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Existence of solutions of implicit integral equations via Z-contraction

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ABSTRACT. The main focus of this work is to assure that the sum of a compact operator with a Z-contraction admits a fixed point. The concept of condensing mapping (in the sense of Hausdorff non-compactness measure) is used to establish the concerned result which generalizes some of the existing state-of-art in the literature. Presented result is used to verify the actuality of solutions of implicit integral equations.

1. INTRODUCTION AND BASIC CONCEPTS

Fixed point approach plays an incontestable role in developing many branches of mathematical sciences. This theory has wide range of applications. The two most applied results in this theory are Banach's principle for contraction mappings and Schauder's theorem for compact operators. Combining these two results, Krasnoselskii established a fixed point result stated as follows:

Theorem 1.1. [14, 21] *Let $S \neq \phi$ be a convex and closed set in a Banach space M. Suppose $A, B : S \to M$ such that*

(i) *A is continuous and compact,*
(ii) *B is contraction,*
(iii) *$Ax + By \in S \; \forall \; x, y \in S$.*

Then $A + B$ admits fixed point.

This celebrated result is readily used in theory of integral and differential equations. However, a large number of improvements have been appeared in the literature over past time modifying the assumptions. These improvements occurred in different directions.

The first direction in achieving improvement is to weaken the condition (iii) in Theorem 1.1. A major breakthrough occurred when the condition $A(S) + B(S) \subset S$ is replaced by

$$(A + B)(S) \subset S. \tag{1.1}$$

This was made possible due to use of non-compactness measure. Let us first recall the following famous notion, called Hausdorff non-compactness measure. Let $B(x, r)$ denotes closed ball with center at $x \in \mathcal{X}$ and radius r.

Definition 1.1. [2] A non-negative number

$$\beta(\mathcal{C}) = \inf\{r > 0 : \mathcal{C} \subset \cup_{i=1}^{N} B(x_i, r), x_i \in \mathcal{X}, i = 1, ..., N\},$$

assigned with each bounded set $\mathcal{C}$ in a metric space $\mathcal{X}$, is called Hausdorff non-compactness measure (in short H-MNC).

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Some basic properties of H-MNC are given below:

(1) $\beta(\mathcal{C}) = 0$ iff $cl(\mathcal{C})$ is compact (i.e. $\mathcal{C}$ is relatively compact), where $cl(\mathcal{C})$ denotes closure of the set $\mathcal{C}$,
(2) $\beta(\mathcal{C}) = \beta(cl(\mathcal{C}))$,
(3) if $\mathcal{P}$ and $\mathcal{Q}$ are bounded, then
  (a) $\mathcal{P} \subset \mathcal{Q}$ implies $\beta(\mathcal{P}) \leq \beta(\mathcal{Q})$ and
  (b) $\beta(\mathcal{P} + \mathcal{Q}) \leq \beta(\mathcal{P}) + \beta(\mathcal{Q})$.

We use notation *n.c.c.b* for 'nonempty convex closed and bounded' in rest of the article. The classical results using non-compactness measure are due to Darbo [9] and Sadovskii [20] which can be stated as follows:

If T is a continuous self mapping on a *n.c.c.b* subset $\mathcal{C}$ of a Banach space $\mathcal{X}$, satisfying one of the following conditions:

(D) $\exists\, 0 \leq \lambda < 1$ such that for any $\mathcal{M} \subset \mathcal{C}$,

$$\beta(T(\mathcal{M})) \leq \lambda\beta(\mathcal{M}),$$

(S) for any $\mathcal{M} \subset \mathcal{C}$ such that $\beta(\mathcal{M}) > 0$,

$$\beta(T(\mathcal{M})) < \beta(\mathcal{M}),$$

then T admits a fixed point.

Any mapping T satisfying condition (D) is called λ-set contraction (due to Darbo [9]) whereas satisfying (S) is called as β-condensing (due to Sadovskii [20]).

Another important aspect in improvement is weakening compactness of mapping A. There are plenty of work being done in this direction (cf. [3,10] and references therein).

