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Mixed problems for degenerate abstract parabolic equations and applications

VELI. B. SHAKHMUROV[1,2] and AIDA SAHMUROVA[3]

ABSTRACT. Degenerate abstract parabolic equations with variable coefficients are studied. Here the boundary conditions are nonlocal. The maximal regularity properties of solutions for elliptic and parabolic problems and Strichartz type estimates in mixed Lebesgue spaces are obtained. Moreover, the existence and uniqueness of optimal regular solution of mixed problem for nonlinear parabolic equation is established. Note that, these problems arise in fluid mechanics and environmental engineering.

1. INTRODUCTION

In this work, the boundary value problems (BVPs) for parameter dependent degenerate differential-operator equations (DOEs) are considered. Namely, linear equations and boundary conditions contain small parameters and are degenerated in some part of boundary. These problems have numerous applications in PDE, pseudo DE, mechanics and environmental engineering. The BVP for DOEs have been studied extensively by many researchers (see e.g. [1–11] and the references therein). The maximal regularity properties for DOEs in Banach space valued function class are investigated e.g. in [2, 3, 5–10]. Nonlinear DOEs studied e.g. in [6, 9].

The main objective of the present paper is to discuses the initial and nonlocal BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(x)\frac{\partial^{[2]}u}{\partial x_k^2}+B\left(\left(t,x,u,D^{[1]}u\right)\right)u=F\left(t,x,u,D^{[1]}u\right), \tag{1.1}$$

where a_k are complex valued functions, B and F are nonlinear operators in a Banach space E and

$$D^{[1]}u=\left(\frac{\partial^{[1]}u}{\partial x_1},\frac{\partial^{[1]}u}{\partial x_2},...,\frac{\partial^{[1]}u}{\partial x_n}\right),\ x=(x_1,x_2,...,x_n)\in G=\prod_{k=1}^{n}(0,b_k),$$

$$D_k^{[i]}u=u_k^{(i)}=\frac{\partial^{[i]}u}{\partial x_k^i}=\left(x_k^{\alpha_k}\frac{\partial}{\partial x_k}\right)^i u(x),\ 0\leq\alpha_k<1.$$

First all of, we consider the nonlocal BVP for the degenerate elliptic DOE with small parameters

$$\sum_{k=1}^{n}a_k(x)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+\lambda u+\sum_{k=1}^{n}A_k(x)\frac{\partial^{[1]}u}{\partial x_k}=f(x), \tag{1.2}$$

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Corresponding author: Veli. B. Shakhmurov; veli.sahmurov@okan.edu.tr

where a_k are complex-valued functions, λ is a complex parameters, $A(x)$ and $A_k(x)$ are linear operators.

We prove that for $f \in L_p(G;E)$, $|\arg \lambda| \leq \varphi$, $0 < \varphi \leq \pi$ and sufficiently large $|\lambda|$, problem (1.2) has a unique solution $u \in W^{[2]}_{p,\alpha}(G;E(A),E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|\frac{\partial^{[i]}u}{\partial x_k^i}\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C\|f\|_{L_p(G;E)}.$$

Then the above result is used to prove the well-posedeness of initial BVP (IBVP) and the uniform Strichartz type estimate for the solution the degenerate abstract parabolic equation with parameters

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x)\frac{\partial^{[2]}u}{\partial x_k^2} + A(x)u = f(x,t), t \in (0,T), x \in G. \tag{1.3}$$

Finally, via maximal regularity properties of (1.3) and contraction mapping argument, the existence and uniqueness of solution of the problem (1.3) is derived.

Note that, the equation and boundary conditions are degenerated with the different rate at different boundary edges, in general.

Let $\gamma = \gamma(x)$ be a positive measurable function on $\Omega \subset R^n$ and E be a Banach space. Let $L_{p,\gamma}(\Omega;E)$ denote the space of strongly measurable E-valued functions defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)\,dx\right)^{\frac{1}{p}}, 1 \leq p < \infty.$$

For $\gamma(x) \equiv 1$ we will denote these spaces by $L_p(\Omega;E)$.

