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# Kottman's constant, packing constant and Riesz angle in some classes of Köthe sequence spaces

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ABSTRACT. We have found a sufficient condition in order that the Kottman constant to be equal to the Riesz angle for Köthe sequence spaces. We have found the ball packing constant in weighted Orlicz sequence spaces, endowed with Luxemburg or $p$–Amemiya norm. We have calculated the Riesz angle for Musielak–Orlicz, Nakano, weighted Orlicz, Orlicz, Orlicz–Lorentz, Lorentz and Cesaro sequence spaces.

## 1. INTRODUCTION

It is well known that only a finite number of spheres can be packed in the unit ball $B_X$ in a finite dimensional space if the spheres are disjoint and have the same radius, no matter how small the radius is. However, for an infinite dimensional Banach space X, there exists a constant $\Gamma_X$, such that an infinite number of disjoint spheres can be packed in a unit ball $B_X$ if the radius is less than $\Gamma_X$. Whereas, only a finite number of disjoint spheres can be packed in the ball $B_X$ if the radius is larger than $\Gamma_X$. This constant is referred to as a packing sphere constant. From the 50's of the previous century, researchers began to investigate the packing spheres problem in Banach spaces [3, 40, 41]. In [28, 29] Kottman finally determined the range of the packing sphere value $\Gamma_X$, where Kottman's constant $K(X)$ was introduced. It measures how big the separation of an inffnite subset of the unit ball can be.

The packing constant is an important and interesting geometric parameter for studying the geometric structure, isometric embedding, noncompactness, and reflexivity in Banach spaces [3, 28, 41, 45].

In order to generalize the technique in [37] for $c_0$ to a larger class of Banach lattices, J. Borwein and B. Sims introduced in [2] the notion of a weakly orthogonal Banach lattice and Riesz angle $a(X)$. Deep results in investigation of fixed point property (FPP) and weak fixed point property (w–FPP) in wide classes of Banach spaces were obtained by using the Riesz angle concept in [2]. The above mentioned result was applied by Nezir to prove that the Riesz angle of the space $\ell^0_{w,\infty}$ with $w = \{1/n^p\}$, where $p \in (0, 1)$ is less than 2 and thus it has the w–FPP [39] and by Cui, Hudzik and Wisła to prove that every reflexive Orlicz sequence space $\ell_{M,p}$ equipped with the $p$–Amemiya norm, has the w–FPP and the FPP as well [11].

Therefore finding of formulas for the calculation or estimation of the Riesz angle, the ball packing constant or Kottman's constant are interesting problems.

Combining the known results about the Riesz angle [49, 53] and Kottman's constant [3, 6, 18] it seems that for a wide class of Köthe sequence spaces with unconditional basis both constants are equal.

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## 2. Preliminaries

We use the standard Banach space terminology from [32].

In what follows $(X, \|\cdot\|)$ is a Banach space, $S_X$ and $B_X$ are the unit sphere and the unit ball of $X$ respectively. By $\mathbb{N}$ we denote the natural numbers and by $\mathbb{R}$ the real numbers. By $\dim(\mathrm{X})$ we will denote the dimension of the space $X$.

The packing constant [28] of $X$ is defined by

$$\Gamma_X = \sup\{r > 0 : \text{there are } x_k \in (1-r)B_X \text{ with } \|x_i - x_j\| \geq 2r \text{ for } i, j \in \mathbb{N}, i \neq j\},$$

i.e. if $r \in (0, \Gamma_X]$, then $B_X$ contains infinitely many disjoint balls with radius $r$, and when $r > \Gamma_X$, then $B_X$ contains only finitely many such balls. The exact value of packing constant $\Gamma_{\ell_p}$ is found in [3], $\Gamma_{L^p_{[0,1]}}$ is found in [46], for Orlicz sequence spaces $\Gamma_{\ell_M}$ equipped with Luxemburg's or Amemiya's norm is found in [6], for Musielak–Orlicz sequence spaces $\Gamma_{\ell_{\{M_i\}}}$ equipped with Luxemburg's norm, for Nakano sequence spaces $\Gamma_{\ell_{\{p_i\}}}$ [18], for Lorentz sequence spaces $\Gamma_{d(w,p)}$ in [50], for an infinite dimensional Hilbert space [41] and for Cesaro sequence spaces $\Gamma_{ces_p}$ in [10]. There are a lot of results, where estimation of the packing constant is obtained for wide classes of spaces [16, 17, 41, 43, 48]

A well known result [6] is that $\Gamma_X \in [\frac{1}{3}, \frac{1}{2}]$, provided that $\dim(\mathrm{X}) = \infty$.

Let us denote $\mathrm{sep}(\{x^{(n)}\}) = \inf\left\{\|x^{(n)} - x^{(m)}\| : n \neq m\right\}$.

Kottman's constant of an infinite dimensional Banach space $X$ is defined in [28] as

$$K(X) = \sup\left\{\mathrm{sep}(\{x^{(n)}\}) : \{x^{(n)}\}_{n=1}^{\infty} \subset S_X\right\}.$$

Clearly $K(X) \in [1, 2]$. The following relationship

$$\Gamma_X = \frac{K(X)}{2 + K(X)} \tag{2.1}$$

is obtained in [28].

