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Solution of the singular Cauchy problem for a general inhomogeneous Euler–Poisson–Darboux equation

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ABSTRACT. In this paper, we solve Cauchy problem for a general form of an inhomogeneous Euler–Poisson–Darboux equation, where Bessel operator acts instead of the each second derivative. In the classical formulation, the Cauchy problem for this equation is not correct. However, for a specially selected form of the initial conditions, the equation has a solution. The general form of the Euler–Poisson–Darboux equation with such conditions we will call the singular Cauchy problem.

1. INTRODUCTION

In this paper we give a solution of the singular Cauchy problem

$$Lu(x,t)=\left[\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}-\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right)\right]u(x,t)=f(x,t), \tag{1.1}$$

$$u(x,0)=\varphi(x), \qquad t^k u_t(x,t)|_{t=0}=\psi(x). \tag{1.2}$$

where $x=(x_1,...,x_n)$, $k\in(0,1)$, $\gamma_i>0$, $x_i>0$, $i=1,2,...,n$, $t>0$. We will call the equation (1.1) **general inhomogeneous Euler–Poisson–Darboux equation**. It is complicated to mention all publications on the Cauchy problem for the equation (1.1) with initial conditions

$$u(x,0)=\varphi(x), \qquad u_t(x,0)=0, \tag{1.3}$$

when γ_i=0 for i=$1,...,n$. We just mention that a solution to (1.1)–(1.3) when f=0, γ_i=0 for i=$1,...,n$ in the classical sense was obtained in [30]–[33] and in [7]–[6] in the distributional sense. Solution to (1.1) when $u(x,0)=0$, $u_t(x,0)=0$, k,γ_i=0 for i=$1,...,n$ in terms of Riezs potential have been established in [21]. When $f=0$ a solution to the equation (1.1) with conditions (1.3) was obtained in [9, 22]. This problem has been extended in [23] for the more general equation $Lu=c^2u$, $c\in\mathbb{R}$. Solution to the equation $Lu=0$ with conditions $u(x,0)$=$\varphi(x)$, $t^k u_t(x,0)$=$\psi(x)$ was obtained in [24]. The abstract Euler–Poisson–Darboux equation (when in the right hand of (1.1) an arbitrary closed linear operator is presented) was studied in [11]–[13]. An equation of the form (1.1) is solved for the first time. It improves results obtaining in [21] when $u(x,0)=0$, $u_t(x,0)=0$, k,γ_i=0 for i=$1,...,n$.

Throughout this paper we make extensive use of the techniques of transmutation operators developed for the Bessel operator $(B_\nu)_t=\frac{\partial^2}{\partial t^2}+\frac{\nu}{t}\frac{\partial}{\partial t}$ in [25]–[14].

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2. Basic definitions

In this section, we give some basic definitions and notions needed for our further considerations.

We deal with the subset of the Euclidean space

$$\mathbb{R}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ \ t>0,\ x_1>0,\ldots,x_n>0\}.$$

Let $x=(x_1,...,x_n)$, $|x|=\sqrt{\sum\limits_{i=1}^n x_i^2}$ and Ω be finite or infinite open set in $\mathbb{R}^{n+1}$ symmetric with respect to each hyperplane $t=0$, x_i=0, $i=1,...,n$, $\Omega_+=\Omega\cap\mathbb{R}^{n+1}_+$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^{n+1}_+$ where

$$\overline{\mathbb{R}}{}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ \ t>0,x_1\geq 0,\ldots,x_n\geq 0\}.$$

Consider the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to t and x_i for any $i=1,...,n$ are continuous up to $t=0$ and x_i=0. Function $f\in C^m(\overline{\Omega}_+)$ we will call *even with respect to* t and x_i, $i=1,...,n$ if $\left.\frac{\partial^{2k+1}f}{\partial t^{2k+1}}\right|_{t=0,x=0}=0$, $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{t=0,x=0}=0$ for all nonnegative integer $k\leq\frac{m-1}{2}$ (see [15], p. 21). Class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions from $C^m(\overline{\Omega}_+)$ even with respect to each variable t and x_i, $i=1,...,n$. In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^{n+1}_+)$ by C^m_{ev}. Let $\overset{\circ}{C}{}^m_{ev}(\overline{\Omega}_+)$ be the space of all functions $f\in C^m(\overline{\Omega}_+)$ with a compact support. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}^{n+1}_+)=C^\infty_{ev}$.

The space S_{ev} is the subspace of the space of rapidly decreasing functions:

$$S_{ev}=S_{ev}(\mathbb{R}^{n+1}_+)=\left\{f\in C^\infty_{ev}:\sup_{(t,x)\in\mathbb{R}^{n+1}_+}\left|t^{\alpha_0}x^\alpha D_t^{\beta_0}D^\beta f(x)\right|<\infty\right\},$$

where $\alpha=(\alpha_1,...,\alpha_n)$, $\beta=(\beta_1,...,\beta_n)$, $\alpha_0,\alpha_1,...,\alpha_n,\beta_0,\beta_1,...,\beta_n$ are integer nonnegative numbers, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $D^\beta=D^{\beta_1}_{x_1}...D^{\beta_n}_{x_n}$, $D_{x_j}=\frac{\partial}{\partial x_j}$.

