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Fixed point theorems for contractions in semicomplete semimetric spaces

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ABSTRACT. We introduce the concept of semicompleteness on semimetric space, which is weaker than completeness. We prove fixed point theorems for contractions in semicomplete semimetric spaces. Also, we generalize Jachymski-Matkowski-Świątkowski's fixed point theorem in semimetric spaces.

1. INTRODUCTION

We begin with the definition of semimetric space.

Definition 1.1. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *semimetric space* if the following hold:

(D1) $d(x, x) = 0$.
(D2) $d(x, y) = 0 \Rightarrow x = y$.
(D3) $d(x, y) = d(y, x)$. (symmetry)

Every metric space satisfies (D1)–(D3) and the following (D4).

(D4) $d(x, z) \leq d(x, y) + d(y, z)$. (subadditivity or triangle inequality)

In this paper, we often assume the following (D5) instead of (D4).

(D5) There exist $\delta > 0$ and $\varepsilon > 0$ such that $d(x, y) < \delta$ and $d(y, z) < \delta$ imply $d(x, z) < \varepsilon$.

Note that (D5) is much weaker than (D4) because (D5) only ensures the existence of δ and ε. Under such a weak condition, Jachymski, Matkowski and Świątkowski in [6] proved the following splendid fixed point theorem.

Theorem 1.1 (Theorem 1 in [6]). *Let (X, d) be a Hausdorff, complete semimetric space satisfying* (D5). *Let T be a Matkowski contraction on X, that is, there exists a function φ from $[0, \infty)$ into itself satisfying the following*:

(i) *φ is nondecreasing.*
(ii) $\lim_n \varphi^n(t) = 0$.
(iii) $d(Tx, Ty) \leq \varphi \circ d(x, y)$.

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

See [7], for Matkowski contractions. See also [8, 10, 13] and others.

The author thinks that the biggest merit of Theorem 1.1 is that we do not have to assume (D4). Considering it, he feels that the assumption of completeness in Theorem 1.1 is a little bit stronger because he feels that the concept of completeness matches (D4).

Motivated by the above, in this paper, we introduce the concept of semicompleteness, which is weaker than completeness. We prove fixed point theorems for contractions in semicomplete semimetric spaces. Also, we generalize Theorem 1.1.

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2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. We give some definitions on semimetric space.

Definition 2.2. Let (X, d) be a semimetric space, let $\{x_n\}$ be a sequence in X and let $x, y \in X$.

- $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$.
- $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup\{d(x_n, x_m) : m > n\} = 0$.
- X is said to be *bounded* if $\sup\{d(u, v) : u, v \in X\} < \infty$.
- X is said to be *Hausdorff* if $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$ imply $x = y$.
- X is said to be *complete* if every Cauchy sequence converges.
- X is said to be *semicomplete* if every Cauchy sequence has a convergent subsequence.

Remark 2.1.

- The definition of 'semicomplete' is new.
- The concept of completeness is stronger than that of semicompleteness.
- In metric spaces, the concepts of completeness and semicompleteness are equivalent.

We give an example of a semicomplete semimetric space which is not complete.

Example 2.1. Put $X = [-1, 1] \cup \{\alpha\}$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned}
&d(x, y) = |x - y| && \text{if } x, y \in X \setminus \{0, \alpha\}, x < y \\
&d(0, y) = y && \text{if } y > 0 \\
&d(0, y) = 1 && \text{if } y \in X \setminus [0, 1] \\
&d(\alpha, y) = |y| && \text{if } y < 0 \\
&d(\alpha, y) = 1 && \text{if } y > 0 \\
&d(x, y) = d(y, x) && \text{if } d(y, x) \text{ is defined by the above} \\
&d(x, x) = 0.
\end{aligned}$$

Then the following hold:

(i) (X, d) is a semimetric space.
(ii) X is not complete.
(iii) X is semicomplete.
(iv) X is Hausdorff.