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embeds into E. Let us consider the Sobolev-Lions type space $W^m_{p,\gamma}(a,b;E_0,E)$, consisting of all functions $u \in L_{p,\gamma}(a,b;E_0)$ that have generalized derivatives $u^{(m)} \in L_{p,\gamma}(a,b;E)$ with the norm

$$\|u\|_{W^m_{p,\gamma}} = \|u\|_{W^m_{p,\gamma}(a,b;E_0,E)} = \|u\|_{L_{p,\gamma}(a,b;E_0)} + \left\|u^{(m)}\right\|_{L_{p,\gamma}(a,b;E)} < \infty.$$

Let $\gamma = \gamma(x)$ be a positive measurable function on $(0,1)$ and

$$W^{[m]}_{p,\gamma} = W^{[m]}_{p,\gamma}(0,1;E_0,E) = \{u : u \in L_p(0,1;E_0),$$

$$\left. u^{[m]} \in L_p(0,1;E), \|u\|_{W^{[m]}_{p,\gamma}} = \|u\|_{L_p(0,1;E_0)} + \left\|u^{[m]}\right\|_{L_p(0,1;E)} < \infty\right\}.$$

Let

$$\alpha_k(x) = x_k^{\alpha_k},\ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

Consider E-valued weighted space defined by

$$W^{[m]}_{p,\alpha}(G,E(A),E) = \{u; u \in L_p(G;E_0),\ \frac{\partial^{[m]}u}{\partial x_k^m} \in L_p(G;E),$$

$$\left. \|u\|_{W^{[m]}_{p,\alpha}} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^{n}\left\|\frac{\partial^{[m]}u}{\partial x_k^m}\right\|_{L_p(G;E)} < \infty\right\}.$$

2. DEGENERATE ABSTRACT ELLIPTIC EQUATIONS

Consider the BVP for the following degenerate partial DOE with parameters

$$\sum_{k=1}^{n} a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+\lambda u+\sum_{k=1}^{n}A_k(x)\frac{\partial^{[1]}u}{\partial x_k}=f(x), \tag{2.4}$$

$$L_{kj}u=\sum_{i=0}^{m_{kj}}\alpha_{kji}u_{x_k}^{[i]}(G_{k0})+\beta_{kji}u_k^{[i]}(G_{kb})=0,\ j=1,2,$$

where a_k are complex-valued functions, $A(x)$ and $A_k(x)$ are linear operators, $u=u(x)$, α_{kji}, β_{kji} are complex numbers, λ is a complex parameter, $m_{kj}\in\{0,1\}$,

$$x=(x_1,x_2,...,x_n)\in G=\prod_{k=1}^{n}(0,b_k),$$

$$G_{k0}=(x_1,x_2,...,x_{k-1},0,x_{k+1},...,x_n),\ p_k\in(1,\infty),$$

$$G_{kb}=(x_1,x_2,...,x_{k-1},b_k,x_{k+1},...,x_n),$$

$$x^{(k)}=(x_1,x_2,...,x_{k-1},x_{k+1},...,x_n)\in G_k=\prod_{j\neq k}(0,b_j).$$

Consider the principal part of (2.4), i.e., consider the problem

$$\sum_{k=1}^{n} a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+\lambda u=f(x), \tag{2.5}$$

$$\sum_{i=0}^{m_{kj}}\alpha_{kji}u_{x_k}^{[i]}(G_{k0})+\beta_{kji}u_k^{[i]}(G_{kb})=0,\ j=1,2.$$

Theorem 2.1. *Assume;*

(1) E is an UMD spacethe Banach space, $0\leq\alpha_k<1-\frac{1}{p_k}$, $p_k\in(1,\infty)$, $\alpha_{km_{k1}}\neq 0$, $\beta_{km_{k2}}\neq 0$;

(2) $A(x)$ is a uniformly R-positive operator in E, $A(x)A^{-1}(\bar{x})\in C\left(\bar{G};L(E)\right)$, $x\in G$;

(3) $a_k\in C^{(m)}([0,b_k])$ and $a_k(x_k)<0$ for $x_k\in[0,b_k]$;

(4) $a_k(G_{j0})=a_k(G_{jb})$, $A(G_{j0})A^{-1}(x_0)=A(G_{jb})A^{-1}(x_0)$, $k,j=1,2,...,n$;

(5) $\eta_k=(-1)^{m_1}\alpha_{k1}\beta_{k2}-(-1)^{m_2}\alpha_{k2}\beta_{k1}\neq 0$.