It is found in [17] that $K(X) = 2$, provided that $X$ is a nonreflexive Banach lattice, and consequently $\Gamma_X = \frac{1}{2}$. We would like to mention that the ball packing constant is found through the application of (2.1) in most articles.

Following [2] a Banach lattice is weakly orthogonal if $\lim_{n\to\infty} \|\,|x_n| \wedge |x|\,\| = 0$ for all $x \in X$, whenever $\{x_n\}_{n=1}^{\infty}$ is a weakly null sequence, where $|x| \wedge |y| = \min(|x|, |y|)$. The Riesz angle $\alpha(X)$ of a Banach lattice $(X, \|\cdot\|)$ is $\alpha(X) = \sup\{\|(|x| \vee |y|)\| : \|x\| \leq 1, \|y\| \leq 1\}$, where $|x| \vee |y| = \max(|x|, |y|)$. Clearly $1 \leq \alpha(X) \leq 2$. If there exists a weakly orthogonal Banach lattice $Y$ such that $d(X, Y).\alpha(Y) < 2$, where $d(X, Y)$ is the Banach-Mazur distance between the Banach spaces $X$ and $Y$, and $\alpha(Y)$ is the Riesz angle of $Y$, then $X$ has the weak fixed point property.

A formula for computing the Riesz angle in Orlicz spaces equipped with Luxemburg's or Amemiya's norm is obtained in [49] and in weighted Orlicz spaces equipped with Luxemburg's or Amemiya's norm is obtained in [53].

Let $\ell^0$ stand for the space of all real sequences i.e. $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$. Let us denote the unit vectors by $\{e_n\}_{n=1}^{\infty}$.

**Definition 2.1.** ([10]) A Banach space $(X, \|\cdot\|)$ is said to be a Köthe sequence space if $X$ is a subspace of $\ell^0$ such that

(i) If $x \in \ell^0$, $y \in X$ and $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \leq \|y\|$;
(ii) There exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.

**Lemma 2.1.** ([49]) *For a Köthe sequences space $(X, \|\cdot\|)$ the Riesz angle $a(X)$ can be expressed as $a(X) = \sup\{\|(|x| \vee |y|)\| : x, y \in S_X, |x| \wedge |y| = 0\}$.*

**Definition 2.2.** ([10]) A Köthe sequence space $(X, \|\cdot\|)$ is called order continuous if for any sequence $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, such that $x^{(n)} \searrow 0$ coordinatewise there holds $\left\|x^{(n)}\right\| \searrow 0$.

A Köthe sequence space $(X, \|\cdot\|)$ is order continuous if and only if for any $x \in X$ there holds $\lim_{n\to\infty} \|(0, 0, \ldots, 0, x_n, x_{n+1}, \ldots)\| = 0$.

**Definition 2.3.** ([10]) A Köthe sequence space $(X, \|\cdot\|)$ has the Fatou property if for any sequence $\left\{x^{(n)}\right\}_{n=1}^{\infty} \subset X$ with $\sup_{n\in\mathbb{N}} \left\|x^{(n)}\right\| < \infty$ and any $x \in \ell^0$, such $x^{(n)} \uparrow x$ coordinatewise there hold $x \in X$ and $\left\|x^{(n)}\right\| \uparrow \|x\|$.

**Definition 2.4.** ([32], p.9) Let $(X, \|\cdot\|)$ be a Banach space. We say that $\{x_n\}_{n=1}^{\infty}$ is a boundedly complete basis if $\sum_{i=1}^{\infty} a_i x_i$ converges whenever there holds the inequality $\sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} a_i x_i\right\| < \infty$.

We will need the next fundamental result, where Kottman's constant can be calculated by using finitely supported norm one vectors in large classes of Köthe sequence spaces. The case (i) is proven in [43] and the case (ii) is proven in [10].

**Theorem 2.1.** *Let $X$ be a Köthe sequence space. Let there hold one of the following:*

*(i) the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a boundedly complete basis of $X$;*
*(ii) $X$ be order continuous with the Fatou property.*

*Then* $K(X) = \sup\left\{\operatorname{sep}\left(\left\{u^{(n)}\right\}_{n=1}^{\infty}\right) : u^{(n)} = \sum_{i=i_{n-1}+1}^{i_n} u_i^{(n)} e_i \in S_X, i_0 < i_1 < \cdots\right\}$.

Let us mention that three different formulas for calculation of $K(X)$ are obtained in [43], but we have stated only the one that coincides with the formula in [10].

Different conditions for a Köthe sequence space to be order continuous are obtained in [12], which are relevant in the context of calculating Kottman's constant by using finitely supported norm one vectors.

Let us point out that Theorem 2.1 holds for any reflexive Köthe sequence space, because reflexivity implies both order continuity and Fatou property of the space.

**Definition 2.5.** ([32], p.15) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space $X$. Then the series $\sum_{n=1}^{\infty} x_n$ is said to converge unconditionally if for any permutation $\pi$ of the integers the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges.