Proof. It is not difficult to prove (i), (iii) and (iv). We will show (ii). Define a sequence $\{x_n\}$ in X by $x_n = (-1/2)^n$. Then it is obvious that $\{x_n\}$ is Cauchy. However, $\{x_n\}$ does not converge. We note that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge to α and 0, respectively. □

3. Fixed Point Theorems

In this section, we prove fixed point theorems for contractions in semicomplete semimetric spaces.

Though the following lemma is well known and very easy to prove, we give a proof because we use this lemma twice in this paper.

Lemma 3.1. *Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Let T be a mapping on X and define a mapping S on X by $S = T^\kappa$, where κ is some positive integer. Then the following hold*:

(i) *If S has a unique fixed point z, then z is a unique fixed point of T.*
(ii) *If $\lim_n d(S^n x, z) = 0$ holds for any $x \in X$, then $\lim_n d(T^n x, z) = 0$ holds for any $x \in X$.*

Proof. We first show (i). Since $S \circ Tz = T \circ Sz = Tz$ holds, Tz is a fixed point of S. Since the fixed point z of S is unique, we have $Tz = z$. Let $w \in X$ be a fixed point of T. Then we have $Sw = T^{\kappa-1}w = \cdots = Tw = w$. Since the fixed point z of S is unique, we have $z = w$.

We next show (ii). For any $j \in \{0, \cdots, \kappa - 1\}$, we have

$$\lim_{n\to\infty} d(T^{\kappa\, n+j}x, z) = \lim_{n\to\infty} d(S^n \circ S^j x, z) = 0.$$

So we obtain the desired result. □

Theorem 3.2. *Let (X, d) be a Hausdorff, semicomplete semimetric space. Let T be a contraction on X, that is, there exists $r \in [0, 1)$ satisfying*

$$d(Tx, Ty) \le r\, d(x, y) \tag{3.1}$$

for all $x, y \in X$. Assume that there exist $M > 0$, $\kappa \in \mathbb{N}$ and $u \in X$ satisfying the following:
(i) $d(u, T^n u) < M$.
(ii) *If a subsequence of $\{T^n u\}$ converges to some $x \in X$, then $d(x, T^\kappa x) < M$ holds.*

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for any $x \in X$.

Proof. Since

$$d(T^m u, T^{m+n} u) \le r^m\, d(u, T^n u) \le r^m\, M$$

holds for any $m, n \in \mathbb{N}$, $\{T^n u\}$ is Cauchy. Since X is semicomplete, there exist $z_0 \in X$ and a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f(n)}u, z_0) = 0$ holds. Since $\{T^{f(n)-1}u\}$ is also Cauchy, there exist $z_1 \in X$ and a subsequence $\{g_1(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f\circ g_1(n)-1}u, z_1) = 0$ holds. Then we have

$$\lim_{n\to\infty} d(T^{f\circ g_1(n)}u, Tz_1) \le \lim_{n\to\infty} r\, d(T^{f\circ g_1(n)-1}u, z_1) = 0.$$

Note that $\{T^{f\circ g_1(n)}u\}$ is a subsequence of $\{T^{f(n)}u\}$. Since X is Hausdorff, we have $Tz_1 = z_0$. Since $\{T^{f\circ g_1(n)-2}u\}$ is also Cauchy, there exist $z_2 \in X$ and a subsequence $\{g_2(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f\circ g_1\circ g_2(n)-2}u, z_2) = 0$ holds. As above, we can prove $Tz_2 = z_1$. Continuing this argument, we can define a sequence $\{z_m\}_{m=0}^\infty$ in X satisfying the following:

- $Tz_m = z_{m-1}$ holds for $m \in \mathbb{N}$.
- For any $m \in \mathbb{N} \cup \{0\}$, there exists a subsequence of $\{T^n u\}$ converging to z_m.