We will deal with the **singular Bessel differential operator** B_ν (see, for example, [15], p. 5):

$$(B_\nu)_t=\frac{\partial^2}{\partial t^2}+\frac{\nu}{t}\frac{\partial}{\partial t}=\frac{1}{t^\nu}\frac{\partial}{\partial t}t^\nu\frac{\partial}{\partial t},\qquad t>0,$$

and the elliptical singular operator or the Laplace-Bessel operator $\triangle_\gamma$:

$$\triangle_\gamma=(\triangle_\gamma)_x=\sum_{i=1}^n(B_{\gamma_i})_{x_i}=\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\right)=\sum_{i=1}^n\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}. \tag{2.4}$$

The operator (2.4) belongs to the class of **B–elliptic** operators by I. A. Kipriyanovs' classification (see [15]). Operator $(\Box_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x$ is **B–hyperbolic** by the same classifications.

The **B–polyharmonic of order** p function $f=f(x)$ is the function $f\in C^{2m}_{ev}(\overline{\mathbb{R}}^n_+)$ such that

$$\Delta^m_\gamma f=0, \tag{2.5}$$

where Δ_γ is operator (2.4). The operator (2.5) was considered in [15]. The B-polyharmonic of order 1 function we will call **B–harmonic**.

The symbol j_ν is used for the **normalized Bessel function**:

$$j_\nu(t) = \frac{2^\nu \Gamma(\nu+1)}{t^\nu} J_\nu(t),$$

where $J_\nu(t)$ is the Bessel function of the first kind of order ν (see [29]). The function $j_\nu(t)$ is even by t. Using formulas 9.1.27 from [1] we obtain

$$(B_\nu)_t j_{\frac{\nu-1}{2}}(\tau t) = -\tau^2 j_{\frac{\nu-1}{2}}(\tau t). \tag{2.6}$$

We deal with multi-index $\gamma{=}(\gamma_1, \ldots, \gamma_n)$ which consists of positive fixed reals $\gamma_i > 0$, $i{=}1, ..., n$, $|\gamma|{=}\gamma_1{+}\ldots{+}\gamma_n$.

The operator ${}^k T_t^\tau$ for $k > 0$ is **generalized translation** acts by a variable t defined by the next formula (see [16], p. 122, formula (5.19))

$${}^k T_t^\tau f(t,x) {=} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)} \int\limits_0^\pi f(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}, x)\ \sin^{k-1}\varphi\, d\varphi \tag{2.7}$$

and ${}^\gamma \mathbf{T}_x^y {=} {}^{\gamma_1} T_{x_1}^{y_1} ... {}^{\gamma_n} T_{x_n}^{y_n}$ is **multidimensional generalized translation**, where each of the one–dimensional generalized translations ${}^{\gamma_i} T_{x_i}^{y_i}$ acts by a variable x_i for $i{=}1, ..., n$ according to the formula (2.7).

Based on the multidimensional generalized translation ${}^\gamma \mathbf{T}_x^y$ the **weighted spherical mean** $M_r^\gamma[f(x)]$ of a suitable function is constructed by the formula

$$M_r^\gamma[f(x)] = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma \mathbf{T}_x^{r\theta} f(x) \theta^\gamma dS, \tag{2.8}$$

where $\theta^\gamma {=} \prod\limits_{i=1}^n \theta_i^{\gamma_i}$, $S_1^+(n){=}\{\theta{:}|\theta|{=}1, \theta{\in}\mathbb{R}_+^n\}$ and $|S_1^+(n)|_\gamma = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}$. It is easy to see that

$$\lim_{r\to 0} M_r^\gamma[f(x)] = f(x), \qquad \lim_{r\to 0} \frac{\partial}{\partial r} M_r^\gamma[f(x)] = 0. \tag{2.9}$$

3. Solution of the Cauchy problems for homogeneous Euler–Poisson–Darboux equation

Here we give solutions to Cauchy problems for linear homogeneous Euler–Poisson–Darboux equation and some auxiliary results.

In [22] the next basic lemma was proven.

Lemma 3.1. *The weighted spherical mean $M_r^\gamma[f(x)]$ is the transmutation operator (see* [25]*) intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_r$ for the twice continuously differentiable function f even with respect to each of the independent variables:*

$$M_r^\gamma[(\Delta_\gamma)_x f(x)] = (B_{n+|\gamma|-1})_r M_r^\gamma[f(x)].$$

Using this lemma, the Hadamard descent method, B–polyharmonic functions and recurrence formulas, a solution to the Cauchy problem for the multidimensional Euler–Poisson–Darboux equation wherein the Bessel operator acts in each of the variables:

$$(\Box_{k,\gamma})_{t,x} u(t,x) = 0, \quad -\infty < k < \infty, \quad u = u(t,x), \quad (t,x) \in \mathbb{R}_+^{n+1} \tag{3.10}$$

was obtained. We will call (3.10) the **general Euler–Poisson–Darboux equation**.