First, we prove the separability properties of the problem (2.5):

Then, problem (2.5) has a unique solution $u\in W_{p,\alpha}^{[2]}(G;E(A),E)$ for $f\in L_p(G;E)$, $|\arg\lambda|\leq\varphi$ with sufficiently large $|\lambda|$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|\frac{\partial^{[i]}u}{\partial x_k^i}\right\|_{L_p(G;E)}+\|Au\|_{L_p(G;E)}\leq C\|f\|_{L_p(G;E)}. \tag{2.6}$$

Proof. Consider the BVP

$$(L+\lambda)u=a_1(x_1)D_{x_1}^{[2]}u(x_1)+(A(x_1)+\lambda)u(x_1)=f(x_1), \tag{2.7}$$

$$L_{1j}u=0,\ j=1,2,\ x_1\in(0,b_1),$$

where L_{1j} are boundary conditions of type (2.5) considered on $(0,b_1)$. By virtue of [7], problem (2.7) has a unique solution $u\in W_{p,\alpha_1}^{[2]}(0,b_1;E(A),E)$ for $f\in L_{p_1}(0,b_1;E)$, $|\arg\lambda|\leq\varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \left\| u^{[j]} \right\|_{L_{p_1}(0,b_1;E)} + \|Au\|_{L_{p_1}(0,b_1;E)} \leq C \|f\|_{L_{p_1}(0,b_1;E)} .$$

Now, let us consider the following BVP

$$\sum_{k=1}^{2} a_k (x_k) D_k^{[2]} u (x_1, x_2) + A (x_1, x_2) u (x_1, x_2) + \lambda u (x_1, x_2) = f (x_1, x_2) , \tag{2.8}$$

$$L_{k1} u = 0, \ L_{k2} u = 0, \ k = 1, 2, x_1, x_2 \in G_2 = (0, b_1) \times (0, b_2) .$$

Let $\alpha (2) = (\alpha_1, \alpha_2)$. Since $L_p (0, b_2; L_p (0, b_1) ; E) = L_p (G_2; E)$, the BVP (2.8) can be expressed as

$$a_2 D_2^{[2]} u (x_2) + (B (x_2) + \lambda) u (x_2) = f (x_2) , \ L_{2j} u = 0, j = 1, 2,$$

for $x_1 \in (0, b_1)$, where B is a differential operator in $L_{p_1} (0, b_1; E)$ for $x_2 \in (0, b_2)$, generated by problem (2.7). By virtue of [1, Theorem 4.5.2], $L_{p_1} (0, b_1; E) \in UMD$ for $p_1 \in (1, \infty)$. Moreover, in view of [10] the operator B is $R-$positive in $L_{p_1} (0, b_1; E)$. Hence, the problem (2.8) has a unique solution $u \in W_{p,\alpha(2)}^{[2]} (G_2; E (A) ; E)$ for $f \in L_p (G_2; E)$, $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ and (2.6) holds for $n = 2$. By continuing this process we obtain the assertion. □

Theorem 2.2. *Let the conditions of Theorem 2.1 are satisfied and* $A_k (x) A^{-\left(\frac{1}{2}-\nu\right)} (x) \in C \left(\bar{G}; L (E)\right)$ *for* $0 < \nu < \frac{1}{2}$. *Then, problem* (2.1) *has a unique solution* $u \in W_{p,\alpha}^{[2]} (G; E (A) , E)$ *for* $f \in L_p (G; E)$, $|\arg \lambda| \leq \varphi$ *with sufficiently large* $|\lambda|$ *and the coercive uniform estimate holds*

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^{[i]} u}{\partial x_k^i} \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C \|f\|_{L_p(G;E)} . \tag{2.9}$$

Proof. By second assumption and by [5, Theorem 3.5] for all $h > 0$ we have the following Ehrling-Nirenberg-Gagliardo type estimate

$$\|L_1 u\|_{L_p(G;E)} \leq h^{\mu} \ \|u\|_{W_{p,\alpha}^{[2]}(G;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(G;E)} . \tag{2.10}$$

Let O denote the operator generated by problem (2.5) and

$$L_1 u = \sum_{k=1}^{n} A_k (x) \frac{\partial^{[1]} u}{\partial x_k}.$$

By using the estimate 2.10 we obtain that there is a $\delta \in (0, 1)$ such that

$$\left\| L_1 (O + \lambda)^{-1} \right\|_{B(X)} < \delta.$$

Hence, from perturbation theory of linear operators we obtain the assertion. □

3. ABSTRACT CAUCHY PROBLEM FOR DEGENERATE PARABOLIC EQUATION

Consider the IBVP for degenerate parabolic equation:

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k\left(x_k\right)\frac{\partial^{[2]}u}{\partial x_k^2}+A\left(x\right)u+du=f\left(x,t\right),t\in\left(0,T\right),x\in G, \tag{3.11}$$

$$L_{kj}u=0$$

$$u\left(x,0\right)=0,\ t\in\left(0,T\right),x^{(k)}\in G_k, \tag{3.12}$$

where $u=u\left(x,t\right)$ is a solution, δ_{ki}, β_{ki} are complex numbers, a_k are complex-valued functions on G, $A\left(x\right)$ is a linear operator in a Banach space E, domains G, $G_{k,}$ G_{k0}, G_{kb}, σ_{ik} and $x^{(k)}$ are defined in the section 2.

