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On some results concerning the polygonal polynomials

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ABSTRACT. In this paper we define the nth polygonal polynomial $P_n(z) = (z-1)(z^2-1)\cdots(z^n-1)$ and we investigate recurrence relations and exact integral formulae for the coefficients of P_n and for those of the Mahonian polynomials $Q_n(z) = (z+1)(z^2+z+1)\cdots(z^{n-1}+\cdots+z+1)$. We also explore numerical properties of these coefficients, unraveling new meanings for old sequences and generating novel entries to the Online Encyclopedia of Integer Sequences (OEIS). Some open questions are also formulated.

1. INTRODUCTION

For a positive integer n, we define the nth *polygonal polynomial* by

$$P_n(z) = (z-1)(z^2-1)\cdots(z^n-1). \tag{1.1}$$

For each $k = 1, \dots, n$, the roots of $z^k - 1$ are the complex coordinates of the vertices of the regular k-gon centered in the origin and having 1 as a vertex. Consequently, the roots of P_n are the vertices, with repetitions, of the regular k-gons, $k = 1, \dots, n$, motivating the name of this polynomial. Clearly, P_n has degree $\frac{n(n+1)}{2}$ and integer coefficients. These polynomials are closely linked to Euler's famous pentagonal number theorem concerning the infinite expansion $(1-x)(1-x^2)(1-x^3)\cdots = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}$, where $|x| < 1$, and the exponents are called (generalised) pentagonal numbers [4].

The polynomials P_n are a special case of the general class of polynomials defined by

$$F_{m_1,\dots,m_n}^{z_1,\dots,z_n}(z) = \prod_{k=1}^{n}(z^{m_k} - z_k),$$

where $n, m_1, m_2, \dots, m_n$ are positive integers and $z_1, z_2, \dots, z_n$ are complex numbers with $|z_1| = |z_2| = \cdots = |z_n| = 1$, introduced and investigated by Andrica and Bagdasar in [3]. This general class also contains the cyclotomic polynomials.

Recall that the nth cyclotomic polynomial Φ_n is defined by

$$\Phi_n(z) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}}(z - \zeta_n^k), \tag{1.2}$$

where $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ denotes the first primitive root of order n of the unity. It is well known that the following formula holds

$$z^n - 1 = \prod_{d|n}\Phi_d(z). \tag{1.3}$$

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Therefore, we obtain the decomposition of P_n as product of cyclotomic polynomials

$$P_n(z) = \prod_{k=1}^{n} \prod_{d|k} \Phi_d(z). \tag{1.4}$$

Because every cyclotomic polynomial is irreducible over $\mathbb{Z}$ ([7], Theorem 1, page 195), it follows that P_n has exactly $\nu(n) = \sum_{k=1}^{n} \tau(k)$ factors irreducible over $\mathbb{Z}$, where $\tau(k)$ denotes the number of divisors of the positive integer k. This property gives a new interpretation for the integer sequence $\nu(n)$ (indexed as A006218 in the Online Encyclopedia of Integer Sequences (OEIS) [12]), in terms of the polynomial P_n. The terms of $\nu(n)$ are

$$1, 3, 5, 8, 10, 14, 16, 20, 23, 27, 29, 35, 37, 41, 45, 50, 52, 58, 60, 66, 70, 74, 76, 84, 87, 91, 95, \ldots$$

The explicit computations for the magnitude of coefficients of cyclotomic polynomials may involve complicated formulations [8, 9], [14, p.258-259]. The integral formulae for the coefficients of the cyclotomic polynomials established by Andrica and Bagdasar [2], opened new perspectives in the study of these coefficients and related sequences.

It is easy to see that the direct link between the Gaussian polynomials defined by

$$\binom{m}{r}_z = \frac{(z^m - 1)\cdots(z^{m-r+1} - 1)}{(z-1)\cdots(z^r - 1)},$$

and the polygonal polynomials is given by the formula

$$\binom{m}{r}_z = \frac{P_m(z)}{P_r(z)P_{m-r}(z)}. \tag{1.5}$$

While the polynomial P_n seems very simple, from many points of view it can be seen as the "father" of the cyclotomic polynomials Φ_d (see formula (1.4) above), and it hides deep algebraic, arithmetic and combinatorial properties. The natural companion to P_n is the Mahonian polynomial Q_n defined in (2.16), with a key role in the theory of partitions. The aim of this paper is to explore the polynomials P_n and Q_n and their coefficients. The coefficients of Q_n are known as Mahonian numbers (A008302 in [12]).

In Section 2 we investigate the coefficients of P_n, for which we establish a recursive formula (subsection 2.1), we deduce an exact integral formula (subsection 2.2) and give a combinatorial interpretation (subsection 2.3). We then analyze the companion polynomial Q_n (subsection 2.4), for whose coefficients we give recurrence formulae (Theorem 2.3, Proposition 2.1). We also show by a direct computation that the polynomial Q_n is unimodal (Proposition 2.2), and we establish a link between the coefficients of polynomials P_n and Q_n (Theorem 2.4).

Section 3 is devoted to some integer sequences related to polynomials P_n and Q_n. First, we examine the number of distinct complex roots of P_n (subsection 3.1). Then, we analyze the sequence of middle coefficients of P_n and deduce an integral formula, and of Q_n (Kendall-Mann numbers), and propose some conjectures (subsection 3.2). Furthermore, in subsection 3.3 we conjecture that every integer n can be a coefficient of some polynomial P_m (property known to hold for cyclotomic polynomials [15]), the result being confirmed numerically for the first 10^5 numbers. We also propose a new integer sequence, defined by the smallest value m, for which the integer $n \geq 1$ appears as a coefficient in P_m (A301701 in [12]). Finally, in subsection 3.4 we discuss the number of positive, negative, and zero coefficients of P_n, adding new entries to the OEIS, namely the sequences A301703, A301704, and A301705.