**Proposition 2.1.** ([32], p.15) *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space $X$. Then the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if and only if for every $\varepsilon > 0$ there exists an integer $N$ so that $\left\|\sum_{n\in\sigma} x_n\right\| < \varepsilon$ for every finite set of integers $\sigma$, which satisfies* $\min\{n \in \sigma\} > N$.

**Definition 2.6.** ([32], p.18) A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space $X$ is called unconditional if every $x \in X$ its expansion $\sum_{n=1}^{\infty} a_n x_n$ in terms of the basis converges unconditionally.

**Definition 2.7.** ([32], p.5) Two bases $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of a Banach space $X$ are called equivalent, provided that a series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

Let $X$ be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. For $x = \sum_{n=1}^{\infty} a_n x_n \in X$ we will denote $\operatorname{supp} x = \{n \in \mathbb{N} : a_n \neq 0\}$ and call it a support of $x$. If $x, y \in X$ are two vectors with bounded supports satisfy the inequality $\max\{i \in \operatorname{supp} x\} < \min\{i \in \operatorname{supp} y\}$ we will write $x < y$. If $x \in X$ is a vector with bounded support, which satisfies the inequality $M < \min\{i \in \operatorname{supp} x\} \leq \max\{i \in \operatorname{supp} x\} < N$ we will write $M < x < N$.

## 3. Main results

**Proposition 3.2.** *Let $(X, \|\cdot\|)$ be a Banach Köthe sequence space, $x, y \in X$ be such that* supp $x \cap$ supp $y = \emptyset$ *and* $\|x\| \geq \|y\|$*. Then for any $z \in X$, such that* supp $z \cap$ supp $(x+y) = \emptyset$ *there holds the inequality* $\|z+x\| \geq \|z+y\|$.

*Proof.* Let us consider the closed ball $B[\|z-x\|] = \{u \in X : \|u\| \leq \|z-x\|\}$. Let us consider the subspaces $Y \subset X$, such that $Y = \{x = \sum_{n\in I} x_n e_n : I = \text{supp } (x+y)\}$ and let us denote the plane $L = Y + z$. The intersection $B[\|z-x\|] \cap L = B_z[\|x\|]$ is a ball in $L$, such that $B_z[\|x\|] = \{u = \sum_{n\in I} x_n e_n \in L : \|z-u\| \leq \|x\|\}$. The vector $z + \frac{\|x\|}{\|y\|} y$ is in the sphere $S_z[\|x\|] = \{u = \sum_{n\in I} x_n e_n \in L : \|z-u\| = \|x\|\}$. Indeed $\left\|z - (z + \frac{\|x\|}{\|y\|} y)\right\| = \|x\|$. From the assumption that $X$ is a Köthe sequence space and the inequalities $\frac{\|x\|}{\|y\|}|y_i| \geq |y_i|$ it follows that $z+y$ is in the open ball $B_z(\|x\|) = \{u = \sum_{n\in I} x_n e_n \in L : \|z-u\| < \|x\|\}$ and therefore $\|z+x\| \geq \|z+y\|$. $\square$

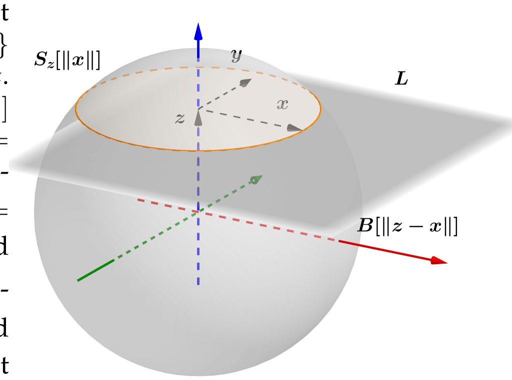

Figure 1."Proposition 3.2"

It is easy to observe that if $\|x\| = \|y\|$ in Proposition 3.2 then $\|z+x\| = \|z+y\|$.

**Theorem 3.2.** *Let $(X, \|\cdot\|)$ be a Banach Köthe sequence space. Let there hold one of the following*
*(a) $X$ be order continuous with the Fatou property;*
*(b) the unit vectors $\{e_n\}_{n=1}^{\infty}$ form an unconditional and boundedly complete basis of $X$.*
*Then $K(X) = \alpha(X)$.*

*Proof.* According to Lemma 2.1 and Theorem 2.1 it is enough to prove the equality

$$\begin{aligned} S_1 &= \sup\left\{\text{sep}(\{u^{(n)}\}) : u^{(n)} = \sum_{i=i_{n-1}+1}^{i_n} u^{(n)}(i) e_i \in S_X, 0 = i_0 < i_1 < \cdots\right\} \\ &= \sup\{\|(|x| \vee |y|\|) : x, y \in S_X, |x| \wedge |y| = 0\}. \end{aligned}$$

It is easy to see that $K(X) \leq \alpha(X)$, because the $\sup\{\|(|x| \vee |y|\|)\}$ is taken among all vectors $x, y \in S_X$, such that supp $x \cap$supp $y = \emptyset$ including those with unbounded supports.