The third important direction of improvement is generalizing the condition (ii) of Theorem 1.1. In this direction, Burton [5] did some revolutionary work and successfully replaced the contraction mapping by *large contractions* which is defined as follows:

Definition 1.2. A self mapping T on a metric space $(\mathcal{X}, m)$ is said to be large contraction if $\forall\, \varepsilon > 0, \exists\, \delta < 1$ such that $m(y,x) \geq \varepsilon$ implies $m(Ty,Tx) \leq \delta m(y,x),\ \forall y,x \in \mathcal{X}$.

Later, Przeradki [19] used the concept of H-MNC and β-condensing mapping to actualize the fixed points for sum of compact operator and *generalized contraction*.

Definition 1.3. The self mapping T on $(\mathcal{X}, m)$ is said to be a generalized contraction if there exists a function $\gamma : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that

$$\sup_{(a \leq m(y,x) \leq b)} \gamma(y,x) < 1 \text{ for all } b \geq a > 0$$

and $\forall\, x, y \in \mathcal{X}$

$$m(Ty,Tx) \leq \gamma(y,x)m(y,x).$$

Przeradzki [19] also shown *generalized contractions* to be real generalizations of the *large contractions*. This way he succeed in improving Krasnoselskii's as well as Burton's result. Park [18] also contributed significantly in this direction. Very recently following the analogy of [19], Wardowski [22] obtained another generalization of Krasnoselskii's result for φ-F-contraction instead of contraction. Further, Burton and C. Kirk [7] combined Schaefer's theorem to the thesis of Krsnoselskii's result. Some authors investigated the case of set-valued mappings (see [4,11,16] and references therein). All these improvements and generalizations make it easy to apply the obtained results.

On the other hand, recently Khojasteh et al. [13] bring a new concept called simulation function into the doctrine of fixed points. The notion is described as:

Definition 1.4. A mapping $\Xi : [0,\infty)^2 \to \mathbb{R}$ is called simulation function if it satisfies

(KSR-1) $\Xi(0,0) = 0$,
(KSR-2) $\Xi(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,
(KSR-3) if $\{s_n\}$ and $\{t_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$, then $\lim_{n\to\infty} \sup \Xi(t_n, s_n) < 0$.

However, de-Hierro and Samet [12] modified the above defined notion slightly and enlarged the simulation functions family by replacing condition (KSR-3) with

(DS-3) if $\{s_n\}$ and $\{t_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$ and $t_n < s_n$ then $\lim_{n\to\infty} \sup \Xi(t_n, s_n) < 0$.

In a parallel development, Argoubi et al. [1] found that the condition (KSR-1) is redundant and can be deduced from (KSR-2) and (KSR-3) or (DS-3). They redefined the simulation function by removing the condition (KSR-1) as:

Definition 1.5. [1] A simulation function is a mapping $\Xi : [0, \infty)^2 \to \mathbb{R}$ satisfying the following:

(ASV-1) $\Xi(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,
(ASV-2) if $\{s_n\}$ and $\{t_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$ and $s_n > t_n$, then $\lim_{n\to\infty} \sup \Xi(t_n, s_n) < 0$.

Let $\mathcal{Z}_{ASV}$ denotes the class of all functions $\Xi : [0, \infty)^2 \to \mathbb{R}$ satisfying (ASV-1) and (ASV-2).

Example 1.1. A function $\Xi_1 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\Xi_1(t, s) = \delta s - t, \ \forall s, t \in [0, \infty)$$

where $0 \leq \delta < 1$, then $\Xi_1 \in \mathcal{Z}_{ASV}$.

Example 1.2. If $\varphi : [0, \infty) \to [0, \infty)$ is a mapping satisfying $\limsup_{t\to r^+} \varphi(t) < 1$, a function $\Xi_2 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\Xi_2(t, s) = \varphi(s)s - t, \ \forall s, t \in [0, \infty)$$

then $\Xi_2 \in \mathcal{Z}_{ASV}$.

Example 1.3. If $\kappa : [0, \infty) \to [0, \infty)$ is an upper semi-continuous mapping satisfying $\kappa(t) < t$, a function $\Xi_3 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\Xi_3(t, s) = \kappa(s) - t, \ \forall s, t \in [0, \infty)$$

then $\Xi_3 \in \mathcal{Z}_{ASV}$.

For more examples of simulation functions, refer to [8, 13] and references therein.