The next theorem gives solution to the (3.10) with conditions $u(0,x){=}\varphi(x)$, $u_t(0,x){=}0$ for any $k \in \mathbb{R}$ (see [22]).

Theorem 3.1. *In the case $k>n+|\gamma|-1$, $\varphi(x)\in C^2_{ev}(\mathbb{R}^n_+)$ solution to*

$$(\Box_{k,\gamma})_{t,x}u(t,x)=0, \tag{3.11}$$

$$u(0,x)=\varphi(x),\qquad u_t(0,x)=0, \tag{3.12}$$

is

$$u(t,x)=A(n,\gamma,k)\,t^{1-k}\int\limits_0^t(t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}M_r^\gamma[\varphi(x)]dr, \tag{3.13}$$

where

$$A(n,\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}.$$

Let $\varphi\in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}(\mathbb{R}^n_+)$. Then the solution to (3.11)–(3.12) *for $k<n+|\gamma|-1$, $k\neq-1,-3,-5,...$ is*

$$u(t,x)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}v(t,x)), \tag{3.14}$$

where m is a minimum integer such that $m\geq\frac{n+|\gamma|-k-1}{2}$ and $v(t,x)$ is the solution to the Cauchy problem

$$(B_{k+2m})_t v=(\Delta_\gamma)_x v, \tag{3.15}$$

$$v(0,v)=\frac{\varphi(x)}{(k+1)(k+3)...(k+2m-1)},\qquad v_t(0,x)=0. \tag{3.16}$$

If φ is B–polyharmonic of order $\frac{1-k}{2}$ and even with respect to each variable then one of the solutions of the Cauchy problem (3.11)–(3.12) *for the $k=-1,-3,-5,...$ is given by*

$$u(t,x)=f(x),\qquad k=-1, \tag{3.17}$$

$$u(t,x)=\varphi(x)+\sum_{h=1}^{-\frac{k+1}{2}}\frac{\Delta_\gamma^h\varphi}{(k+1)...(k+2h-1)}\,\frac{t^{2h}}{2\cdot4\cdot....\cdot2h},\qquad k=-3,-5,... \tag{3.18}$$

The solution to (3.11)–(3.12) *is unique for $k\geq0$.*

The next theorem gives solution to the (3.10) with conditions $u(0,x)=0$, $t^ku_t(0,x)=\psi(x)$ for any $k<1$ (see [24]).

Theorem 3.2. *If $\psi\in C_{ev}^{\left[\frac{n+|\gamma|+k-1}{2}\right]}(\mathbb{R}^n_+)$ then the solution $u=u(t,x))$ of*

$$(\Box_{k,\gamma})_{t,x}u(t,x)=0, \tag{3.19}$$

$$u(0,x)=0,\qquad t^ku_t(0,x)=\psi(x) \tag{3.20}$$

is given by

$$u(t,x)=\frac{\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\sum_{s=0}^q\frac{C_q^s\,t^{1-k+2s}}{2^s\Gamma\left(\frac{3-k}{2}+s\right)}\times$$

$$\times\int\limits_0^1(1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}}r^{n+|\gamma|-1}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_{tr}^\gamma[\psi(x)]dr. \tag{3.21}$$

if $n+|\gamma|+k$ is not an odd integer and

$$u(t,x)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\left(t^{n+|\gamma|-2}M_t^\gamma[\psi(x)]\right).$$

if $n+|\gamma|+k$ is an odd integer, where $q\geq 0$ is the smallest positive integer number such that $2-k+2q\geq n+|\gamma|-1$.

4. Riesz hyperbolic B-potential and the solution to the Cauchy problems for nonhomogeneous Euler–Poisson–Darboux equation

In this section we deal with Riesz hyperbolic B-potential and its analytic continuations, associated with the operator

$$(\Box_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x,\qquad (B_k)_t=\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t},\qquad (\Delta_\gamma)_x=\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right).$$

Using hyperbolic Riesz B–potential we give the solution to the Cauchy problem for the inhomogeneous general Euler–Poisson–Darboux equation in a unique formula implying an analytic continuation with respect to the parameter α. The main difficulty concerning the analytic continuation was to prove that hyperbolic Riesz B–potential for $\alpha=0$ is the identity operator.

The negative real powers $(\Box_{k,\gamma})^{-\frac{\alpha}{2}}, \alpha>0$ will be Riesz potential $I^\alpha_{\Box_{k,\gamma}}$ with Lorentz distance generated by generalized translation operator

$$(I^\alpha_{\Box_{k,\gamma}}f)(t,x)=\frac{1}{H_{n,k,\gamma}(\alpha)}\int\limits_{K^+}(\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}}({}^kT_t^\tau\,{}^\gamma\mathbf{T}_x^y f(t,x))\tau^k y^\gamma d\tau dy, \tag{4.22}$$

where $y^\gamma=\prod\limits_{i=1}^n y_i^{\gamma_i}$,

$$H_{n,k,\gamma}(\alpha)=\frac{2^{\alpha-n-1}}{\pi}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right),$$

$K^+=\{(t,y)\in\mathbb{R}^{n+1}_+\,:\,t^2\geq|y|^2\}$. The potential (4.22) is called the **hyperbolic Riesz B–potential**.