For $\mathbf{p}=\left(p_0,p\right)$, $G_T=\left(0,T\right)\times G$, $L_{p,\gamma}\left(G_T;E\right)$ will denote the space of all E-valued weighted $\mathbf{p}$-summable functions with mixed norm.

Theorem 3.3. *Suppose all conditions of Theorem 2.1 are satisfied for $\varphi>\frac{\pi}{2}$. Then, for $f\in L_{\mathbf{p}}\left(G_T;E\right)$ and sufficiently large $d>0$ problem (3.11) - (3.12) has a unique solution belonging to $W_{\mathbf{p},\alpha}^{1,[2]}\left(G_T;E\left(A\right),E\right)$ and the following coercive uniform estimate holds*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p}}\left(G_T;E\right)}+\sum_{k=1}^{2}\left\|\frac{\partial^{[2]}u}{\partial x_k^2}\right\|_{L_{\mathbf{p}}\left(G_T;E\right)}+\left\|Au\right\|_{L_{\mathbf{p}}\left(G_T;E\right)}\leq C\left\|f\right\|_{L_{\mathbf{p}}\left(G_T;E\right)}.$$

Proof. The problem (3.11) can be expressed as the following abstract Cauchy problem

$$\frac{du}{dt}+\left(O+d\right)u\left(t\right)=f\left(t\right),\ u\left(0\right)=0. \tag{3.13}$$

By virtue of [9], O is R-positive in $X=L_p\left(G;E\right)$, i.e O is a generator of an analytic semigroup in X. Then by virtue of [10, Theorem 4.2] problem (3.13) has a unique solution $u\in W_{p_0}^1\left(0,T;D\left(O\right),X\right)$ for $f\in L_{p_0}\left(0,T;X\right)$ and sufficiently large $d>0$. Moreover, the following uniform estimate holds

$$\left\|\frac{du}{dt}\right\|_{L_{p_0}\left(0,T;X\right)}+\left\|Ou\right\|_{L_{p_0}\left(0,T;X\right)}\leq C\left\|f\right\|_{L_{p_0}\left(0,T;X\right)}.$$

Since $L_{p_0}\left(G_T;X\right)=L_{\mathbf{p}}\left(G_T;E\right)$, by Theorem 2.1 we have

$$\left\|\left(O+d\right)u\right\|_{L_{p_0}\left(\left(0,T\right);X\right)}=D\left(O\right).$$

Hence, the assertion follows from the above estimate. □

4. NONLINEAR DEGENERATE ABSTRACT PARABOLIC PROBLEM

In this section, we consider IBVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k\left(x_k\right)\frac{\partial^{[2]}u}{\partial x_k^2}+B\left(\left(t,x,u,D^{[1]}u\right)\right)u=F\left(t,x,u,D^{[1]}u\right), \tag{4.14}$$

$$L_{kj}u=0,u\left(x,0\right)=0,\ t\in\left(0,T\right),x\in G,x^{(k)}\in G_k, \tag{4.15}$$

where $u=u\left(x,t\right)$ is a solution, α_{kji}, β_{kji} are complex numbers, a_k are complex-valued functions on $\left[0,b_k\right]$; domains G, G_k, G_{k0}, G_{kb} and σ_{ik}, $x^{(k)}$ are defined in the section 2 and

$$D_k^{[i]}u=\frac{\partial^{[i]}u}{\partial x_k^i}=\left(x_k^{\alpha_k}\frac{\partial}{\partial x_k}\right)^iu\left(x,t\right),0\leq\alpha_k<1.$$

Let $G_T = (0,T) \times G$. Moreover, we let

$$\alpha = \alpha(x) = \prod_{k=1}^{n} x_k^{\alpha_k}.$$

$$G_0 = \prod_{k=1}^{n} (0, b_{0k}), \ G = \prod_{k=1}^{n} (0, b_k), b_k \in (0, b_{0k}),$$

$$T \in (0, T_0), \ B_{ki} = \left(W^{2,p}(G_k, E(A), E), L^p(G_k; E)\right)_{\eta_{ik},p},$$

$$\eta_{ik} = \frac{m_{ki} + \frac{1}{p(1-\alpha_k)}}{2}, \ B_0 = \prod_{k=1}^{n}\prod_{i=0}^{1} B_{ki}.$$