From the assumption, we have $d(z_m, T^\kappa z_m) < M$ for $m \in \mathbb{N}$. Hence, putting $z = z_0$, we have

$$d(z, T^\kappa z) = d(T^m z_m, T^{m+\kappa} z_m) \le r^m\, d(z_m, T^\kappa z_m) \le r^m\, M$$

for any $m \in \mathbb{N}$. Since $m \in \mathbb{N}$ is arbitrary, we obtain $d(z, T^\kappa z) = 0$. Thus, z is a fixed point of T^κ. We let $w \in X$ be a fixed point of T^κ. Then we have

$$d(z, w) = d(T^\kappa z, T^\kappa w) \le r^\kappa\, d(z, w).$$

Since $r^\kappa < 1$ holds, we obtain $d(z, w) = 0$, thus, $z = w$ holds. So we have shown that z is a unique fixed point of T^κ. By Lemma 3.1 (i), z is also a unique fixed point of T. Let $x \in X$ be arbitrary. Then we have

$$\lim_{n\to\infty} d(T^n x, z) = \lim_{n\to\infty} d(T^n x, T^n z) \le \lim_{n\to\infty} r^n\, d(x, z) = 0.$$

Thus, $\{T^n x\}$ converges to z. □

Using Theorem 3.2, we prove two fixed point theorems. We can tell that the first one is a generalization of Proposition 1 in [6].

Theorem 3.3. *Let* (X, d) *be a Hausdorff, bounded, semicomplete semimetric space. Let* T *be a contraction on* X*. Then* T *has a unique fixed point* z*. Moreover,* $\{T^n x\}$ *converges to* z *for any* $x \in X$*.*

Proof. Since X is bounded, all the assumptions of Theorem 3.2 holds. Thus, the same conclusion holds. □

Remark 3.2. Theorem 3.3 can be also proved by Theorem 3.4.

The following is our main result.

Theorem 3.4. *Let* (X, d) *be a Hausdorff, semicomplete semimetric space satisfying* (D5)*. Let* T *be a contraction on* X*. Then* T *has a unique fixed point* z*. Moreover,* $\{T^n x\}$ *converges to* z *for any* $x \in X$*.*

Proof. Without loss of generality, we may assume $\delta \le \varepsilon$. Let $r \in [0, 1)$ satisfy (3.1) for all $x, y \in X$. Let $\kappa \in \mathbb{N}$ satisfy $\varepsilon\, r^\kappa < \delta$. Define a function f from X into $[0, \infty)$ by

$$f(x) = \max\left\{d(T^i x, T^j x) : i, j \in \{0, \cdots, \kappa\}\right\},$$

where T^0 is the identity mapping on X. It is obvious that $f(Tx) \le r\, f(x)$ holds. Hence $\lim_n f(T^n x) = 0$ holds.

Fix $u \in X$ and choose $\mu \in \mathbb{N}$ satisfying $f(T^\mu u) < \delta$. We will show

$$d(T^\mu u, T^{\mu+n} u) < \varepsilon \tag{3.2}$$

for any $n \in \mathbb{N}$. It is obvious (3.2) holds in the case where $n \in \{1, \cdots, \kappa\}$. We fix $n \in \mathbb{N}$ with $n > \kappa$ and assume that (3.2) holds for positive integers less than n. Then we have

$$d(T^{\mu+\kappa} u, T^{\mu+n} u) \le r^\kappa\, d(T^\mu u, T^{\mu+n-\kappa} u) \le r^\kappa\, \varepsilon < \delta.$$

We also have

$$d(T^\mu u, T^{\mu+\kappa} u) \le f(T^\mu u) < \delta.$$

By (D5), we obtain (3.2). So (i) of Theorem 3.2 holds with $u := T^\mu u$ and $M := \varepsilon$. For $i, j \in \mathbb{N}$ with $\mu + \kappa \le i \le j$, we have by (3.2)

$$d(T^i u, T^j u) \le r^{i-\mu}\, d(T^\mu u, T^{\mu+j-i} u) \le r^\kappa\, \varepsilon < \delta. \tag{3.3}$$