2. Formulae and properties of the coefficients

Recall that a polynomial $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ of degree m is called

- palindromic (or reciprocal) if $f(z) = z^m f\left(\frac{1}{z}\right)$, i.e., $a_j = a_{m-j}$, $j = 0, \ldots, m$;
- antipalindromic (or antireciprocal) if $f(z) = -z^m f\left(\frac{1}{z}\right)$, i.e., $a_j = -a_{m-j}$;
- unimodal if the sequence of its coefficients is unimodal, i.e., there exists an integer t (called mode), such that $a_1 \leq a_2 \leq \cdots \leq a_t$ and $a_t \geq a_{t+1} \geq \cdots \geq a_m$.

The algebraic form of the polynomial P_n in (1.1) is

$$P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j. \tag{2.6}$$

An easy computation shows that

$$P_n(z) = (-1)^n z^{\frac{n(n+1)}{2}} P_n\left(\frac{1}{z}\right),$$

hence P_n is palindromic for n even, i.e., we have $c_j^{(n)} = c_{\frac{n(n+1)}{2}-j}^{(n)}$, $j = 0, \ldots, \frac{n(n+1)}{2}$. Also, it is antipalindromic for n odd, i.e., we have $c_j^{(n)} = -c_{\frac{n(n+1)}{2}-j}^{(n)}$, $j = 0, \ldots, \frac{n(n+1)}{2}$.

2.1. The recurrence formula for the coefficients of P_n.

The coefficients of the polynomial P_n can be obtained recursively. Notice that

$$P_n(z) = \prod_{k=1}^{n} (z^k - 1) = P_{n-1}(z)\left(z^n - 1\right). \tag{2.7}$$

Using the coefficients of P_n and P_{n-1}, one obtains

$$P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j = \left(\sum_{j=0}^{\frac{n(n-1)}{2}} c_j^{(n-1)} z^j\right)\left(z^n - 1\right). \tag{2.8}$$

This indicates the following formula:

$$c_j^{(n)} = \begin{cases} -c_j^{(n-1)} & \text{if} \quad j \in \{0, \ldots, n-1\}, \\ c_{j-n}^{(n-1)} - c_j^{(n-1)} & \text{if} \quad j \in \left\{n, \ldots, \frac{n(n-1)}{2}\right\}, \\ c_j^{(n-1)} & \text{if} \quad j \in \left\{\frac{n(n-1)}{2} + 1, \ldots, \frac{n(n+1)}{2}\right\}. \end{cases} \tag{2.9}$$

Using the recurrence (2.9), we obtain the numbers in Table 1.

The values of the coefficients $c_j^{(n)}$ correspond to $(-1)^n T(n,j)$ of A231599 in OEIS.

2.2. The integral formula for the coefficients $c_j^{(n)}$.

Writing the polynomial P_n in the algebraic form, one obtains

$$P_n(z) = \sum_{j=0}^{\frac{n(n+1)}{2}} c_j^{(n)} z^j = (-1)^n (1-z)(1-z^2)\cdots(1-z^n),$$

therefore

$$c_j^{(n)} + \sum_{k \neq j} c_k^{(n)} z^{k-j} = (-1)^n z^{-j} (1-z)(1-z^2)\cdots(1-z^n).$$

$c_j^{(1)}$	-1, 1
$P_1(z)$	$-1+z$
$c_j^{(2)}$	1, -1, -1, 1
$P_2(z)$	$1-z-z^2+z^3$
$c_j^{(3)}$	-1, 1, 1, 0, -1, -1, 1
$P_3(z)$	$-1+z+z^2-z^4-z^5+z^6$
$c_j^{(4)}$	1, -1, -1 , 0, 0, 2 , 0, 0, -1 , -1, 1
$P_4(z)$	$1-z-z^2+2z^5-z^8-z^9+z^{10}$
$c_j^{(5)}$	-1, 1, 1, 0, 0, -1, -1, -1, 1, 1, 1, 0, 0, -1, -1, 1
$P_5(z)$	$-1+z+z^2-z^5-z^6-z^7+z^8+z^9+z^{10}-z^{13}-z^{14}+z^{15}$
$c_j^{(6)}$	1, -1, -1, 0, 0, 1, 0, 2, 0, -1, -1, -1, -1, 0, 2, 0, 1, 0, 0, -1, -1, 1
$P_6(z)$	$1-z-z^2+z^5+2z^7-z^9-z^{10}-z^{11}-z^{12}+2z^{14}+z^{16}-z^{19}-z^{20}+z^{21}$
$c_j^{(7)}$	-1, 1, 1, 0, 0, -1, 0, -1, -1, 0, 1, 1, 2, 0, 0, 0, -2, -1, -1, 0, 1, 1, 0, 1, 0, 0, -1, -1, 1
$P_7(z)$	$-1+z+z^2-z^5-z^7-z^8+z^{10}+z^{11}+2z^{12}-2z^{16}-z^{17}-z^{18}+z^{20}+z^{21}+$ $+z^{23}-z^{26}-z^{27}+z^{28}$

TABLE 1. Polynomials P_n and their coefficients for $n=1,2,3,4,5,6,7$.