It is easy to see that when $f\in S_{ev}$ the integral in (4.22) converges absolutely for $\alpha>n+k+|\gamma|-1$. (cf. [22]).

Lemma 4.2. *Let $\lambda>0$, $p=1,2,...,$ $(t,x_1,...,x_n)\in\mathbb{R}^{n+1}_+$ and $(\Box_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x$, then*

$$((\Box_{k,\gamma})_{t,x})^p\,(t^2-|x|^2)^{\lambda+p}=$$

$$=4^p(\lambda+1)...(\lambda+p)\left(\frac{n+|\gamma|+k+1}{2}+\lambda\right)...\left(\frac{n+|\gamma|+k+1}{2}+\lambda+p-1\right)(t^2-|x|^2)^\lambda. \tag{4.23}$$

Proof. Let verify the formula (4.23) for $p=1$:

$$((B_k)_t-(\Delta_\gamma)_x)\,(t^2-|x|^2)^{\lambda+1}=(B_k)_t(t^2-|x|^2)^{\lambda+1}-(\Delta_\gamma)_x(t^2-|x|^2)^{\lambda+1}=$$

$$=\frac{1}{t^k}\frac{\partial}{\partial t}t^k\frac{\partial}{\partial t}(t^2-|x|^2)^{\lambda+1}-\sum_{i=1}^n\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}(t^2-|x|^2)^{\lambda+1}=$$

$$=2(\lambda+1)\frac{1}{t^k}\frac{\partial}{\partial t}t^{k+1}(t^2-|x|^2)^\lambda+\sum_{i=1}^n 2(\lambda+1)\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i+1}(t^2-|x|^2)^\lambda=$$

$$= 2(\lambda+1)\frac{1}{t^k}((k+1)t^k(t^2-|x|^2)^\lambda + 2\lambda t^{k+2}(t^2-|x|^2)^{\lambda-1})+$$
$$+\sum_{i=1}^n 2(\lambda+1)\frac{1}{x_i^{\gamma_i}}((\gamma_i+1)x_i^{\gamma_i}(t^2-|x|^2)^\lambda - 2\lambda x_i^{\gamma_i+2}(t^2-|x|^2)^{\lambda-1}) =$$
$$= 2(\lambda+1)\Big[(k+1)(t^2-|x|^2)^\lambda + 2\lambda t^2(t^2-|x|^2)^{\lambda-1}+$$
$$+\sum_{i=1}^n(\gamma_i+1)(t^2-|x|^2)^\lambda - 2\lambda x_i^2(t^2-|x|^2)^{\lambda-1})\Big] =$$
$$= 2(\lambda+1)\big[(k+1)(t^2-|x|^2)^\lambda + 2\lambda t^2(t^2-|x|^2)^{\lambda-1}+$$
$$+(n+|\gamma|)(t^2-|x|^2)^\lambda - 2\lambda|x|^2(t^2-|x|^2)^{\lambda-1})\big] =$$
$$= 2(\lambda+1)((n+|\gamma|+k+1+2\lambda)(t^2-|x|^2)^\lambda).$$

So we get

$$((B_k)_t - (\Delta_\gamma)_x)(t^2-|x|^2)^{\lambda+1} = 4(\lambda+1)\left(\frac{n+|\gamma|+k+1}{2}+\lambda\right)(t^2-|x|^2)^\lambda. \tag{4.24}$$

Applying formula (4.24) p–times we obtain (4.23). □

Lemma 4.3. *If $n+|\gamma|-2<\alpha$ and $p\in\mathbb{N}$, then*

$$I^{\alpha+2p}_{\square_{k,\gamma}}(\square_{k,\gamma})^p f = I^{\alpha}_{\square_{k,\gamma}} f, \tag{4.25}$$

for any $f\in S_{ev}$, such that $\frac{\partial^m f}{\partial t^m}=0$ $\frac{\partial^m f}{\partial x_i^m}=0$ when $(t,x_1,...,x_n)=(0,0,...,0)$, $m=0,..,2p$.

Proof. From the integral representation (4.22) using the formula 1.8.3 from [15] of the form ${}^{\gamma_i}T^{y_i}_{x_i}(B_{\gamma_i})_{x_i} = (B_{\gamma_i})^{\gamma_i}_{y_i}T^{y_i}_{x_i}$ we obtain

$$(I^{\alpha+2p}_{\square_{k,\gamma}}(\square_{k,\gamma})^p f)(t,x) =$$
$$= \frac{1}{H_{n,k,\gamma}(\alpha+2p)}\int\limits_{K^+}(\tau^2-|y|^2)^{\frac{\alpha+2p-n-1-k-|\gamma|}{2}}({}^kT^\tau_t\,{}^\gamma\mathbf{T}^y_x(\square_{k,\gamma})^p_{t,x}f(t,x))\tau^k y^\gamma d\tau dy =$$
$$= \frac{1}{H_{n,k,\gamma}(\alpha+2p)}\int\limits_{K^+}(\tau^2-|y|^2)^{\frac{\alpha+2p-n-1-k-|\gamma|}{2}}\left[(\square_{k,\gamma})^p_{\tau,y}({}^kT^\tau_t\,{}^\gamma\mathbf{T}^y_x f(t,x))\right]\tau^k y^\gamma d\tau dy.$$