Theorem 4.4. *Assume the following hold:*

(1) E is an UMD space and $0 \leq \alpha_1, \alpha_2 < 1 - \frac{1}{p}$, $p \in (1, \infty)$;

(2) a_k *are continuous functions on* $[0, b_k]$, $a_k(x_k) < 0$, *for all* $x \in [0, b_k]$, $\alpha_{km_{k1}} \neq 0$, $\beta_{km_{k2}} \neq 0$, $k = 1, 2, ..., n$;

(3) there exist $\Phi_{ki} \in B_{ki}$ *such that the operator* $B(t, x, \Phi)$ *for* $\Phi = \{\Phi_{ki}\} \in B_0$ *is R-positive in E uniformly with respect to* $x \in G_0$ *and* $t \in [0, T_0]$; *moreover,*

$$B(t, x, \Phi) B^{-1}\left(t^0, x^0, \Phi\right) \in C\left(\bar{G}; L(E)\right), \ t^0 \in (0, T), \ x^0 \in G;$$

(4) $A = B\left(t^0, x^0, \Phi\right)$: $G_T \times B_0 \to L(E(A), E)$ *is continuous. Moreover, for each positive r there is a positive constant* $L(r)$ *such that*

$\left\|\left[B(t, x, U) - B\left(t, x, \bar{U}\right)\right] v\right\|_E \leq L(r) \left\|U - \bar{U}\right\|_{B_0} \|Av\|_E$

for $t \in (0, T)$, $x \in G$, $U, \bar{U} \in B_0, \bar{U} = \{\bar{u}_{ki}\}, \bar{u}_{ki} \in B_{ki}$, $\|U\|_{B_0}, \left\|\bar{U}\right\|_{B_0} \leq r, v \in D(A)$;

(5) the function F: $G_T \times B_0 \to E$ *such that* $F(., U)$ *is measurable for each* $U \in B_0$ *and* $F(t, x, .)$ *is continuous for a.a.* $t \in (0, T)$, $x \in G$. *Moreover,* $\left\|F(t, x, U) - F\left(t, x, \bar{U}\right)\right\|_E \leq \Psi_r(x) \left\|U - \bar{U}\right\|_{B_0}$ *for a.a.* $t \in (0, T)$, $x \in G$, $U, \bar{U} \in B_0$ *and* $\|U\|_{B_0}, \left\|\bar{U}\right\|_{B_0} \leq r$; $f(.) = F(., 0) \in L_p(G_T; E)$.

Then there is a $T \in (0, T_0)$ *and a* $b_k \in (0, b_{0k})$ *such that problem (4.14) - (4.15) has a unique solution belonging to* $W_{p,\alpha}^{1,[2]}(G_T; E(A), E)$.

Proof. Consider the following linear problem

$$\frac{\partial w}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} w}{\partial x_k^2} + du = f(x, t), \ x \in G, \ t \in (0, T),$$

$$L_{kj} w = 0, w(x, 0) = 0, t \in (0, T), x \in G, x^{(k)} \in G_k, \ d > 0. \tag{4.16}$$

By Theorem 3.3 there is a unique solution $w \in W_{p,\alpha}^{1,[2]}(G_T; E(A), E)$ of the problem (5.3) for $f \in L_p(G_T; E)$ and sufficiently large $d > 0$ and it satisfies the following coercive estimate

$$\|w\|_{W_{p,\alpha}^{1,[2]}(G_T; E(A), E)} \leq C_0 \|f\|_{L_p(G_T; E)},$$

uniformly with respect to $b \in (0 \ , b_0]$, i.e., the constant C_0 does not depends on $f \in L_p(G_T; E)$ and $b \in (0 \ b_0]$ where

$$A(x) = B(x, 0), \ f(x) = F(x, 0), \ x \in (0, b).$$

We want to solve the problem (4.14) - (4.15) locally by means of maximal regularity of the linear problem (4.16) via the contraction mapping theorem. For this purpose, let w be a solution of the linear BVP (4.16) Consider a ball