It remains to prove that $\alpha(X) \leq K(X)$.

Let $\varepsilon > 0$ be arbitrary chosen. There are $x, y \in S_X$, so that $\|(|x| \vee |y|)\| > \alpha(X) - \varepsilon$ and $|x| \wedge |y| = 0$.

Let us denote $z = \sum_{i=1}^{\infty} z_i e_i = x + y$.

If $X$ is order continuous or $\{e_n\}_{n=1}^{\infty}$ is unconditional basis then there exists $q \in \mathbb{N}$ such that

$$\left\|\sum_{i=q+1}^{\infty} x_i e_i\right\| < \varepsilon, \quad \left\|\sum_{i=q+1}^{\infty} y_i e_i\right\| < \varepsilon \text{ and } \left\|\sum_{i=q+1}^{\infty} z_i e_i\right\| < \varepsilon.$$

Let us denote $u = \sum_{i=1}^{q} x_i e_i$, $v = \sum_{i=1}^{q} y_i e_i$ and $w = u + v$. By $|x| \wedge |y| = 0$ it follows that $|u| \wedge |v| = 0$. Then

$$1 \geq \|u\| = \left\|\sum_{i=1}^{\infty} x_i e_i - \sum_{i=q+1}^{\infty} x_i e_i\right\| \geq \left\|\sum_{i=1}^{\infty} x_i e_i\right\| - \left\|\sum_{i=q+1}^{\infty} x_i e_i\right\| \geq 1 - \varepsilon,$$

$$1 \geq \|v\| = \left\| \sum_{i=1}^{\infty} y_i e_i - \sum_{i=q+1}^{\infty} y_i e_i \right\| \geq \left\| \sum_{i=1}^{\infty} y_i e_i \right\| - \left\| \sum_{i=q+1}^{\infty} y_i e_i \right\| \geq 1 - \varepsilon$$

and

$$\begin{aligned} \alpha(X) - \varepsilon &\geq \|w\| = \left\| \textstyle\sum_{i=1}^{\infty} z_i e_i - \sum_{i=q+1}^{\infty} z_i e_i \right\| \\ &\geq \|\textstyle\sum_{i=1}^{\infty} z_i e_i\| - \left\| \sum_{i=q+1}^{\infty} z_i e_i \right\| \geq \alpha(X) - 2\varepsilon \end{aligned}$$

For any vector $x \in X$ with bounded support and any $N \in \mathbb{N}$, such that $\text{supp}\, x < N$ we denote by $\overline{x}_N$ a vector with bounded support such that $N \leq \overline{x}_N$ and $\|x\| = \|\overline{x}_N\|$.

There are $N, M \in \mathbb{N}$, such that $\text{supp}\,(u+v) \cap \text{supp}\, \overline{u}_N = \emptyset$, $\text{supp}\,(u+v) \cap \text{supp}\, \overline{v}_M = \emptyset$ and $\text{supp}\, \overline{v}_M \cap \text{supp}\, \overline{u}_N = \emptyset$. Indeed we can choose $N > \text{supp}\,(u+v)$, then we choose $\overline{u}_N$, then we choose $M > \text{supp}\,(u+v+\overline{u}_N)$ and finally we choose $\overline{u}_M$.

WLOG we may assume that $\|u\| \geq \|v\|$.

We will prove that $\max\{\|u+\overline{u}_N\|, \|v+\overline{v}_M\|\} \geq \|u+v\| - \varepsilon$. From Proposition 3.2 and the inequality $\|v\| \leq \|\overline{u}_N\| = \|u\|$ it follows that $\|u+\overline{u}_N\| \geq \|u+v\|$ and therefore

$$\max\{\|u+\overline{u}_N\|, \|v+\overline{v}_M\|\} \geq \|u+\overline{u}_N\| \geq \|u+v\|.$$

Consequently we may assume without loss of generality that there are $N \in \mathbb{N}$ and $\overline{u}_N$ such that $\|u+\overline{u}_N\| \geq \|u+v\|$.

There exist $\overline{u}_n$, $\overline{u}_m$ with $n, m > N$ such that $\text{supp}\, u < \text{supp}\, \overline{u}_N < \text{supp}\, \overline{u}_n < \text{supp}\, \overline{u}_m$. We will show that the inequality $\|\overline{u}_n - \overline{u}_m\| \geq \|u-v\|$ holds. From Proposition 3.2 we get $\|\overline{u}_n + \overline{u}_m\| \geq \|\overline{u}_n + \overline{u}_N\| \geq \|u+\overline{u}_N\| \geq \|u+v\| - \varepsilon$. Consequently we get that

$$K(X) \geq \inf\{\|\overline{u}_n - \overline{u}_m\| : n, m \in \mathbb{N}\} \geq \|u-v\| \geq \alpha(X) - 3\varepsilon$$

and by the arbitrary choice of $\varepsilon > 0$ it follows that $K(X) \geq \alpha(X)$. □

**Corollary 3.1.** *Let $(X, \|\cdot\|)$ be a Köthe sequence space. Let there hold one of the following*

*(a) $X$ be order continuous with the Fatou property;*

*(b) the unit vectors $\{e_n\}_{n=1}^{\infty}$ form an unconditional and boundedly complete basis of $X$.*

*Then there holds the equality $\Gamma_X = \frac{\alpha(X)}{\alpha(X)+2}$.*

The proof follows from the fact that for any Banach space $X$ there holds the equality $\Gamma_X = \frac{K(X)}{K(X)+2}$.