Definition 1.6. A self mapping T on $(\mathcal{X}, d)$ is called a Z-contraction if there exists $\Xi \in \mathcal{Z}_{ASV}$ such that

$$\Xi(d(Ty, Tx), d(y, x)) \geq 0 \tag{1.2}$$

for all $y, x \in \mathcal{X}$.

On a complete metric space, every Z-contraction has unique fixed point [13]. We wish to show that Z-contraction is not a particular instance of a generalized contraction. For this we give following example where mapping is Z-contraction but not later one.

Example 1.4. Let $X = \{x_n = n\sqrt{2} + 2^{1-n} : n \in \mathbb{N}\}$. Then (X, d) is complete metric space with metric $d(x, y) = |x - y|$, $x, y \in X$. Consider the mapping $T : X \to X$ defined as

$$Tx_n = \begin{cases} x_1 & \text{if } n = 1, \\ \\ x_{n-1} & \text{if } n \neq 1. \end{cases}$$

First to show that T fails to be generalized contraction. In fact, for any $n \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = \sqrt{2} - 2^{-n}.$$

Hence

$$\frac{2\sqrt{2} - 1}{2} \leq |x_{n+1} - x_n| < \sqrt{2}, \ \forall\, n \in \mathbb{N}.$$

Assuming to the contrary, that T is generalized contraction, there exists a function γ satisfying second condition in Definition 1.3. Then for $n \geq 2$, we get

$$\gamma(x_{n+1}, x_n) \geq \frac{d(Tx_{n+1}, Tx_n)}{d(x_{n+1}, x_n)} = \frac{|x_n - x_{n-1}|}{|x_{n+1} - x_n|} = \frac{\sqrt{2} - 2^{1-n}}{\sqrt{2} - 2^{-n}}.$$

So by first condition of Definition 1.3, we get

$$1 > \sup_{(\frac{2\sqrt{2}-1}{2} < d(x,y) < \sqrt{2})} \gamma(x, y) \geq \gamma(x_{n+1}, x_n) \geq \frac{\sqrt{2} - 2^{1-n}}{\sqrt{2} - 2^{-n}},$$

letting $n \to \infty$, we get contradiction. Now to show T to be a Z-contraction, we need to consider a mapping $\varphi : [0, \infty) \to [0, \infty)$ defined as

$$\varphi(t) = \begin{cases} 1 + \dfrac{2^{-n} - 2^{-m}}{t + 1} & \text{if } (n - m)\sqrt{2} - 1 < t < (n - m)\sqrt{2}, \ n > m \geq 2, \\ \\ 1 + \dfrac{2^{-n} - \sqrt{2}}{t + 1} & \text{if } (n - 1)\sqrt{2} - 1 < t < (n - 1)\sqrt{2}, \ n \geq 3 \text{ and } m = 1. \end{cases}$$

Then one can observe that $\limsup\limits_{t \to r^+} \varphi(t) < 1$ for any $t \geq 0$. Now for any $m, n \in \mathbb{N}$, $n > m \geq 2$, we have

$$|x_m - x_n| = (n - m)\sqrt{2} + 2^{1-n} - 2^{1-m}$$

and

$$(n - m)\sqrt{2} - 1 < |x_m - x_n| < (n - m)\sqrt{2}.$$

Hence, we obtain

$$\begin{aligned} |Tx_m - Tx_n| &= \left(1 + \frac{2^{1-n} - 2^{1-m}}{|x_m - x_n|}\right)|x_m - x_n| \\ &< \left(1 + \frac{2^{-n} - 2^{-m}}{|x_m - x_n| + 1}\right)|x_m - x_n| \\ &= \varphi(|x_m - x_n|)|x_m - x_n|. \end{aligned}$$

Also, for any $n \geq 3$ we get

$$|x_n - x_1| = (n - 1)\sqrt{2} - 1 + 2^{1-n}$$

and

$$(n - 1)\sqrt{2} - 1 < |x_n - x_1| < (n - 1)\sqrt{2},$$

which returns us with

$$\begin{aligned}|Tx_n - Tx_1| &= \left(1 + \frac{2^{1-n} - \sqrt{2}}{|x_n - x_1|}\right)|x_n - x_1| \\ &< \left(1 + \frac{2^{-n} - \sqrt{2}}{|x_n - x_1|} + 1\right)|x_n - x_1| \\ &< \varphi(|x_n - x_1|)|x_n - x_1|.\end{aligned}$$

If we take $\Xi(s,t) = \varphi(s)s - t$ then clearly Ξ is simulation function and hence T is a Z-contraction.