Letting $z=\cos 2t+i\sin 2t$, by the well-known de Moivre formula, we obtain

$$c_j^{(n)}+\sum_{k\neq j}c_k^{(n)}z^{k-j}=(-1)^n\left(\cos 2jt-i\sin 2jt\right)\prod_{s=1}^{n}\left(2\sin^2 st-2i\sin st\cos st\right)$$
$$=2^n i^n(\cos 2jt-i\sin 2jt)\prod_{s=1}^{n}(\cos st+i\sin st)\prod_{s=1}^{n}\sin st$$
$$=2^n\left[\cos\left(\frac{n(n+1)-4j}{2}t+\frac{n\pi}{2}\right)+i\sin\left(\frac{n(n+1)-4j}{2}t+\frac{n\pi}{2}\right)\right]\prod_{s=1}^{n}\sin st.$$

Integrating on the interval $[0,\pi]$ we get

$$c_j^{(n)}=\frac{2^n}{\pi}\int_0^{\pi}\cos\left(\frac{n(n+1)-4j}{2}t+\frac{n\pi}{2}\right)\sin t\sin 2t\cdots\sin nt\,\mathrm{d}t$$

$$(2.10)\qquad =\begin{cases}\frac{(-1)^{\frac{n}{2}}2^n}{\pi}\int_0^{\pi}\cos\left(\frac{n(n+1)-4j}{2}t\right)\sin t\sin 2t\cdots\sin nt\,\mathrm{d}t & \text{if n is even}\\ \frac{(-1)^{\frac{n+1}{2}}2^n}{\pi}\int_0^{\pi}\sin\left(\frac{n(n+1)-4j}{2}t\right)\sin t\sin 2t\cdots\sin nt\,\mathrm{d}t & \text{if n is odd.}\end{cases}$$

In addition, from the proof of the integral formula (2.10) it follows that

$$\int_0^{\pi}\sin\left(\frac{n(n+1)-4j}{2}t+\frac{n\pi}{2}\right)\sin t\sin 2t\cdots\sin nt\,\mathrm{d}t=0.$$

2.3. **The combinatorial interpretation of the coefficients $c_j^{(n)}$.** The calculation of these polynomial coefficients involves tuples with fixed sum. Let s,k,n be positive integers. We denote by $\alpha(s,k,n)$ the number of integer s-tuples $(i_1,\ldots,i_s)$ with the properties

$$(2.11)\qquad i_1+i_2+\cdots+i_s=k,\quad 1\le i_1<i_2<\cdots<i_s\le n.$$

The link with the coefficients $c_j^{(n)}$ is given in the following theorem.

Theorem 2.1. *The following formula holds :*

$$(2.12)\qquad c_k^{(n)}=(-1)^{n-1}\left(\alpha(1,k,n)-\alpha(2,k,n)+\alpha(3,k,n)-\cdots\right).$$

Proof. The coefficient $c_k^{(n)}$ of z^k in the expansion $(z-1)(z^2-1)\cdots(z^n-1)$ involves s distinct terms chosen from the set $\{1,\ldots,n\}$ with the property that their sum is k, and also $n-s$ terms equal to (-1), for $s=0,\ldots,n$. Explicitly, $c_k^{(n)}$ is given by the expression

$$c_k^{(n)} = (-1)^{n-1}\alpha(1,k,n) + (-1)^{n-2}\alpha(2,k,n) + (-1)^{n-3}\alpha(3,k,n) + \cdots .$$

This ends the proof. □

Clearly, $\alpha(s,k,n)$ is an increasing function with n. When n is large enough, the function is stationary to a value not depending on n, simply denoted by $\alpha(s,k)$.

The following result shows a link between the polynomial P_s and integers $\alpha(s,k)$.

Theorem 2.2. *If s is a positive integer, then for all $z \in \mathbb{C}$ such that $|z| < 1$ we have*

$$\frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)} = \lim_{n\to\infty} \sum_{1\le i_1<i_2<\cdots<i_s\le n} z^{i_1+i_2+\cdots+i_s} = \sum_{k=0}^{\infty} \alpha(s,k) z^k. \tag{2.13}$$

Proof. Clearly, we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{1\le i_1<\cdots<i_s\le n} z^{i_1+\cdots+i_s} = \sum_{1\le i_1<\cdots<i_s} z^{i_1+\cdots+i_s} = \sum_{1\le i_1<\cdots<i_{s-1}} z^{i_1+\cdots+i_{s-1}} \cdot \frac{z^{i_{s-1}+1}}{1-z}\\
&= \frac{z}{1-z} \sum_{1\le i_1<\cdots<i_{s-1}} z^{i_1+\cdots+2i_{s-1}} = \frac{z}{1-z} \sum_{1\le i_1<\cdots<i_{s-2}} z^{i_1+\cdots+i_{s-2}} \cdot \frac{z^{2(i_{s-2}+1)}}{1-z^2}\\
&= \frac{z^{1+2}}{(1-z)(1-z^2)} \sum_{1\le i_1<\cdots<i_{s-2}} z^{i_1+\cdots+3i_{s-2}} = \cdots = \frac{z^{1+\cdots+(s-1)}}{(1-z)\cdots(1-z^{s-1})} \sum_{1\le i_1} z^{si_1}\\
&= \frac{z^{\frac{s(s+1)}{2}}}{(1-z)\cdots(1-z^s)} = \frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)}.
\end{aligned}$$

□

Notice also that by (2.13) it follows that

$$(-1)^s z^{\frac{s(s+1)}{2}} = P_s(z)\left(\sum_{k=0}^{\infty} \alpha(s,k) z^k\right) = \left(\sum_{j=0}^{\frac{s(s+1)}{2}} c_j^{(s)} z^j\right)\left(\sum_{k=0}^{\infty} \alpha(s,k) z^k\right),$$

hence we obtain the following result.