$$B_r = \{v \in Y, \ v - w \in Y_1, \ \|v - w\|_Y \leq r\}.$$

For given $\upsilon \in B_r$, consider the following linearized problem

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n} a_k\left(x_k\right)\frac{\partial^{[2]}u}{\partial x_k^2}+A\left(x\right)=F\left(x,V\right)+\left[B\left(x,0\right)-B\left(x,V\right)\right]\upsilon, \tag{4.17}$$

$$L_{kj}u=0,\, u\left(x,0\right)=0,\ t\in\left(0,T\right),x\in G,x^{(k)}\in G_k. \tag{4.18}$$

where $V=\{\upsilon_{ki}\},\ \upsilon_{ki}\in B_{ki}$. Define a map Q on B_r by $Q\upsilon=u$, where u is solution of (4.17). We want to show that $Q\left(B_r\right)\subset B_r$ and that Q is a contraction operator provided T and b_k are sufficiently small, and r is chosen properly. In view of separability properties of the problem (4.16) we have

$$\left\|Q\upsilon-w\right\|_Y=\left\|u-w\right\|_Y\leq C_0\left\{\left\|F\left(x,V\right)-F\left(x,0\right)\right\|_X+\right.$$

$$\left.\left\|\left[B\left(0,W\right)-B\left(x,V\right)\right]\upsilon\right\|_X\right\}.$$

By assumption (4) we have

$$\left\|\left[B\left(0,W\right)\upsilon-B\left(x,V\right)\right]\upsilon\right\|_X\leq\sup_{x\in[0,b]}\left\{\left\|\left[B\left(0,W\right)-B\left(x,W\right)\right]\upsilon\right\|_{L(E_0,E)}\right.$$

$$\left.+\left\|B\left(x,W\right)-B\left(x,V\right)\right\|_{L(E_0,E)}\left\|\upsilon\right\|_Y\right\}\leq$$

$$\left[\delta\left(b\right)+L\left(R\right)\left\|W-V\right\|_{\infty,E_0}\right]\left[\left\|\upsilon-w\right\|_Y+\left\|w\right\|_Y\right]\leq$$

$$\{\delta\left(b\right)+L\left(R\right)\left[C_1\left\|\upsilon-w\right\|_Y+\left\|\upsilon-w\right\|_Y\right]$$

$$\left[\left\|\upsilon-w\right\|_Y+\left\|w\right\|_Y\right]\}\leq\delta\left(b\right)+L\left(R\right)\left[C_1r+r\right]\left[r+\left\|w\right\|_Y\right],$$

where

$$\delta\left(b\right)=\sup_{x\in[0,b]}\left\|\left[B\left(0,W\right)-B\left(x,W\right)\right]\right\|_{B(E_0,E)}.$$

In view of above estimates, by suitable choice of μ_R, L_R and for sufficiently small $T\in(0,T_0)$ and $b_k\in(0\ ,b_{0k}]$ we have

$$\left\|Q\upsilon-w\right\|_Y\leq r,$$

i.e.

$$Q\left(B_r\right)\subset B_r.$$

Moreover, in a similar way we obtain

$$\left\|Q\upsilon-Q\bar{\upsilon}\right\|_Y\leq C_0\left\{\mu_RC_1+M_a+L\left(R\right)\left[\left\|\upsilon-w\right\|_Y+C_1r\right]+\right.$$

$$\left.L\left(R\right)C_1\left[r+\left\|w\right\|_Y\right]\left\|\upsilon-\bar{\upsilon}\right\|_Y\right\}+\delta\left(b\right).$$

By suitable choice of μ_R, L_R and for sufficiently small $T\in(0,T_0)$ and $b_k\in(0,b_{0k})$ we obtain $\left\|Q\upsilon-Q\bar{\upsilon}\right\|_Y<\eta\left\|\upsilon-\bar{\upsilon}\right\|_Y,\ \eta<1$, i.e. Q is a contraction operator. Eventually, the contraction mapping principle implies a unique fixed point of Q in B_r which is the unique strong solution $u\in W_{p,\alpha}^{1,[2]}\left(G_T;E\left(A\right),E\right).$ □

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[1]OKAN UNIVERSITY
MECHANICAL ENGINEERING
AKFIRAT, TUZLA 34959 ISTANBUL, TURKEY

[2]KHAZAR UNIVERSITY
MATHEMATICS
BAKU, AZERBAIJAN
E-mail address: veli.sahmurov@okan.edu.tr

[3]OKAN UNIVERSITY
ENVIRONMENTAL ENGINEERING
AKFIRAT, TUZLA 34959 ISTANBUL, TURKEY
E-mail address: aida.sahmurova@okan.edu.tr