Recall that $\frac{\partial^m f}{\partial t^m}=0$ $\frac{\partial^m f}{\partial x_i^m}=0$ when $(t,x_1,...,x_n)=(0,0,...,0)$, $m=0,..,2p$. Then applying the integration by parts formula we find

$$(I^{\alpha+2p}_{\square_{k,\gamma}}(\square_{k,\gamma})^p f)(t,x) =$$
$$= \frac{1}{H_{n,k,\gamma}(\alpha+2p)}\int\limits_{K^+}\left[(\square_{k,\gamma})^p_{\tau,y}(\tau^2-|y|^2)^{\frac{\alpha+2p-n-1-k-|\gamma|}{2}}\right]({}^kT^\tau_t\,{}^\gamma\mathbf{T}^y_x f(t,x))\tau^k y^\gamma d\tau dy.$$

Applying (4.23) we get

$$(\square_{k,\gamma})^p_{\tau,y}(\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}+p} = 4^p\left(\frac{\alpha-n-1-k-|\gamma|}{2}+1\right)...$$
$$...\left(\frac{\alpha-n-1-k-|\gamma|}{2}+p\right)\left(\frac{\alpha}{2}\right)...\left(\frac{\alpha}{2}+p-1\right)(t^2-r^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}}.$$

The result (4.25) follows from the

$$\frac{4^p\left(\frac{\alpha-n-1-k-|\gamma|}{2}+1\right)...\left(\frac{\alpha-n-1-k-|\gamma|}{2}+p\right)\left(\frac{\alpha}{2}\right)...\left(\frac{\alpha}{2}+p-1\right)}{H_{n,k,\gamma}(\alpha+2p)} =$$

$$= \frac{\pi 4^p \left(\frac{\alpha-n-1-k-|\gamma|}{2}+1\right)\ldots\left(\frac{\alpha-n-1-k-|\gamma|}{2}+p\right)\left(\frac{\alpha}{2}\right)\ldots\left(\frac{\alpha}{2}+p-1\right)}{2^{\alpha+2p-n-1}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}+p\right)\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}+p\right)} = \frac{1}{H_{n,k,\gamma}(\alpha)}.$$

□

Now we prove that the integral $I^{\alpha}_{\Box_{k,\gamma}}$ can be analytically continued to all values $\alpha > -1$ and that for these values it is a holomorphic function of α for $f \in S_{ev}$. Moreover we show that $I^{0}_{\Box_{k,\gamma}}$ is the identity operator for $f \in S_{ev}$.

Theorem 4.3. *Let $f \in S_{ev}$, $n+|\gamma|-k>0$, k is not odd, then hyperbolic Riesz B–potential*

$$(I^{\alpha}_{\Box_{k,\gamma}} f)(t,x) = \frac{1}{H_{n,k,\gamma}(\alpha)} \int\limits_{K^+} (\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}} ({}^kT^{\tau}_t\, {}^{\gamma}\mathbf{T}^y_x f(t,x))\tau^k y^{\gamma} d\tau dy$$

can be analytically continued to all values $\alpha > -1$ and $(I^{0}_{\Box_{k,\gamma}} f)(t,x) = f(t,x)$.

Proof. Let $(\tau, y) \in \mathbb{R}^{n+1}_{+}$, $\delta > 0$ and $\tau + |y| = \delta$ is a part of a cone with the vertex at $(\delta, 0, ..., 0)$. We denote K^+_δ the area bounded by a part of a cone $\tau + |y| = \delta$ from above by $\tau = |y|$ from below and by $\tau = 0, y_i = 0, i = 1, \ldots n$, including boundary. Then $K^+ = K^+_\delta \cup (K^+ \setminus K^+_\delta)$.

Let consider first $(I^{\alpha}_{\Box_{k,\gamma}} f)(O)$, where $O = (0, .., 0) \in \overline{\mathbb{R}}^{n+1}_{+}$. Dividing domain K^+ into two parts K^+_δ and $K^+ \setminus K^+_\delta$ we can write

$$(I^{\alpha}_{\Box_{k,\gamma}} f)(O) = \frac{1}{H_{n,k,\gamma}(\alpha)} \int\limits_{K^+} (\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}} f(\tau,y)\tau^k y^{\gamma} d\tau dy = I^{\alpha}_1 + I^{\alpha}_2,$$

where

$$I^{\alpha}_1 = \frac{1}{H_{n,k,\gamma}(\alpha)} \int\limits_{K^+_\delta} (\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}} f(\tau,y)\tau^k y^{\gamma} d\tau dy,$$

$$I^{\alpha}_2 = \frac{1}{H_{n,k,\gamma}(\alpha)} \int\limits_{K^+\setminus K^+_\delta} (\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}} f(\tau,y)\tau^k y^{\gamma} d\tau dy.$$

We will show that I^{α}_1 and I^{α}_2 are holomorphic for $\alpha > -1$ and $I^0_1 = f(O)$, $I^0_2 = 0$ which gives that $(I^{\alpha}_{\Box_{k,\gamma}} f)(O) = f(O)$.