Corollary 2.1. *Considering that $c_j^{(s)} = 0$ for $j > \frac{s(s+1)}{2}$, we have*

$$\sum_{j=0}^{n} \alpha(s,n-j) c_j^{(s)} = (-1)^s \delta_{n,\frac{s(s+1)}{2}}, \tag{2.14}$$

where $\delta_{u,v}$ denotes the Kronecker symbol.

Corollary 2.2. *Let $\beta(s,k)$ be the number of distinct solutions to the equation*

$$j_1 + 2j_2 + \cdots + sj_s = k, \tag{2.15}$$

such that the numbers $j_1, j_2, \ldots, j_s \ge 1$ are not necessarily distinct. We have $\alpha(s,k) = \beta(s,k)$.

Proof. Indeed, for $j \in \{1,\ldots,s\}$ and $|z|<1$ one has $\frac{z^j}{1-z^j} = z^j + z^{2j} + \cdots$.
By taking the product over $j \in \{1,\ldots,s\}$, one obtains

$$\frac{(-1)^s z^{\frac{s(s+1)}{2}}}{P_s(z)} = (z+z^2+\cdots)(z^2+z^4+\cdots)\cdots(z^s+z^{2s}+\cdots) = \sum_{k=0}^{\infty} \beta(s,k) z^k.$$

The result follows by (2.13). Clearly, $\alpha(s,k)=\beta(s,k)=0$ for $0\le k\le \frac{s(s+1)}{2}-1$. □

Recent results concerning the number of ordered partitions of the integer k into s parts each of size at least 0 but no larger than n have been obtained in [13] and [16].

2.4. **The connection to the Mahonian polynomial Q_n.** The polygonal polynomial can also be written as $P_n(z)=(z-1)^nQ_n(z)$, where

$$Q_n(z)=(z+1)(z^2+z+1)\cdots(z^{n-1}+z^{n-2}+\cdots+z+1)=\sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)}z^j. \tag{2.16}$$

We call Q_n the Mahonian polynomial, which presents interest in its own right and has been investigated in many papers [11] (related to coin-tossing), or Margolius [10]. The coefficients $a_k^{(n)}$ are called Mahonian numbers, representing the number of permutations of the set $\{1,\ldots,n\}$ with k inversions, indexed as A008302 in OEIS. If $z^k=z^{k_1}\cdots z^{k_{n-1}}$ with z^{k_j} coming from the factor $1+z+\cdots+z^j$ in (2.16), then a direct interpretation of the coefficient $a_k^{(n)}$ is the number of partitions of the integer $k=k_1+\cdots+k_{n-1}$, with the constraints $0\le k_j\le j,\quad 1\le j\le n-1$. These numbers are related to the Mahonian distribution, which interestingly, are used in the mixing of diffusing particles [5].

$a_j^{(2)}$	$1,1$
$Q_2(z)$	$1+z$
$a_j^{(3)}$	$1,2,2,1$
$Q_3(z)$	$1+2z+2z^2+z^3$
$a_j^{(4)}$	$1,3,5,6,5,3,1$
$Q_4(z)$	$1+3z+5z^2+6z^3+5z^4+3z^5+z^6$
$a_j^{(5)}$	$1,4,9,15,20,22,20,15,9,4,1$
$Q_5(z)$	$1+4z+9z^2+15z^3+20z^4+22z^5+20z^6+15z^8+9z^8+4z^9+z^{10}$
$a_j^{(6)}$	$1,5,14,29,49,71,90,101,101,90,71,49,29,14,5,1$
$Q_6(z)$	$1+5z+14z^2+29z^3+49z^4+71z^5+90z^6+101z^7+101z^8+90z^9+71z^{10}+49z^{11}+$ $29z^{12}+14z^{13}+5z^{14}+z^{15}$

TABLE 2. Polynomials Q_n and their coefficients for $n=2,3,4,5,6$.

For $n=10$ the following formula is obtained for the polynomial Q_{10}:

$$\begin{aligned}Q_{10}(z)&=1+9z+44z^2+155z^3+440z^4+1068z^5+2298z^6+4489z^7+8095z^8+13640z^9\\&+21670z^{10}+32683z^{11}+47043z^{12}+64889z^{13}+86054z^{14}+110010z^{15}+135853z^{16}\\&+162337z^{17}+187959z^{18}+211089z^{19}+230131z^{20}+243694z^{21}+250749z^{22}+250749z^{23}\\&+243694z^{24}+230131z^{25}+211089z^{26}+187959z^{27}+162337z^{28}+135853z^{29}\\&+110010z^{30}+86054z^{31}+64889z^{32}+47043z^{33}+32683z^{34}+21670z^{35}+13640z^{36}\\&+8095z^{37}+4489z^{38}+2298z^{39}+1068z^{40}+440z^{41}+155z^{42}+44z^{43}+9z^{44}+z^{45}.\end{aligned}$$

We also give a recurrence formula for $a_j^{(n)}$, as a function of the coefficients of Q_j.

