infty)$, so

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq 2K\left(f, \frac{\gamma(x)}{2(b_n+\beta)}\right). \tag{3.24}$$

By using the relation between Peetre's K-functional and second modulus of smoothness and choosing $\delta = \frac{\gamma(x)}{2(b_n+\beta)}$ (3.24) becomes

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq 2M\left\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\right\}.$$

□

Now, we prove Voronovskaja type result for our operators $T^*_{n,\alpha,\beta}$ given by (1.10):

Theorem 3.7. *For $f \in C_B^2[0,\infty)$, we have the following formula:*

$$\lim_{n\to\infty} a_n\big(T^*_{n,\alpha,\beta}(f;x) - f(x)\big) = \left(\lambda + \alpha + 1 + \frac{A'(1)}{A(1)}\right)f'(x) + \left(1 + \frac{H''(1)}{2}\right)xf''(x),$$

for every $x \in [0,a]$.

Proof. For a fixed $x \in [0,\infty)$ and for all $t \in [0,\infty)$, by the Taylor formula, we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \varphi(t,x)(t-x)^2 \tag{3.25}$$

where $\varphi(t,x)$ is a function belonging to the space $C_B[0,\infty)$ and $\lim\limits_{t\to x}\varphi(t,x) = 0$. By Lemma 2.2 (1) and (3.25), we can write

$$\begin{aligned} a_n\big(T^*_{n,\alpha,\beta}(f;x) - f(x)\big) =& a_nT^*_{n,\alpha,\beta}\big((t-x);x\big)f'(x) + \frac{1}{2}a_nT^*_{n,\alpha,\beta}\big((t-x)^2;x\big)f''(x) \\ &+ a_nT^*_{n,\alpha,\beta}\big(\varphi(t,x)(t-x)^2;x\big), \end{aligned} \tag{3.26}$$

for every $n \in \mathbb{N}$. Using (2.11) and (2.12), we have

$$\lim_{n\to\infty} a_nT^*_{n,\alpha,\beta}\big((t-x);x\big) = \lambda + \alpha + 1 + \frac{A'(1)}{A(1)}, \tag{3.27}$$

$$\lim_{n\to\infty} a_nT^*_{n,\alpha,\beta}\big((t-x)^2;x\big) = (2 + H''(1))x. \tag{3.28}$$

Applying the Cauchy-Schwartz inequality for the third term on the right hand side of (3.26), we get

$$a_n T^*_{n,\alpha,\beta}\big(\varphi(t,x)(t-x)^2;x\big) \leq \sqrt{a_n^2 T^*_{n,\alpha,\beta}\big((t-x)^4;x\big)}\sqrt{T^*_{n,\alpha,\beta}\big(\varphi^2(t,x);x\big)}. \tag{3.29}$$

From Lemma 2.2, we can find

$$\lim_{n\to\infty} a_n^2 T^*_{n,\alpha,\beta}\big((t-x)^4;x\big) = \left(12(1+H''(1))+3(H''(1))^2+8(2+H''(1))\frac{A'(1)}{A(1)}\right)x^2. \tag{3.30}$$

Since for the function $\psi(t,x) = \varphi^2(t,x),\ x \geq 0$, we have $\psi(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x}\psi(t,x)=0$. Then it follows from Theorem 3.2 that

$$\lim_{n\to\infty} T^*_{n,\alpha,\beta}\big(\varphi^2(t,x);x\big) = \lim_{n\to\infty} T^*_{n,\alpha,\beta}\big(\psi(t,x);x\big) = \psi(x,x) = 0, \tag{3.31}$$

uniformly with respect to $x \in [0,a]$. So, considering (3.29)-(3.31), we obtain

$$\lim_{n\to\infty} a_n T^*_{n,\alpha,\beta}\big(\varphi(t,x)(t-x)^2;x\big) = 0. \tag{3.32}$$

Now, taking the limit as $n \to \infty$ in (3.26) and using (3.27), (3.28) and (3.32), we have

$$\lim_{n\to\infty} a_n\big(T^*_{n,\alpha,\beta}(f;x)-f(x)\big) = \left(\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right)f'(x)+\frac{1}{2}(2+H''(1))xf''(x).$$

Thus the proof is completed. □

4. Approximation properties in weighted spaces

Theorem 4.8. *Let $T^*_{n,\alpha,\beta}$ be the sequence of positive linear operators defined by* (1.10)*. Then for each function $f \in C^*_\rho[0,\infty)$,*

$$\lim_{n\to\infty} \|T^*_{n,\alpha,\beta}f - f\|_\rho = 0.$$

Proof. It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 2.1 are satisfied. From Lemma 2.2 (1), it is immediate that

$$\lim_{n\to\infty} \|T^*_{n,\alpha,\beta}(1;x)-1\|_\rho = 0, \tag{4.33}$$

and by Lemma 2.2 (2), we have

$$\sup_{x\in[0,\infty)} \frac{|T^*_{n,\alpha,\beta}(t;x)-x|}{1+x^2} \leq \left|\frac{a_n}{b_n+\beta}-1\right| \sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{1}{b_n+\beta}\left|\lambda+\alpha+1+\frac{A'(1)}{A(1)}\right| \sup_{x\in[0,\infty)}\frac{1}{1+x^2}.$$