Consider I^{α}_1. Expressing integral by y in spherical coordinates $y = \rho\theta$ we obtain

$$I^{\alpha}_1 = \frac{1}{H_{n,k,\gamma}(\alpha)} \int\limits_{K^+_\delta} \rho^{n+|\gamma|-1}(\tau^2-\rho^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}} f(\tau,\rho\theta)\tau^k \theta^{\gamma} dS d\rho d\tau.$$

Changing variables τ and ρ by formulas

$$\rho = \frac{1}{2}\sigma(1-\chi), \qquad \tau = \frac{1}{2}\sigma(1+\chi), \tag{4.26}$$

noticing that $\frac{\partial(\tau,\rho)}{\partial(\sigma,\chi)} = \frac{1}{2}\sigma$ and $(\tau, y) = (\tau, \rho\theta) = \sigma(b + \chi c)$, where $b, c \in \mathbb{R}^{n+1}_{+}$ we obtain

$$I^{\alpha}_1 = \frac{1}{2^{n+k+|\gamma|}H_{n,k,\gamma}(\alpha)} \int\limits_{S^+_1(n)} \theta^{\gamma} dS \int\limits_0^{\delta} \sigma^{\alpha-1} d\sigma \times$$

$$\times \int_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}f(\sigma(b+\chi c))d\chi.$$

We develop $f(\sigma(b+\chi c))$ by the Teylor formula in χ:

$$f(y)=f(\sigma(b+\chi c))=\sum_{p=0}^{N-1}\frac{\chi^p}{p!}F_p(\sigma,\theta)+R_N(\chi),$$

where

$$F_p(\sigma,\theta)=\frac{\partial^p}{\partial\chi^p}f(\sigma(b+\chi c))\Big|_{\chi=0}$$

and

$$R_N(\chi)=\frac{1}{(N-1)!}\int_0^{\chi}\frac{\partial^N}{\partial\chi^N}f(\sigma(b+\widetilde{\chi}c))\,(\chi-\widetilde{\chi})^{N-1}d\widetilde{\chi}.$$

Then

$$I_1^\alpha=\frac{1}{2^{n+k+|\gamma|}H_{n,k,\gamma}(\alpha)}\left(\sum_{p=0}^{N-1}\frac{1}{p!}\int_{S_1^+(n)}\theta^\gamma dS\int_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma\times\right.$$

$$\times\int_0^1\chi^{\frac{\alpha-n-1-k-|\gamma|}{2}+p}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}d\chi+$$

$$\left.+\int_{S_1^+(n)}\theta^\gamma dS\int_0^\delta\sigma^{\alpha-1}d\sigma\int_0^1\chi^{\frac{\alpha-n-1-k-|\gamma|}{2}}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}R_N(\chi)d\chi\right).\tag{4.27}$$

Integral

$$\int_0^1\chi^{\frac{\alpha-n-1-k-|\gamma|}{2}+p}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}d\chi$$

is integral representation of the Gauss hypergeometric function

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

for $b=\frac{\alpha-n+1-k-|\gamma|}{2}+p>0$, $c-b=n+|\gamma|>0$. It is known that ${}_2F_1(a,b;c;z)$ is defined for $z=-1$ when $c-a-b>0$. Since in our case $c-a-b=n+|\gamma|+k>0$ it can be analytically continued to $\frac{\alpha-n+1-k-|\gamma|}{2}+p\leq-1$ as the power series. So we have

$$\int_0^1\chi^{\frac{\alpha-n-1-k-|\gamma|}{2}+p}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}d\chi=\frac{\Gamma\left(\frac{\alpha-n+1-k-|\gamma|}{2}+p\right)\Gamma(n+|\gamma|)}{\Gamma\left(\frac{\alpha+n+1+|\gamma|-k}{2}+p\right)}\times$$

$$\times\,{}_2F_1\left(-k,\frac{\alpha-n+1-k-|\gamma|}{2}+p;\frac{\alpha+n+1+|\gamma|-k}{2}+p;-1\right).$$

It means that we have analytic continuation of I_1^α to all $\alpha>0$. Integrating by parts the integral by σ in (4.27) we obtain analytic continuation of I_1^α to all $\alpha>-1$.