Theorem 2.3. *Let $n\ge k+1$. The coefficient $a_k^{(n)}$ is given by the recursive formula*

$$a_k^{(n)}=\sum_{j=0}^{k}\binom{n-1-j}{k-j}a_j^{(k)}. \tag{2.17}$$

Proof. Notice that $a_k^{(n)}$ is the coefficient of z^k in $Q_k(z)(1+z+\cdots+z^k)^{n-k}$. Clearly:

$$(1+z+\cdots+z^k)^{n-k} = \left(\frac{1-z^{k+1}}{1-z}\right)^{n-k} = \left(1-z^{k+1}\right)^{n-k}\left(\frac{1}{1-z}\right)^{n-k}.$$

For $|z|<1$, the geometric series summation yields

$$\frac{1}{1-z} = 1+z+z^2+\cdots+z^s+\cdots.$$

By differentiating $(n-k-1)$ times one obtains

$$(n-k-1)!\left(\frac{1}{1-z}\right)^{n-k} = \sum_{s=0}^{\infty}(s+1)(s+2)\cdots(s+n-k-1)z^s,$$

hence

$$\left(\frac{1}{1-z}\right)^{n-k} = \sum_{s=0}^{\infty}\frac{(s+1)(s+2)\cdots(s+n-k-1)}{(n-k-1)!}z^s = \sum_{s=0}^{\infty}\binom{s+n-1-k}{s}z^s.$$

The coefficient of z^k in

$$\left(\sum_{j=0}^{\frac{k(k-1)}{2}} a_j^{(k)}\right)\left(1-z^{k+1}\right)^{n-k}\left(\frac{1}{1-z}\right)^{n-k},$$

is then given by

$$a_k^{(n)} = \sum_{s=0}^{k} a_{k-s}^{(k)}\binom{s+n-k-1}{s} = \sum_{j=0}^{k} a_j^{(k)}\binom{n-j-1}{k-j}.$$

□

Example 2.1. First, one may show that $a_0^{(n)}=1$ and $a_1^{(n)}=n-1$. By Theorem 2.3 one obtains:

- $k=2$: For $n\geq 3$ we have
$$a_2^{(n)} = \binom{n-1}{2}a_0^{(2)}+\binom{n-2}{1}a_1^{(2)}+\binom{n-3}{0}a_2^{(2)} = \frac{(n-2)(n+1)}{2};$$
- $k=3$: For $n\geq 4$ we have
$$a_3^{(n)} = \binom{n-1}{3}a_0^{(3)}+\binom{n-2}{2}a_1^{(3)}+\binom{n-3}{1}a_2^{(3)}+\binom{n-4}{0}a_3^{(3)} = \frac{n(n^2-7)}{6};$$
- $k=4$: For $n\geq 5$ we have
$$a_4^{(n)} = \binom{n-1}{4}a_0^{(4)}+\binom{n-2}{3}a_1^{(4)}+\binom{n-3}{2}a_2^{(4)}+\binom{n-4}{1}a_3^{(4)}+\binom{n-5}{0}a_4^{(4)}$$
$$= \frac{n(n+1)(n^2+n-14)}{24};$$
- $k=5$: For $n\geq 6$ we have
$$a_5^{(n)} = \binom{n-1}{5}a_0^{(5)}+\binom{n-2}{4}a_1^{(5)}+\binom{n-3}{3}a_2^{(5)}+\binom{n-4}{2}a_3^{(5)}+\binom{n-5}{1}a_4^{(5)}+$$
$$+\binom{n-6}{0}a_5^{(5)} = \frac{1}{120}(n-1)(n+6)(n^3-9n-20);$$

- $k=6$: For $n \geq 7$ we have
$$a_6^{(n)} = \frac{1}{720} n \left(n^5 + 9n^4 - 5n^3 - 165n^2 - 356n + 516\right).$$
For $n=10$ we have $a_5^{(10)} = 1068$ and $a_6^{(10)} = 2298$, confirming the values of Q_{10}.

The coefficients of the polynomial Q_n can also be obtained recursively. One may write

$$Q_n(z) = Q_{n-1}(z)\left(z^{n-1} + \cdots + z + 1\right). \tag{2.18}$$

Using the coefficients of Q_n and Q_{n-1}, one obtains

$$Q_n(z) = \sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)} z^j = \left(\sum_{j=0}^{\frac{(n-2)(n-1)}{2}} a_j^{(n-1)} z^j\right)\left(z^{n-1} + \cdots + z + 1\right). \tag{2.19}$$

Proposition 2.1. *The following formula holds:*

$$a_j^{(n)} = \begin{cases} a_j^{(n-1)} + \cdots + a_0^{(n-1)} & \text{if} \quad j \in \{0, \ldots, n-1\}, \\ a_j^{(n-1)} + \cdots + a_{j-(n-1)}^{(n-1)} & \text{if} \quad j \in \left\{n, \ldots, \frac{n(n-1)}{2}\right\}. \end{cases} \tag{2.20}$$

One can prove that Q_n is a Λ-polynomial, i.e., it is both palindromic and unimodal with nonnegative coefficients. This follows from a general result by Andrews [1], and the fact that Q_n is a product of the Λ-polynomials $z+1$, z^2+z+1,..., $z^{n-1}+\cdots+z+1$. This property can be seen in Table 1. We also present a direct proof of the unimodality Q_n.