In the present article, first we show that every Z-contraction is β-condensing. Then using Sadovskii's theorem we obtain a result concerning actuality of fixed points for sum of a compact mappings with a Z-contractions. Later we apply the obtained result to verify existence of solutions for implicit integral equations.

2. Main results

We enunciate with showing Z-contraction to be a β-condensing map.

Theorem 2.2. *Every Z-contraction T on a metric space $(\mathcal{X}, d)$ is β-condensing.*

Proof. Let $\phi \neq \mathcal{C} \subseteq \mathcal{X}$ such that $\beta(\mathcal{C}) > 0$. Let $\{t_n\}$ and $\{s_n\}$ be two sequences defined by $t_n = \beta(\mathcal{C}) - \epsilon_n > 0$ and $s_n = \beta(\mathcal{C}) + \epsilon_n > 0$ where $\{\epsilon_n\}$ is such that $\epsilon_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \beta(\mathcal{C}) > 0$ and $t_n < s_n$. Then by (ASV-2), we have

$$\limsup_{n\to\infty} \Xi(t_n, s_n) < 0. \tag{2.3}$$

Choosing $\epsilon = \sup\{\epsilon_n\}$ sufficiently small, from (2.3), $\exists\ \Delta < 0$ such that

$$\Xi(t, s) < \Delta, \tag{2.4}$$

where $t \in [\beta(\mathcal{C}) - \epsilon, \beta(\mathcal{C}))$ and $s \in (\beta(\mathcal{C}), \beta(\mathcal{C}) + \epsilon]$.

Let $R = \beta(\mathcal{C}) + \epsilon$. Assume that $\mathcal{C}$ has a R-net, i.e.

$$\mathcal{C} \subset \cup_{i=1}^{k} B(x_i, R),\ x_1, x_2, ..., x_k \in \mathcal{X}. \tag{2.5}$$

Assume that $R' = \beta(\mathcal{C}) - \epsilon$. We show that $T(\mathcal{C})$ has a R'-net. For this, let $y \in T(\mathcal{C})$. Let $\exists$ $x \in \mathcal{C}$ such that $Tx = y$. Then from (2.5), $\exists\ x_i$ such that $d(x_i, x) < R$.

Now if $Tx = Tx_i$, then $d(Tx_i, Tx) < R'$ (obviously).

Suppose $Tx \neq Tx_i$, then we have following two cases:

(1) If $0 < d(x_i, x) < R'$, then as T is Z-contraction we get $0 \leq \Xi(d(Tx_i, Tx), d(x_i, x))$, which in return by (ASV-1) yields us $d(Tx_i, Tx) < d(x_i, x) < R'$.

(2) If $R' < d(x_i, x) < R$, then either $d(Tx_i, Tx) < R'$ or $d(Tx_i, Tx) \geq R'$. Suppose $d(Tx_i, Tx) \geq R'$, then from (2.4), we have

$$\Xi(d(Tx_i, Tx), d(x_i, x)) < \Delta,$$

which is contrary to the truth (i.e. T is Z-contraction).

So $d(Tx_i, Tx) < R'$ holds in both cases. Thus $T(\mathcal{C})$ has a R'-net, which implies that $\beta(T(\mathcal{C})) \leq R' < \beta(\mathcal{C})$. □ □

Combining above result with Sadovskii's theorem leads us to the following Krasnoselskii type result.