## 4. Ball packing constant in weighted Orlicz sequence spaces

We recall that $M$ is an Orlicz function if $M$ is even, convex, $M(0) = 0$, $M(t) > 0$ for $t > 0$. The Orlicz function $M(t)$ is said to satisfy the$\Delta_2$–condition at zero if there exists a constant $c$ such that $M(2t) \leq cM(t)$ for every $t \in [0,1]$. To every Orlicz function $M$ the following number is associated (see [32], p. 143)

$$\beta_M = \inf\{p : \inf\{M(uv)/u^p M(v) : u, v \in (0,1]\} > 0\},$$

called an upper Matuszewska–Orlicz index and defined in [36]. An Orlicz function $M$ satisfies the $\Delta_2$–condition at zero iff $\beta_M < \infty$, which implies of course $M(uv) \geq u^q M(v)$, $u, v \in [0,1]$ for some $q \geq \beta_M$ (see [32] p.140). If $M$ satisfies the $\Delta_2$–condition at zero we will use the notation $M \in \Delta_2$.

Following [24] let us recall the definition of a Musielak–Orlicz sequenc space. A sequence $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ of Orlicz functions is called a Musielak–Orlicz function or MO function in short.

Given an MO function $\Phi$ we define for any $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$ the function $\widetilde{\Phi}(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)$. The Musielak–Orlicz sequence space $\ell_\Phi$ or MO sequence space in short, generated by a MO function $\Phi$ is the set of all real sequences $x \in \ell^0$ such that $\widetilde{\Phi}(\lambda x) = \sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$.

The space $\ell_\Phi$ is a Banach space if endowed with norm $\|x\|_\Phi = \inf\left\{r > 0 : \widetilde{\Phi}(x/r) \le 1\right\}$, called Luxemburg norm.

Following [8] for $p \in [1, +\infty]$ and any $u \ge 0$ we define $s_{M,p}(x) = s_p\left(\widetilde{\Phi}(x)\right)$, where

$$s_p(u) = \begin{cases} (1+u^p)^{1/p}, & \text{for } 1 \le p < \infty \\ \max\{1, u\}, & \text{for } p = \infty. \end{cases}$$

For any $x \in \ell_\Phi$ we define the $p$–Amemiya norm by $|||x|||_{\Phi,p} = \inf_{k>0} \frac{s_{\Phi,p}(kx)}{k}$. The 1–Amemiya norm is called just Amemiya norm. It is known [8] that $\|x\|_\Phi = |||x|||_{\Phi,\infty}$. The inequalities $\frac{|||x|||_\Phi}{2} \le \|x\|_\Phi \le |||x|||_{\Phi,p} \le 2^{1/p}\|x\|_\Phi \le 2^{1/p}|||x|||_\Phi$ hold for $p \in [1, +\infty)$.

Just to simplify the notations we will denote by $\|\cdot\|$ the Luxemburg norm and by $|||\cdot|||$ the Amemiya norm.

We denote by $h_\Phi$ the closed linear subspace of $\ell_\Phi$, generated by all $x = \{x_i\}_{i=1}^{\infty} \in \ell_\Phi$, such that the inequality $\widetilde{\Phi}(\lambda x) < \infty$ holds for every $\lambda > 0$.

We will write $\ell_\Phi$, when the statement holds for the MO sequence space equipped with both norms – Luxemburg or $p$–Amemiya.

**Definition 4.8.** ([21]) We say that the MO function $\Phi$ satisfies the $\delta_2$ condition at zero if there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that for every $n \in \mathbb{N}$ holds the inequality

$$\Phi_n(2t) \le K\Phi_n(t) + c_n,$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces $\ell_\Phi$ and $h_\Phi$ coincide iff $\Phi$ has $\delta_2$ condition at zero [26, 32, 47].

Any $MO$ sequence space is a Köthe sequence space with the Fatou property [27]. When the condition $\delta_2$ at zero is satisfied [32], the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a boundedly complete unconditional basis in $\ell_\Phi$ and $\ell_\Phi$ is order continuous.

If $\Phi$ satisfies the $\delta_2$ condition at zero we will use the notation $M \in \delta_2$.

An extensive study of Orlicz and MO spaces can be found in [32, 38, 42].

The exact value of Kottman's constant in MO sequence spaces endowed with the Luxemburg norm is obtained in [18].

**Definition 4.9.** ([18]) We say that the MO function $\Phi$ satisfies condition $(+)$ if for any $c > 0$ and any $\varepsilon \in (0, c)$ there is $\delta > 0$ such that $\Phi_n((1+\delta)t) \le c$, whenever $\Phi_n(t) \le c - \varepsilon$ for $n = 1, 2, \ldots$ and $t > 0$.