Let

$$K_p(\alpha)=\frac{\pi}{2^{k+|\gamma|+\alpha-1}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right)}\times$$

$$\times\frac{\Gamma\left(\frac{\alpha-n+1-k-|\gamma|}{2}+p\right)\Gamma(n+|\gamma|)}{\Gamma\left(\frac{\alpha+n+1+|\gamma|-k}{2}+p\right)}\times{}_2F_1\left(-k,\frac{\alpha-n+1-k-|\gamma|}{2}+p;\frac{\alpha+n+1+|\gamma|-k}{2}+p;-1\right).$$

The most important term in (4.27) is the term with $p=0$ has the form

$$\frac{K_0(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_{S_1^+(n)}\theta^\gamma dS\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma. \tag{4.28}$$

Using formula 15.1.21 from [1] of the form

$${}_2F_1(a,b;a-b+1;-1)=\frac{\sqrt{\pi}\Gamma(a-b+1)}{2^a\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(\frac{a+1}{2}\right)},\qquad a-b+1\neq 0,-1,-2,...$$

and taking into account the Euler's reflection formula

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin\left(\pi z\right)},\qquad z\notin\mathbb{Z}$$

we get for $\frac{k+1}{2}\notin\mathbb{Z}$

$$K_0(0)=\frac{\sqrt{\pi}\Gamma(n+|\gamma|)}{2^{|\gamma|-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}. \tag{4.29}$$

Now we carry out the analytic continuation of the expression (4.28). The factor $K_0(\alpha)$ has no singularity at $\alpha=0$ and for $\frac{k+1}{2}\notin\mathbb{Z}$ and the formula (4.29) is valid for $K_0(0)$. Integrating $\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma$ by parts we get

$$\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma=\frac{1}{\alpha\Gamma\left(\frac{\alpha}{2}\right)}\left(f(\sigma b)\sigma^\alpha\Big|_{\sigma=0}^{\delta}-\int\limits_0^\delta f'_\sigma(\sigma b)\sigma^\alpha d\sigma\right)$$

$$=\frac{1}{2\Gamma\left(\frac{\alpha}{2}+1\right)}\left(f(\delta b)\delta^\alpha-\int\limits_0^\delta f'_\sigma(\sigma b)\sigma^\alpha d\sigma\right).$$

Then, since

$$\int\limits_{S_1^+(n)}\theta^\gamma dS=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}$$

using the formula

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z)$$

we obtain

$$\lim_{\alpha\to0}\frac{K_0(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_{S_1^+(n)}\theta^\gamma dS\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma=\frac{K_0(0)}{2}\left(f(\delta b)-\int\limits_0^\delta f'_\sigma(\sigma b)d\sigma\right)\int\limits_{S_1^+(n)}\theta^\gamma dS=$$

$$= f(\delta b) - f(\delta b) + f(0) = f(0).$$

Now we show that for $p = 1, 2, ...$ all summands in (4.27) are equal to zero. Applying formula $\Gamma(z+m+1) = z(z+1)\cdots(z+m)\Gamma(z)$, $m \in \mathbb{N}$ to $\Gamma\left(\frac{\alpha-n+1-k-|\gamma|}{2}+p\right)$ we obtain

$$K_p(\alpha) = \frac{\pi\Gamma(n+|\gamma|)\left(1-\frac{n+1+k+|\gamma|-\alpha}{2}\right)\left(2-\frac{n+1+k+|\gamma|-\alpha}{2}\right)...\left(p-\frac{n+1+k+|\gamma|-\alpha}{2}\right)}{2^{k+|\gamma|+\alpha-1}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{\alpha+n+1+|\gamma|-k}{2}+p\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times$$

$$\times\, {}_2F_1\left(-k, \frac{\alpha-n+1-k-|\gamma|}{2}+p; \frac{\alpha+n+1+|\gamma|-k}{2}+p; -1\right).$$

That means that $K_p(0)$ has no singularity at $\alpha = 0$. For any positive integer p we have

$$\frac{\partial^p}{\partial\tau^p} f(\sigma(b+\tau c)) = \sigma^p\left(\sum_{k=1}^{n} c^k\partial_k\right)^p f, \tag{4.30}$$

where $\partial_k = \frac{\partial}{\partial y_k}$, $y = \sigma(b+\tau c)$. Hence all intergals

$$\int\limits_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma, \qquad \int\limits_0^\delta \sigma^{\alpha-1}d\sigma\int\limits_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}R_N(\chi)d\chi$$

converge for $\alpha > -1$ when $p = 1, 2, ...$, $K_p(0)$ is finite and $\lim\limits_{\alpha\to 0}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} = 0$ we get that all summands in (4.27) for $p = 1, 2, ...$ are equal to zero.

Now we consider I_2^α. Expressing integral by y in spherical coordinates $y = \rho\theta$ we obtain

$$I_2^\alpha = \frac{1}{H_{n,k,\gamma}(\alpha)}\int\limits_{K^+\setminus K_\delta^+}\rho^{n+|\gamma|-1}(\tau^2-\rho^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}}f(\tau,\rho\theta)\tau^k\theta^\gamma dSd\rho d\tau.$$

Making the change of variables (4.26) in the last expression we obtain

$$I_2^\alpha = \frac{1}{2^{n+k+|\gamma|}H_{n,k,\gamma}(\alpha)}\int\limits_{S_1^+(n)}\theta^\gamma dS\times$$

$$\times\int\limits_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}d\chi\int\limits_\delta^\infty \sigma^{\alpha-1}f(\sigma(b+\chi c))d\sigma.$$