Proposition 2.2. *The polynomial Q_n is unimodal.*

Proof. Let us denote for convenience $m = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$ and $M = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$. We shall prove the result by induction. Assume that the polynomial Q_n is unimodal. Observe that

(a) if $m = \frac{n(n-1)}{4}$, then the sequence $\{a_j^{(n)}\}_{j=0}^{n(n-1)/2}$ has a single maximum value at $j=m$, and also $a_{m-j}^{(n)} = a_{m+j}^{(n)}$, $j=0,\ldots,m$, by the unimodality of Q_n;

(b) if $m \neq \frac{n(n-1)}{4}$, then sequence $\{a_j^{(n)}\}_{j=0}^{n(n-1)/2}$ has maximum values at $j=m, m+1$, and also $a_{m-j}^{(n)} = a_{m+1+j}^{(n)}$, $j=0,\ldots,m$, by the unimodality of Q_n.

Clearly, $Q_2(z) = 1+z$ is unimodal. We show that the sequence $a_j^{(n+1)}$, $j=0,\ldots,M$, is increasing. First, by formula (2.20), if $0 \leq j \leq n-1$, then $a_{j+1}^{(n+1)} - a_j^{(n+1)} = a_{j+1}^{(n)} \geq 1$. Then, if $n \leq j \leq m-1$, one has $a_{j+1}^{(n+1)} - a_j^{(n+1)} = a_{j+1}^{(n)} - a_{j-n}^{(n)} > 0$, as Q_n is unimodal. Finally, we prove that the inequality $a_{j+1}^{(n+1)} - a_j^{(n+1)} > 0$ holds whenever $m \leq j \leq M-1$. Denote for convenience $p = j-m$, for $p \in \{0,\ldots,M-m-1\}$.

Case 1. If $n=4k$, then $M-m=2k$. For $p \in \{0,\ldots,2k-1\}$ (even n), one obtains
$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+p+1}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$
since $p+1 < n-p$. Also, one may notice that for $p=2k$, one obtains
$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+2k+1}^{(n)} - a_{m-2k}^{(n)} = a_{m+2k+1}^{(n)} - a_{m+2k}^{(n)} = a_{M+1}^{(n)} - a_M^{(n)} < 0,$$
hence sequence $a_j^{(n+1)}$, $j=0,\ldots,M$ is increasing, and has the single mode $j=M$.

Case 2. If $n=4k+1$, then $M-m=2k$. For $p \in \{0,\ldots,2k-1\}$ (odd n) we have
$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+p}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p < n-p$. Also, for $p = 2k$ one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+2k}^{(n)} - a_{m-2k}^{(n)} = a_{M+1}^{(n)} - a_{M}^{(n)} = 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \ldots, M$ is increasing, and has modes $j = M, M+1$.

Case 3. If $n = 4k+2$, then $M - m = 2k+1$. For $p \in \{0, \ldots, 2k\}$ (even n), one has

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+p+1}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p+1 < n-p$. Also, one may notice that for $p = 2k+1$, one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{m+2k+2}^{(n)} - a_{m-(2k+1)}^{(n)} = a_{m+2k+2}^{(n)} - a_{m+2k+1}^{(n)} = a_{M+1}^{(n)} - a_{M}^{(n)} < 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \ldots, M$ is increasing, and has the single mode $j = M$.

Case 4. If $n = 4k+3$, then $M - m = 2k+2$. For $p \in \{0, \ldots, 2k+1\}$ (odd n) we have

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+p}^{(n)} - a_{m-(n-p)}^{(n)} > 0,$$

since $p < n-p$. Also, for $p = 2k+2$ one obtains

$$a_{j+1}^{(n)} - a_{j-n}^{(n)} = a_{(m+1)+2k+2}^{(n)} - a_{m-2k-2}^{(n)} = a_{M+1}^{(n)} - a_{M}^{(n)} = 0,$$

hence sequence $a_j^{(n+1)}$, $j = 0, \ldots, M$ is increasing, and has modes $j = M, M+1$. $\square$

Using the definition of the polynomial Q_n, we obtain another interpretation of the coefficients of polynomial P_n, in terms of Kandall-Mann numbers. Indeed, from

$$P_n(z) = \left[z^n \binom{n}{0} - \binom{n}{1} z^{n-1} + \binom{n}{2} z^{n-2} - \cdots + (-1)^n \binom{n}{n} \right] \left(\sum_{j=0}^{\frac{(n-1)n}{2}} a_j^{(n)} z^j \right),$$

one obtains a link between the coefficients of polynomial P_n and those of Q_n.

Theorem 2.4. *The following formula holds*

$$c_j^{(n)} = \begin{cases} (-1)^n \left(a_j^{(n)} \binom{n}{0} - a_{j-1}^{(n)} \binom{n}{1} + \cdots + (-1)^j a_0^{(n)} \binom{n}{j} \right) & \text{if} \quad j \in \{0, \ldots, n-1\}, \\ (-1)^n \left(a_j^{(n)} \binom{n}{0} - a_{j-1}^{(n)} \binom{n}{1} + \cdots + (-1)^n a_{j-n}^{(n)} \binom{n}{n} \right) & \text{if} \quad j \in \left\{ n, \ldots, \frac{n(n-1)}{2} \right\}. \end{cases}$$

Example 2.2. Applying the formula in Theorem 2.4 for $j = 1, \ldots, 6$ and $n \geq j+1$ we obtain