Definition 4.9 was defined first for $c = 1$ in [24].

Let $\Phi$ be a MO function. Let us denote

$$c(x,m,n) = \inf\left\{c > 0 : \sum_{i=n}^{n+m} \Phi_i\left(\frac{x_i}{c}\right) \le \frac{1}{2} : x = \{x_i\} \in S_{\ell_\Phi}, n, m \in \mathbb{N}\right\},$$

$d(x,n) = \lim_{m\to\infty} c(x,m,n)$, $d(n) = \sup\{d(x,n) : x = \{x_i\} \in S_\Phi\}$, $n \in \mathbb{N}$, $d_\Phi = \lim_{n\to\infty} d(n)$.

**Theorem 4.3.** ([18]) Let $\Phi$ be a MO function, that satisfies the $\delta_2$ condition at zero and the condition (+). Then $K((\ell_\Phi, \|\cdot\|)) = d_\Phi$ and $\Gamma_X = \frac{d_\Phi}{2+d_\Phi}$.

If the MO function $\Phi$ consists of one and the same function $M$ one obtains the Orlicz sequence spaces $\ell_M$ and $h_M$. A weight sequence $w = \{w_i\}_{i=1}^{\infty}$ is a sequence of positive reals. If $\Phi_i = w_i M$, where $M$ is an Orlicz function and $w = \{w_i\}_{i=1}^{\infty}$ is a weight sequence we get the weighted Orlicz sequence spaces $\ell_M(w)$ and $h_M(w)$.

Following [15] we say that $w = \{w_i\}_{i=1}^{\infty}$ is from the class $\Lambda$ if there exists a subsequence $w = \{w_{i_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} w_{i_k} = 0$ and $\sum_{k=1}^{\infty} w_{i_k} = \infty$. If the weight sequence $w$ is from the class $\Lambda$ we will use the notation $w \in \Lambda$.

When the weight sequence $w$ is the constant sequence $w_i = 1$ for every $i \in \mathbb{N}$ we get the Orlicz sequence spaces.

It is easy to observe that if $M \in \Delta_2$, then the MO function $\Phi$, defined by $\Phi_i = w_i M$ satisfies the $\delta_2$ condition at zero and the condition (+). Therefore $K((\ell_M(w), \|\cdot\|)) = d_\Phi$ and $\Gamma_X = \frac{d_\Phi}{2+d_\Phi}$.

We will present a formula, which is different from that in Theorem 4.3, for calculating of Kottman's constant and the ball packing constant in weighted Orlicz sequence spaces, when the weighted sequence $w$ belongs to the class $\Lambda$.

The next lemma is a direct corollary of Lemma 2 from [18] for weighted Orlicz sequence spaces. As far as its proof is omitted in [18], just for completeness we will prove it.

**Lemma 4.2.** *Let $M \in \Delta_2$ be an Orlicz function $w \in \Lambda$ be a weight sequence. Then for any $\delta > 0$ and any sequence $x = \left\{x^{(n)}\right\}_{n=1}^{\infty} \subset S_{\ell_M(w)}$ there exist a subsequence $y = \left\{y^{(n)}\right\}_{n=1}^{\infty}$ of $x$ and a subsequence $\{p_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that:*

*(1) $\sum_{i=p_k+1}^{\infty} w_i M\left(y_i^{(k)}\right) < \delta$ for every $k \in \mathbb{N}$;*

*(2) $\sum_{i=1}^{p_{k-1}} w_i M\left(y_i^{(n)} - y_i^{(m)}\right) < \delta$ for every $k \in \mathbb{N}$, $m, n \geq k$;*

*(3) $\sum_{i=p_{k-1}+1}^{p_k} w_i M\left(y_i^{(n)}\right) < \delta$ for every $k \geq 2$, $n \geq k$.*

*Proof.* Since for any $n, i \in \mathbb{N}$ there holds the inequality $\left|x_i^{(n)}\right| \leq M^{-1}(w_i)$ we may assume by the diagonal method, that there is a subsequence $\left\{x^{(n_k)}\right\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty} x_i^{(n_k)} = \alpha_i$ for every $i \in \mathbb{N}$. Just to simplify the notation we will denote the subsequence $\left\{x^{(n_k)}\right\}_{k=1}^{\infty}$ with $\left\{x^{(n)}\right\}_{n=1}^{\infty}$. It is easy to observe that $\sum_{i=1}^{\infty} w_i M(\alpha_i) \leq \lim_{n\to\infty} \widetilde{M_w}\left(x^{(n)}\right) = 1$.

Let us put $y^{(1)} = x^{(1)}$. We can choose $p_1 \in \mathbb{N}$ such that $\sum_{i=p_1+1}^{\infty} w_i M\left(y^{(1)}\right) < \delta$ and $\sum_{i=p_1+1}^{\infty} w_i M(\alpha_i) < \delta$.

From $\lim_{n\to\infty} x_i^{(n)} = \alpha_i$ it follows that there exists $N_1 \in \mathbb{N}$ so that for any $n, m > N_1$ the inequality $\sum_{i=1}^{p_1} w_i M\left(x_i^{(n)} - x_i^{(m)}\right) < \delta$ holds.