Theorem 2.3. *Let $\mathcal{C}$ be an n.c.c.b subset of a Banach space $\mathcal{X}$. If $A : \mathcal{C} \to \mathcal{X}$ is a Z-contraction and $B : \mathcal{C} \to \mathcal{X}$ is a compact mapping such that $(A + B)(\mathcal{C}) \subset \mathcal{C}$, then $A + B$ admits fixed point.*

Proof. Let $\mathcal{K} \subset \mathcal{C}$ having nonzero H-MNC. Then we have

$$\beta((A+B)(\mathcal{K})) \leq \beta(A(\mathcal{K})+B(\mathcal{K})) \leq \beta(A(\mathcal{K}))+\beta(B(\mathcal{K})). \tag{2.6}$$

Using the fact B is compact (i.e. $\beta(B(\mathcal{K}))=0$) and conclusion of Theorem 2.2 in (2.6), we get

$$\beta((A+B)(\mathcal{K})) \leq \beta(A(\mathcal{K})) < \beta(\mathcal{K}).$$

Thus $A+B$ is β-condensing map. Therefore, Sadovskii's theorem implies $A+B$ has a fixed point. □ □

Following corollaries are some consequences of above theorem.

Corollary 2.1. *Let $\mathcal{C}$ be an n.c.c.b subset of a Banach space $\mathcal{X}$. If $A, B : \mathcal{C} \to \mathcal{X}$ are a contraction and compact mapping, respectively, satisfying $(A+B)(\mathcal{C}) \subset \mathcal{C}$, then $(A+B)$ admits fixed point.*

Proof. If we take $\Xi(t,s) = ks - t$ for $k \in [0,1)$ in Theorem 2.3, then proof follows. □ □

Corollary 2.2. *Let $\mathcal{C}$ be an n.c.c.b subset of a Banach space $\mathcal{X}$. If $A, B : \mathcal{C} \to \mathcal{X}$ are a φ-contraction and compact mapping, respectively, satisfying $(A+B)(\mathcal{C}) \subset \mathcal{C}$, then $(A+B)$ admits a fixed point.*

3. Applications

Consider the integral equation of implicit form given by

$$V(t, x(t)) = F(t, \int_0^t G(t,s,x(s))ds), \tag{3.7}$$

where $V, F : [-\mu,\mu] \times [-\mu,\mu] \to \mathbb{R}$ and $G : [-\mu,\mu] \times [-\mu,\mu] \times [-\mu,\mu] \to \mathbb{R}$ are continuous, $\mu > 0$. Our intension in this section is to apply the main theorem to verify the existence of solutions for above equation. We wish to find the solution of (3.7) in a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$ of continuous function $\Psi : [-\nu,\nu] \to \mathbb{R}$, $0 < \nu < \mu$ endowed with the supremum norm. Subset $\mathcal{C}$ can be described in the form

$$\mathcal{C} = \{\Psi \in \mathcal{X} : \Psi(0) = 0, \|\Psi\| \leq \mu\}.$$

Theorem 3.4. *If $F(0,0) = V(t,0) = 0$ for all $t \in [-\mu,\mu]$, $\mu > 0$ and the operator $(A\Psi)(t) = \Psi(t) - V(t,\Psi(t))$ is a Z-contraction on $\mathcal{C}$, then* (3.7) *has a solution in $\mathcal{C}$.*

Proof. We have $A(0) = 0 - V(t,0) = 0$ for all $t \in [-\mu,\mu]$. Taking any $\Psi \in \mathcal{C}$ with $A\Psi \neq 0$, since A is Z-contraction, we have

$$\|A\Psi\| < \|\Psi\|.$$

We claim that for any $\Psi \in \mathcal{C}$

$$\|A\Psi\| \leq \gamma < \mu, \text{for some } \gamma > 0. \tag{3.8}$$

Suppose this does not hold, then there exists sequence $\{\Psi_n\}$ in $\mathcal{C}$ such that $\{\|A\Psi_n\|\}$ is increasing and $\lim\limits_{n\to\infty} \|A\Psi_n\| = \mu$. Since $\Psi_n \in \mathcal{C}$, we have $\|A\Psi_n\| < \|\Psi_n\| \leq \mu$ and thus $\lim\limits_{n\to\infty} \|\Psi_n\| = \mu$. Then by (ASV-2) we have

$$0 \leq \limsup_{n\to\infty} \Xi(\|A\Psi_n\|, \|\Psi_n\|) < 0,$$

a contradiction. Thus our claim is true.