$c_0^{(n)} = (-1)^n$,
$c_1^{(n)} = (-1)^n (a_1^{(n)} - a_0^{(n)} \binom{n}{1}) = (-1)^n \left((n-1) - n\right) = (-1)^{n+1}$,
$c_2^{(n)} = (-1)^n (a_2^{(n)} - a_1^{(n)} \binom{n}{1} + a_0^{(n)} \binom{n}{2}) = (-1)^{n+1}$,
$c_3^{(n)} = (-1)^n (a_3^{(n)} - a_2^{(n)} \binom{n}{1} + a_1^{(n)} \binom{n}{2} - a_0^{(n)} \binom{n}{3}) = 0$,
$c_4^{(n)} = (-1)^n (a_4^{(n)} - a_3^{(n)} \binom{n}{1} + a_2^{(n)} \binom{n}{2} - a_1^{(n)} \binom{n}{3} + a_0^{(n)} \binom{n}{4}) = 0$,
$c_5^{(n)} = (-1)^n (a_5^{(n)} - a_4^{(n)} \binom{n}{1} + a_3^{(n)} \binom{n}{2} - a_2^{(n)} \binom{n}{3} + a_1^{(n)} \binom{n}{4} - a_0^{(n)} \binom{n}{5}) = (-1)^n$,
$c_6^{(n)} = (-1)^n (a_6^{(n)} - a_5^{(n)} \binom{n}{1} + a_4^{(n)} \binom{n}{2} - a_3^{(n)} \binom{n}{3} + a_2^{(n)} \binom{n}{4} - a_1^{(n)} \binom{n}{5} + a_0^{(n)} \binom{n}{6}) = 0$.

3. Associated integer sequences

3.1. The number of distinct roots. The number of distinct roots of P_n is given by

$$A(n) = \varphi(1) + \varphi(2) + \cdots + \varphi(n). \tag{3.21}$$

Indeed, by (1.1) one has $P_{n+1}(z) = P_n(z)(z^{n+1} - 1)$. The new roots added by $z^{n+1} - 1$ to the set of roots of P_n, are those given by the primitive roots of order $n+1$, whose number is $\varphi(n+1)$. The result follows by induction.

The sequence $A(n)$ is indexed as A002088 in the OEIS, starting with the values:

$1, 2, 4, 6, 10, 12, 18, 22, 28, 32, 42, 46, 58, 64, 72, 80, 96, 102, 120, 128, 140, 150, 172, 180, 200, \ldots$

The asymptotic formula for $A(n)$ is given in [4, Theorem 3.7, page 72]:

$$A(n) \sim \frac{3n^2}{\pi^2} + O(n \log n).$$

Proposition 3.3. *Let $1 \le k \le n$ be an integer. If $z_{p,k} = e^{2\pi i \frac{p}{k}}$ is a k^{th} primitive root, then the multiplicity of root $z_{p,k}$ in the polynomial P_n is $\left\lfloor \frac{n}{k} \right\rfloor$. Consequently, one recovers the identity*

$$\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

Proof. The root $z_{p,k}$ appears for the first time in polynomial P_k from the factor $z^k - 1$. Each of the $\varphi(k)$ roots appears as a non-primitive root of every multiple of k smaller than n. $\square$

3.2. The middle coefficients of P_n and Q_n. Formula (2.10) provides an insight into the sequence of middle terms. Denote by $m = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$.

- If $n = 4k$, then for $m = \frac{n(n+1)}{4}$ we have $\cos\left(\frac{n(n+1)-4m}{2} t\right) = 1$ and

$$c_m^{(n)} = \frac{2^n}{\pi} \int_0^{\pi} \sin t \sin 2t \cdots \sin nt \, dt. \tag{3.22}$$

  This sequence recovers A269298 in OEIS, having the starting values

  $2, 2, 4, 6, 8, 16, 28, 50, 100, 196, 388, 786, 1600, 3280, 6780, 14060, 29280, 61232, \ldots$

- If $n = 4k+1$, then $n(n+1) - 4m = 2$ and $\sin\left(\frac{n(n+1)-4m}{2} t\right) = \sin t$, hence

$$c_m^{(n)} = -\frac{2^n}{\pi} \int_0^{\pi} \sin^2 t \sin 2t \cdots \sin nt \, dt = -c_{m+1}^{(n)}. \tag{3.23}$$

  The sequence of terms multiplied by (-1) is not currently indexed in OEIS:

  $1, 1, 1, 1, 2, 2, 3, 4, 6, 10, 17, 28, 52, 94, 176, 339, 651, 1268, 2505, 4965, 9916, 19926, \ldots$

- If $n = 4k+2$, then $n(n+1) - 4m = 2$ and $\cos\left(\frac{n(n+1)-4m}{2} t\right) = \cos t$, hence

$$c_m^{(n)} = -\frac{2^n}{\pi} \int_0^{\pi} \cos t \sin t \sin 2t \cdots \sin nt \, dt = c_{m+1}^{(n)}. \tag{3.24}$$

  The sequence of terms multiplied by (-1) is not currently indexed in OEIS:

  $1, 1, 2, 3, 5, 10, 19, 34, 68, 135, 269, 544, 1111, 2274, 4694, 9729, 20237, 42260, 88538, \ldots$

- If $n = 4k+3$ then for $m = \frac{n(n+1)}{4}$ we have $\sin\left(\frac{n(n+1)-4m}{2} t\right) = 0$ and $c_m^{(n)} = 0$.