Let us put $y^{(2)} = x^{(N_1)}$. Thereafter we choose a natural number $p_2 > p_1$ such that $\sum_{i=p_2+1}^{\infty} w_i M\left(y_i^{(2)}\right) < \delta$. Since

$$\sum_{i=p_1+1}^{p_2} w_i M(\alpha_i) \leq \sum_{i=p_1+1}^{\infty} w_i M(\alpha_i) < \delta \text{ and } \lim_{n\to\infty} x_i^{(n)} = \alpha_i$$

we can choose $N_2 \in \mathbb{N}$, so that $N_2 > N_1$ and for any $n, m \geq N_2$ to hold the inequalities $\sum_{i=1}^{p_1} w_i M\left(x_i^{(n)} - x_i^{(m)}\right) < \delta$ and $\sum_{i=p_1+1}^{p_2} w_i M\left(x_i^{(n)}\right) < \delta$.

Let us put $y^{(3)} = x^{(N_2)}$. There is a natural number $p_3 > p_2$ so that $\sum_{i=p_3+1}^{\infty} w_i M\left(y_i^{(3)}\right) < \delta$. Since $\lim_{n\to\infty} x_i^{(n)} = \alpha_i$ and $\sum_{i=p_2+1}^{p_3} w_i M(\alpha_i) \leq \sum_{i=p_2+1}^{\infty} w_i M(\alpha_i) < \delta$ we can choose $N_3 \in \mathbb{N}$, so that $N_3 > N_1$ and for any $n, m \geq N_3$ to hold $\sum_{i=1}^{p_2} w_i M\left(x_i^{(n)} - x_i^{(m)}\right) < \delta$ and $\sum_{i=p_2+1}^{p_3} w_i M\left(x_i^{(n)}\right) < \delta$.

We continue this process by induction to obtain the sequences $\left\{y^{(n)}\right\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$. □

Let $\ell_\Phi$ be a MO sequence space, generated by a MO function $\Phi$. For any $x = \{x_i\}_{i=1}^{\infty} \in \ell_\Phi$ we will use the notation $\widetilde{\Phi}(x) = \sum_{i=1}^{\infty} \Phi(x_i)$. If $x \in \ell_M(w)$ we will use the notation $\widetilde{M_w}(x) = \widetilde{\Phi}(x)$, where $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ is defined by $\Phi_i(t) = w_i M(t)$.

**Lemma 4.3.** *Let $M \in \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then for any $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that $\|x\| \le \varepsilon$, provided that $\widetilde{M_w}(x) \le \delta$ .*

*Proof.* By the assumption that $M \in \Delta_2$ it follows that $M(uv) \ge u^q M(v)$ for $u, v \in [0,1]$ and for some $q \ge \beta_M$. Let us put $\delta = \varepsilon^q$. from the inequalitites $0 \le \delta\varepsilon \le 1$ and $\widetilde{M_w}\left(x\right) \le \delta$ it follows that $\|x\| \le 1$. Using the inequalities

$$\begin{array}{rcl} \delta & \ge & \widetilde{M_w}\left(\|x\|\frac{x}{\|x\|}\right) = \sum_{i=1}^{\infty} w_i M\left(\|x\|\frac{x_i}{\|x\|}\right) \ge \sum_{i=1}^{\infty} w_i \|x\|^q M\left(\frac{x_i}{\|x\|}\right) \\ & = & \|x\|^q \widetilde{M_w}\left(\frac{x}{\|x\|}\right) = \|x\|^q \end{array}$$

we get that $\|x\| \le \delta^{1/q} = \varepsilon$. □

The next lemma is a generalization of ([22], Lemma 5).

**Lemma 4.4.** *Let $M \in \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then for any $L > 0$ and any $\varepsilon > 0$ there exists $\delta > 0$ so that the inequality $|\widetilde{M_w}(u+v) - \widetilde{M_w}(u)| < \varepsilon$ holds whenever there hold the inequalities $\widetilde{M_w}(u) \le L$ and $\widetilde{M_w}(v) \le \delta$.*

A result more general than that of Lemma 4.4 is obtained in ([24], Proposition 10), where the MO function $\Phi$ satisfies $\delta_2$ condition at zero and Definition 4.9 for $c = 1$. Lemma 4.4 is a particular case of ([9], Lemma 2.1), where it is proven for a modular $\rho$ ($\widetilde{M_w}$ is a particular case of a modular). Just for completeness and to skip the definitions of modulars and modular spaces, which are not part of the present work, we will prove Lemma 4.4 as stated above.