In order to prove (ii) of Theorem 3.2, we assume that $\lim_n d(T^{f(n)} u, z) = 0$ holds for some $z \in X$ and some subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$. Choose $\lambda \in \mathbb{N}$ satisfying

$$d(T^{f(\lambda)} u, z) < \delta \quad \text{and} \quad f(\lambda) \ge \mu + 2\,\kappa. \tag{3.4}$$

By (D5), (3.3) and (3.4), we have $d(T^j u, z) < \varepsilon$ for $j \in \mathbb{N}$ with $\mu + \kappa \le j$. Hence

$$d(T^j u, T^\kappa z) \le r^\kappa\, d(T^{j-\kappa} u, z) \le r^\kappa\, \varepsilon < \delta \tag{3.5}$$

holds for $j \in \mathbb{N}$ with $\mu + 2\,\kappa \le j$. By (D5), (3.4) and (3.5), we have $d(z, T^\kappa z) < \varepsilon$. Therefore we have shown (ii) of Theorem 3.2. □

4. PRELIMINARIES, PART 2

Throughout this section, we let η be a function from $[0, \infty)$ into itself. We list the following two conditions:

(H1) For any sequence $\{a_n\}$ in $[0, \infty)$, $\lim_n \eta(a_n) = 0$ iff $\lim_n a_n = 0$.
(H2) $\sup\{\eta(a) : a \in [0, \alpha]\} < \infty$ for any $\alpha > 0$.

The proofs of the following lemmas are obvious.

Lemma 4.2. *The following are equivalent*:

(i) η *satisfies* (H1).
(ii) *The conjunction of the following holds*:
  (a) *For any* $\varepsilon' > 0$*, there exists* $\varepsilon > 0$ *such that* $t < \varepsilon$ *implies* $\eta(t) < \varepsilon'$.
  (b) *For any* $\delta > 0$*, there exists* $\delta' > 0$ *such that* $\eta(t) < \delta'$ *implies* $t < \delta$.

Lemma 4.3 (Lemma 2.1 in [12]). *Let* η *be a continuous, strictly increasing function from* $[0, \infty)$ *into itself with* $\eta(0) = 0$*. Then* (H1) *and* (H2) *hold.*

Lemma 4.4 (Lemma 2.2 in [12]). *Let* η *satisfy* (H1)*. Then* $\eta^{-1}(0) = \{0\}$ *holds, that is,* $\eta(\alpha) = 0 \Leftrightarrow \alpha = 0$.

Proposition 4.1. *Let* (X, d) *be a semimetric space and let* η *satisfy* (H1)*. Define a function* p *from* $X \times X$ *into* $[0, \infty)$ *by* $p = \eta \circ d$*. Let* $\{x_n\}$ *be a sequence in* X *and let* $x \in X$*. Then the following hold*:

(i) (X, p) *is a semimetric space.*
(ii) $\{x_n\}$ *converges to* x *in* (X, d) *iff* $\{x_n\}$ *converges to* x *in* (X, p).
(iii) $\{x_n\}$ *is Cauchy in* (X, d) *iff* $\{x_n\}$ *is Cauchy in* (X, p).
(iv) (X, d) *is semicomplete iff* (X, p) *is semicomplete.*
(v) (X, d) *is Hausdorff iff* (X, p) *is Hausdorff.*
(vi) *If* (X, d) *satisfies* (D5) *and* η *satisfies* (H2)*, then* (X, p) *also satisfies* (D5).

Proof. (i) follows from Lemma 4.4. (ii) follows from (H1). (iii) follows from Lemma 4.2. (iv) follows from (ii) and (iii). (v) follows from (ii).