Conjecture 1. *Let n be an integer and $m = \left\lfloor \frac{n(n+1)}{4} \right\rfloor$. Then the middle coefficient $c_m^{(n)}$ is*

- *positive for $n = 4k$ (mentioned in A231599, without a proof);*

- *negative for $n = 4k+1$ and $n = 4k+2$.*

The sequence of middle coefficients of Q_n gives the Kendall-Mann numbers indexed as A000140 in OEIS, representing the number of permutations of the set $\{1, \dots, n\}$ having the maximum number of inversions. The first terms are:

$$1, 1, 2, 6, 22, 101, 573, 3836, 29228, 250749, 2409581, 25598186, 296643390, 3727542188, \dots$$

The asymptotic behavior of this sequence was conjectured in A000140.

Conjecture 2. *Let n be an integer and $m = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$. Then the sequence of middle coefficients $a_m^{(n)}$ of Q_n satisfies the asymptotic formula*

$$a_m^{(n)} \sim \frac{6n^{n-1}}{e^n}.$$

We are not aware of the existence of any proof at the moment.

For the cyclotomic polynomial Φ_n, the sequence of middle coefficients was studied by Dresden [6], and corresponds to the sequence A094754 in OEIS.

3.3. **First occurrence of n as a coefficient of a polygonal polynomial.** It was proved by Suzuki [15], that every integer n is a coefficient for some cyclotomic polynomial Φ_m. We formulate the following open question regarding polygonal polynomials.

Conjecture 3. *Every integer n appears as a coefficient of some polygonal polynomial.*

Notice that 1 is a coefficient of P_1, while 2 first appears as a coefficient in P_4. For an integer $n \geq 0$, the sequence $a(n)$ defined by the smallest number m for which n is a coefficient of P_m produces sequence A301701, recently added by the authors to OEIS.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19 ...
$a(n)$	3	1	4	10	12	17	16	19	20	22	22	23	24	25	25	25	24	26	26	28 ...

The conjecture was recently checked for the first 10^5 integers.

3.4. **Number of non-zero terms of P_n.** The sequence of non-zero coefficients of P_n is indexed as A086781 in OEIS and starts with the terms

$$1, 2, 4, 6, 7, 12, 14, 18, 25, 32, 36, 42, 53, 68, 64, 84, 97, 108, 126, 146, 161, 170 \dots$$

Recently, we have added the sequences below to the OEIS.

- A301703, representing the number of positive coefficients of P_n:

$$1, 2, 3, 3, 6, 6, 9, 13, 16, 18, 21, 27, 34, 32, 42, 47, 54, 62, 73, 79, 85, 96, 104, 113, 123, 140, 150, \dots$$

- A301704, representing the number of negative coefficients of P_n:

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 21, 26, 34, 32, 42, 50, 54, 64, 73, 82, 85, 96, 104, 116, 123, 134, 150, \dots$$

- A301705, representing the number of zero coefficients of P_n:

$$0, 0, 1, 4, 4, 8, 11, 12, 14, 20, 25, 26, 24, 42, 37, 40, 46, 46, 45, 50, 62, 62, 69, 72, 80, 78, 79, 74, \dots$$

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References

[1] Andrews, G. E., *A Theorem on Reciprocal Polynomials with Applications to Permutations and Compositions*, Amer. Math. Monthly, **82** (1975), 830–833
[2] Andrica, D. and Bagdasar, O., On cyclotomic polynomial coefficients, Malays. J. Math. Sci., in: *Proceedings of Groups, Group Rings, and Related Topics - 2017* (GGRRT 2017), 19 - 22 Nov 2017, Khorfakan, UAE (to appear) (2019)
[3] Andrica, D. and Bagdasar, O., *Some remarks on a general family of complex polynomials* (2018) (submitted)
[4] Apostol, T. M., *Introduction to Analytic Number Theory*, Springer-Verlag, New York Heidelberg Berlin, 1976
[5] Ben-Naim, E., *Mixing of diffusing particles*, Phys. Rev. E, **82** (2010), 061103
[6] Dresden, G., *On the middle coefficient of a cyclotomic polynomial*, Amer. Math. Monthly, **111** (2004), No. 6, 531–533
[7] Ireland, K. and Rosen, M., *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, Springer, 2002
[8] Ji, C. G. and Li, W. P., *Values of coefficients of cyclotomic polynomials*, Discrete Math., **308** (2008), No. 23, 5860–5863
[9] Lehmer, E., *On the magnitude of the coefficients of the cyclotomic polynomial*, Bull. Amer. Math. Soc., **42** (1936), No. 6, 389–392
[10] Margolius, B. H., *Permutations with Inversion*, J. Integer Seq., **4** (2001), Article 01.2.4
[11] Moritz, R. H. and Williams, R. C., *A Coin-Tossing Problem and Some Related Combinatorics*, Math. Mag., **61** (1988), 24–29
[12] OEIS Foundation Inc. (2011). The On-Line Encyclopedia of Integer Sequences. `http://oeis.org`
[13] Ratsaby, J., *Estimate of the number of restricted integer-partitions*, Appl. Anal. Discrete Math., **2** (2008), 222–233
[14] Sándor, J. and Crstici, B. *Handbook of Number Theory II*, Kluwer, Dordrecht, Netherlands, 2004
[15] Suzuki, J., *On coefficients of cyclotomic polynomials*, Proc. Japan Acad. Ser. A Math. Sci., **63** (1987), 279–280
[16] Tani, N. B. and Bouroubi, S., *Enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes*, J. Integer Seq., **14** (2011), Article 11.3.6, 12p

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