Since $f \in S_{ev}$ and $\delta > 0$ then the function $G(\chi,\theta,\alpha) = \int\limits_\delta^\infty \sigma^{\alpha-1}f(\sigma(b+\chi c))d\sigma$ is in S_{ev} by χ as well as by θ and holomorphic in α. Assuming

$$\frac{\pi}{2^{\alpha+k+|\gamma|-1}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}(1+\chi)^k(1-\chi)^{n+|\gamma|-1}G(\chi,\theta,\alpha)=W(\chi)$$

we get

$$I_2^\alpha = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right)}\int\limits_{S_1^+(n)}\theta^\gamma dS\int\limits_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}}W(\chi)d\chi.$$

The expression $\frac{1}{\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right)}\int\limits_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}} W(\chi)d\chi$ can be continued analytically as a holomorphic function of α to any $\alpha>\alpha_0$ by integrating by parts, where α_0 is arbitrary. So

$$\frac{1}{\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right)}\int\limits_{S_1^+(n)}\theta^\gamma dS\int\limits_0^1 \chi^{\frac{\alpha-n-1-k-|\gamma|}{2}} W(\chi)d\chi$$

is a holomorphic function for $\alpha>-1$ and since I_2^α contains a factor $\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}$ it vanishes when $\alpha\to 0$. This completes the proof of the fact that $(I^\alpha_{\square_{k,\gamma}}f)(0)=f(0)$. Taking $g(t,x)=({}^kT_t^\tau\,{}^\gamma\mathbf{T}_x^y f(t,x))$ instead of $f(t,x)$ we can write $(I^\alpha_{\square_{k,\gamma}}f)(t,x)=f(t,x)$ that means that $I^0_{\square_{k,\gamma}}$ is the identity operator. □

Now consider the Cauchy problem when $n+|\gamma|-k>0$, k is not odd

$$(\square_{k,\gamma})_{t,x}u(t,x)=f(t,x),\qquad f\in S_{ev},\tag{4.31}$$

$$u(0,x)=0,\qquad u_t(0,x)=0.\tag{4.32}$$

Applying operator $I^{\alpha+2}_{\square_{k,\gamma}}$ to the (4.31), using Lemma 4.3 and Theorem 4.3 and passing to the limit with $\alpha\to 0$ we obtain that the solution to the problem (4.31)–(4.32) is

$$u(t,x)=(I^2_{\square_{k,\gamma}}f)(t,x),$$

where f is from the Lemma 4.3

So we can write next theorem.

Theorem 4.4. *The solution $u\in S_{ev}$ of the problem*

$$(\square_{k,\gamma})_{t,x}u(t,x)=f(t,x),\tag{4.33}$$

$$u(x,0)=\varphi(x),\qquad t^k u_t(x,0)=\psi(x),\tag{4.34}$$

where $k\in(0,1)$, $(t,x)\in\mathbb{R}^{n+1}_+$ and f is from the Lemma 4.3 is

$$u(t,x)=u_1(t,x)+u_2(t,x)+(I^2_{\square_{k,\gamma}}f)(t,x)$$

where $u_1(t,x)$ is given in Theorem 3.1, $u_2(t,x)$ is given in Theorem 3.2 and $I^2_{\square_{k,\gamma}}$ is hyperbolic Riesz B–potential (4.22) *or its analytic continuation given by Theorem 4.3.*

Example. Consider a Cauchy problem

$$\left(\frac{\partial^2}{\partial t^2}+\frac{2}{t}\frac{\partial^2}{\partial t^2}-(\Delta_\gamma)_x\right)u(t,x)=t^2e^{-t}\mathbf{j}_\gamma(x;b),\tag{4.35}$$

$$u(0,x)=3\mathbf{j}_\gamma(x;b),\qquad u_t(0,x)=0,\tag{4.36}$$

where $n+|\gamma|\geq 2$, $\mathbf{j}_\gamma(x;\xi)=\prod\limits_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$ $\gamma_1>0,...,\gamma_n>0$, $b=(b_1,...,b_n)$, $|b|=1$.

A solution to (4.35)–(4.36) is

$$u(t,x)=\frac{1}{2}e^{-t}(t^2+3t+3)\mathbf{j}_\gamma(x;b).$$

Checking we have

$$(\Delta_\gamma)_x e^{-t}(t^2+3t+3)\mathbf{j}_\gamma(x;b)=-\frac{1}{2}e^{-t}(t^2+3t+3)\mathbf{j}_\gamma(x;b),$$

$$(B_2)_t\,e^{-t}(t^2+3t+3)\mathbf{j}_\gamma(x;b)=\frac{1}{2}e^{-t}(t^2-3t-3)\mathbf{j}_\gamma(x;b)$$

and

$$((B_2)_t - (\Delta_\gamma)_x)\, e^{-t}(t^2+3t+3)\mathbf{j}_\gamma(x;b) = t^2 e^{-t}\mathbf{j}_\gamma(x;b),$$
$$u(0,x) = 3\mathbf{j}_\gamma(x;b), \qquad u_t(0,x) = 0.$$

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