Let us prove (vi). Choose δ and ε appearing in (D5). From (H2), we can put

$$\varepsilon' := \sup\{\eta(a) : a \in [0, \varepsilon]\} + 1 \in (0, \infty).$$

Also we can choose $\delta' > 0$ appearing in Lemma 4.2 (ii) (b). We let $x, y, z \in X$ satisfy $p(x, y) < \delta'$ and $p(y, z) < \delta'$. Then we have $d(x, y) < \delta$ and $d(y, z) < \delta$. By (D5), $d(x, z) < \varepsilon$ holds. Hence $p(x, z) < \varepsilon'$ holds. We have shown (vi). □

Lemma 4.5 (Theorem 3 in [2]). *Let* T *be a Matkowski contraction on a semimetric space* (X, d)*. Then* T^2 *is a Browder contraction* [1]*, that is, there exists a function* φ *from* $[0, \infty)$ *into itself satisfying the following*:

(i) φ *is nondecreasing and right continuous.*
(ii) $\varphi(t) < t$ *for any* $t > 0$.
(iii) $d(Tx, Ty) \le \varphi \circ d(x, y)$.

Proof. The original proof in [2] can work in this setting. □

The following lemma can be easily proved by Theorem 2 in [3]. Please note that F defined in Lemma 3 in [3] is strictly increasing. See also [4, 5, 9, 11] and others.

Lemma 4.6 ([3]). *Let* T *be a Browder contraction in a semimetric space* (X, d)*. Then for any* $r \in (0, 1)$*, there exists a continuous, strictly increasing function* η *from* $[0, \infty)$ *into itself satisfying* $\eta(0) = 0$ *and* $\eta \circ d(Tx, Ty) \le r\, \eta \circ d(x, y)$ *for all* $x, y \in X$.

5. Deduced theorems

The following theorems are deduced by Theorem 3.4.

Theorem 5.5. *Let* (X, d) *be a Hausdorff, semicomplete semimetric space satisfying* (D5). *Let* η *be a function from* $[0, \infty)$ *into itself satisfying* (H1) *and* (H2). *Let* T *be a mapping on* X *satisfying the following*:

- *There exist some* $\kappa \in \mathbb{N}$ *and* $r \in [0, 1)$ *such that*

$$\eta \circ d(T^\kappa x, T^\kappa y) \leq r\, \eta \circ d(x, y)$$

holds for all $x, y \in X$.

Then T *has a unique fixed point* z. *Moreover,* $\lim_n d(T^n x, z) = 0$ *holds for any* $x \in X$.

Proof. Define a function p by $p = \eta \circ d$. Then by Proposition 4.1, (X, p) is a Hausdorff, semicomplete semimetric space satisfying (D5).

Define a mapping S on X by $S = T^\kappa$. Then S is a contraction as a mapping on (X, p). By Theorem 3.4, S has a unique fixed point z. By Lemma 3.1 (i), z is a unique fixed point of T. Moreover, $\lim_n p(S^n x, z) = 0$ holds for any $x \in X$. By (H1), $\lim_n d(S^n x, z) = 0$ holds for any $x \in X$. By Lemma 3.1 (ii), we obtain $\lim_n d(T^n x, z) = 0$ for any $x \in X$. □

Now we generalize Theorem 1.1.

Theorem 5.6. *Let* (X, d) *be a Hausdorff, semicomplete semimetric space satisfying* (D5). *Let* T *be a Matkowski contraction on* X. *Then* T *has a unique fixed point* z. *Moreover,* $\{T^n x\}$ *converges to* z *for any* $x \in X$.

Proof. By Lemma 4.5, T^2 is a Browder contraction. By Lemma 4.6, there exists a strictly increasing, continuous function η satisfying $\eta(0) = 0$ and

$$\eta \circ d(T^2 x, T^2 y) \leq (1/2)\, \eta \circ d(x, y)$$

for all $x, y \in X$. By Lemma 4.3 and Theorem 5.5, T has a unique fixed point z. Moreover, $\lim_n d(T^n x, z) = 0$ holds for any $x \in X$. □

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