For rest of the proof, we follow Burton ([5], Theorem 3). We define $B : \mathcal{C} \to \mathcal{C}$ by $(B\Psi)(t) = F(t, \int_0^t G(t,s,\Psi(s))ds)$. Using continuity of F, the fact $F(0,0) = 0$ and the continuity of G on its compact domain one can show that there exists $0 < \nu < \mu$ such that $\Psi \in \mathcal{C}$ and $|t| < \nu$ imply

$$|(B\Psi)(t)| \leq \mu - \gamma. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\|A\Psi + B\Psi\| \leq \mu.$$

As $(A\Psi)(0) + (B\Psi)(0) = 0$ implies $(A+B)(\Psi) \in \mathcal{C}$.
The compactness of B can be achieved by showing that $B(\mathcal{C})$ is an equicontinuous set.

All the hypotheses of Theorem 2.3 are satisfied. Thus Ψ is a fixed point of $A+B$. □ □

As there may be difficulty in verifying Z-contractivity of operator A. This can be comfortably done if we restrict our search to nonnegative solutions. In light of the above proof technique, we can easily prove the following analogue of the Theorem 3.4.

Theorem 3.5. *If $F(0,0) = V(t,0) = 0$ for all $t \in [-\mu,\mu]$ and the operator $(A\Psi)(t) = \Psi(t) - V(t,\Psi(t))$ is a Z-contraction on $\mathcal{C}^+ = \{\Psi \in \mathcal{C} : \Psi \geq 0\}$, then* (3.7) *has a solution in $\mathcal{C}^+$.*

Example 3.5. Consider the differential equation of the form

$$2kt(t^4-1)x + (t^4+1)(t^4-kt^2+1)x' = (2t + H(t,x))(t^4+1)^2, \tag{3.10}$$

where $k \in [0,1)$ and $H : [-\mu,\mu] \times [-\mu,\mu] \to \mathbb{R}^+$ is continuous, $\mu > 0$. We will show the existence of solution for this differential equation in $\mathcal{C}^+$. The eq. (3.10) can be expressed as

$$\begin{aligned}
2t + H(t,x) &= \frac{2kt(t^4-1)x + (t^4-kt^2+1)(t^4+1)x'}{(t^4+1)^2}\\
&= \frac{(2kt^5-2kt)x + (t^4-kt^2+1)(t^4+1)x'}{(t^4+1)^2}\\
&= \frac{(4t^3-2kt+4t^7-2kt^5-4t^3-4t^7+4kt^5)x + (t^4+1)(t^4-kt^2+1)x'}{(t^4+1)^2}\\
&= \frac{(t^4+1)(4t^3-2kt)x - 4t^3(t^4-kt^2+1)x + (t^4-kt^2+1)(t^4+1)x'}{(t^4+1)^2}. \qquad (3.11)
\end{aligned}$$

Then (3.11) can be equivalently written as

$$\frac{(t^4-kt^2+1)x}{(t^4+1)} = t^2 + \int_0^t H(s,x)ds. \tag{3.12}$$

This is in the form of (3.7) for

$$V(t,x) = \frac{(t^4-kt^2+1)x}{(t^4+1)}$$

and

$$F(t,z) = t^2 + z.$$

In order to satisfy the assumptions of Theorem 3.4 we show that the operator $A(t,x) = x - V(t,x) = \dfrac{kt^2x}{(t^4+1)}$ is a Z-contraction on $\mathcal{C}^+$.

Let $\eta(p) = \delta p$ where $\delta \in [0,1)$. Then we can see that $\eta(p) < p$ and $\eta(p+q) \leq \eta(p)+\eta(q)$. In addition η is increasing and non-convex on $\mathbb{R}^+$.

Now let $\varphi, \psi \in \mathcal{C}^+$ then

$$\begin{aligned}
|A\varphi - A\psi| &= \Big|\frac{kt^2\varphi}{(t^4+1)} - \frac{kt^2\psi}{(t^4+1)}\Big|\\
&\leq \frac{kt^2}{(t^4+1)}|\varphi - \psi|\\
&\leq \eta(|\varphi-\psi|).
\end{aligned}$$

If we choose $\Xi(t,s) = \eta(s) - t$ for all $s, t > 0$, A is Z-contraction. Thus the solution exists by Theorem 3.5.

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