Which implies that

$$\lim_{n\to\infty} \|T^*_{n,\alpha,\beta}(t;x)-x\|_\rho = 0. \tag{4.34}$$

By means of Lemma 2.2 (3), we get

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|T^*_{n,\alpha,\beta}(t^2;x)-x^2|}{1+x^2} \leq & \left|\frac{a_n^2}{(b_n+\beta)^2}-1\right| \sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}+\frac{a_n}{(b_n+\beta)^2}\Big|2(\lambda+\alpha+2)+H''(1),\\ &+2\frac{A'(1)}{A(1)}\Big| \sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{1}{(b_n+\beta)^2}\Big|\alpha^2+(\lambda+1)(\lambda+2)\\ &+2(\lambda+2)\frac{A'(1)}{A(1)}+\frac{A''(1)}{A(1)}\Big| \sup_{x\in[0,\infty)}\frac{1}{1+x^2}.\end{aligned}$$

that follows

$$\lim_{n\to\infty} \|T^*_{n,\alpha,\beta}(t^2;x)-x^2\|_\rho = 0. \tag{4.35}$$

From (4.33)-(4.35), for $k \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|T^*_{n,\alpha,\beta}(t^k;x) - x^k\|_\rho = 0.$$

Applying Theorem 2.1, we obtain the desired result. □

Theorem 4.9. *For $f \in C^*_\rho[0,\infty)$, the following inequality*

$$\sup_{x\geq 0} \frac{|T^*_{n,\alpha,\beta}(f;x) - f(x)|}{(1+x^2)^3} \leq K\Omega\left(f, \frac{1}{\sqrt{b_n+\beta}}\right)$$

is satisfied for a sufficiently large n, where K is a constant independent of a_n and b_n.

Proof. From (2.18), we can write

$$\begin{aligned}
&|T^*_{n,\alpha,\beta}(f;x) - f(x)| \\
= \;& 2\Omega(f,\delta_n)(1+x^2)(1+\delta_n^2)\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k} \\
&\times\left(1+\frac{1}{\delta_n}\left|\frac{b_nt+\alpha}{b_n+\beta}-x\right|\right)\left(1+\left(\frac{b_nt+\alpha}{b_n+\beta}-x\right)^2\right)dt \\
\leq \;& 4\Omega(f,\delta_n)(1+x^2)\Bigg\{1+\frac{1}{\delta_n}\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k}\left|\frac{b_nt+\alpha}{b_n+\beta}-x\right|dt \\
&+\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k}\left(\frac{b_nt+\alpha}{b_n+\beta}-x\right)^2 dt \\
&+\frac{1}{\delta_n}\frac{e^{-a_nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(a_nx)\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-b_nt}t^{\lambda+k}\left|\frac{b_nt+\alpha}{b_n+\beta}-x\right|\left(\frac{b_nt+\alpha}{b_n+\beta}-x\right)^2 dt\Bigg\},
\end{aligned}$$

for any $\delta_n > 0$. Applying Cauchy-Schwartz inequality, we get

$$|T^*_{n,\alpha,\beta}(f;x) - f(x)| \leq 4\Omega(f,\delta_n)(1+x^2)\left(1+\frac{2}{\delta_n}\sqrt{\phi_1}+\phi_1+\frac{1}{\delta_n}\sqrt{\phi_1\phi_2}\right), \tag{4.36}$$

where $\phi_1 = T^*_{n,\alpha,\beta}((t-x)^2;x)$ and $\phi_2 = T^*_{n,\alpha,\beta}((t-x)^4;x)$ given by (2.12) and (2.13), respectively. Using the conditions on $\{a_n\}$ and $\{b_n\}$, we get

$$\begin{aligned}
\phi_1 &= O\left(\frac{1}{b_n+\beta}\right)(x^2+x), \\
\phi_2 &= O\left(\frac{1}{b_n+\beta}\right)(x^4+x^3+x^2+x).
\end{aligned}$$

Substituting the above equalities in (4.36), we have

$$\begin{aligned}
&|T^*_{n,\alpha,\beta}(f;x) - f(x)| \\
\leq\;& 4\Omega(f,\delta_n)(1+x^2)\left(1+\frac{2}{\delta_n}\sqrt{O\left(\frac{1}{b_n+\beta}\right)(x^2+x)}\right. \\
&\left.+O\left(\frac{1}{b_n+\beta}\right)(x^2+x)+\frac{1}{\delta_n}O\left(\frac{1}{b_n+\beta}\right)\sqrt{(x^4+x^3+x^2+x)(x^2+x)}\right),
\end{aligned}$$

and choosing $\delta_n = \dfrac{1}{\sqrt{b_n+\beta}}$, for sufficiently large n, we obtain the desired result. □

5. Conclusion

In this paper, we give a generalization of integral operators which involves Sheffer polynomials introduced by Sucu and Büyükyazici and studied their approximation and convergence properties. For the function $f(x) = x^2 \exp(-3x)$, we have plotted the results for both operators $T_n^*(f;x)$ and $T_{n,0,0}^*(f;x)$. It is clear from figure that our modified operator gives better approximate to the curve.

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