*Proof.* Let us put $h = \sup\left\{\widetilde{M_w}(2u+2v) : \widetilde{M_w}(u) \le L, \widetilde{M_w}(v) \le 1\right\}$. From the assumptions $M \in \Delta_2$ and $w \in \Lambda$ it follows that $L < h < \infty$. WLOG we may assume that $L > 1$ and $\varepsilon < 1$. Let us put $\beta = \frac{\varepsilon}{h}$. From Lemma 4.3 it follows that there exists $\delta > 0$ so that the inequality $\widetilde{M_w}(v) \le \delta$ implies the inequality $\|v\| \le \min\left\{\frac{\beta}{2}, \frac{\varepsilon}{2}\right\}$. Thus there holds $\widetilde{M_w}\left(\frac{2v}{\beta}\right) = \widetilde{M_w}\left(\frac{2\|v\|}{\beta} \cdot \frac{v}{\|v\|}\right) \le \frac{2\|v\|}{\beta}\widetilde{M_w}\left(\frac{v}{\|v\|}\right) = \frac{2\|v\|}{\beta}$. Hence if there hold $\widetilde{M_w}(u) \le L$ and $\widetilde{M_w}(v) \le \delta$ then we can write the chain of inequalities

$$\begin{array}{rcl} \widetilde{M_w}(u+v) & = & \widetilde{M_w}\left((1-\beta)u + \beta\left(u + \frac{v}{\beta}\right)\right) \le (1-\beta)\widetilde{M_w}\left(u\right) + \beta\widetilde{M_w}\left(u + \frac{v}{\beta}\right) \\ & \le & (1-\beta)\widetilde{M_w}\left(u\right) + \frac{\beta}{2}\left(\widetilde{M_w}\left(2u\right) + \widetilde{M_w}\left(\frac{2v}{\beta}\right)\right) \\ & \le & \widetilde{M_w}\left(u\right) + \frac{\beta h}{2} + \|v\| \le \widetilde{M_w}\left(u\right) + \varepsilon. \end{array}$$

Replacing $u$ and $v$ with $u+v$ and $-v$ respectively in the above inequalities we get

$$\begin{array}{rcl} \widetilde{M_w}(u) & = & \widetilde{M_w}(u+v-v) = \widetilde{M_w}\left((1-\beta)(u+v) + \beta\left(u+v-\frac{v}{\beta}\right)\right) \\ & \le & (1-\beta)\widetilde{M_w}\left(u+v\right) + \beta\widetilde{M_w}\left(u+v-\frac{v}{\beta}\right) \\ & \le & (1-\beta)\widetilde{M_w}\left(u+v\right) + \frac{\beta}{2}\left(\widetilde{M_w}\left(2u+2v\right) + \widetilde{M_w}\left(\frac{2v}{\beta}\right)\right) \\ & \le & \widetilde{M_w}\left(u+v\right) + \frac{\beta h}{2} + \|v\| \le \widetilde{M_w}\left(u\right) + \varepsilon. \end{array}$$

Thus there holds $\left|\widetilde{M_w}(u+v) - \widetilde{M_w}(v)\right| < \varepsilon$. □

We will need the next lemma. Let us point out that it is stated in [51] for the case when $v = \{v_i\}_{i=1}^{\infty} \in \Lambda$ is a subsequence of $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. The proof when $v = \{v_i\}_{i=1}^{\infty} \in \Lambda$ is an arbitrary sequence is literary the same.

**Lemma 4.5.** ([51]) *Let $w = \{w_i\}_{i=1}^{\infty}$ and $v = \{v_i\}_{i=1}^{\infty} \in \Lambda$. Then there exist sequences of naturals $\left\{m_i^{(s)}\right\}_{i=1}^{\infty}$, $\left\{k_i^{(s)}\right\}_{i=1}^{\infty}$, $s \in \mathbb{N}$, such that*

$$1 \le m_1^{(1)} \le k_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)},\ \ m_i^{(s)} \le k_i^{(s)},\ \ k_i^{(s)} < m_{i+1}^{(s-1)},\ \ \textit{for}\ \ n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$$

*and for every $i \in \mathbb{N}$ there holds*

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i.$$

Let $M \in \Delta_2$, $w \in \Lambda$ and $x \in S_{\ell_M(w)}$. Then for any $n \in \mathbb{N}$ there exists a unique $c_{x,n} > 0$, such that $\widetilde{M_w}\left(\frac{x}{c_{x,n}}\right) = \frac{1}{n}$. Following the notations from [6] let us denote $d_n = \sup\left\{c_{x,n} : x \in S_{\ell_M(w)}\right\}$. For each sequence $x = \left\{x^{(n)}\right\} \subset S_{\ell_M(w)}$ we set

$$D_n(x) = \inf\left\{\left\|x^{(1)} + \varepsilon_2 x^{(2)} + \cdots + \varepsilon_n x^{(n)}\right\| : x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in x, \varepsilon_i = \pm 1\right\}$$

and $D_n = \sup\left\{D_n(x) : x = \left\{x^{(n)}\right\} \in S_{\ell_M(w)}\right\}$.

**Lemma 4.6.** *Let $M \in \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then $d_n = D_n$.*

*Proof.* Let $\varepsilon \in (0, d_m)$ be arbitrary. Let us choose $y \in S_{\ell_M(w)}$, such that $c_{y,m} > d_m - \varepsilon$. It is easy to observe that by using a diagonal argument any sequence $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ can be split into countably many sequences $u^{(j)} = \{w_k\}_{k \in I^j}$, so that $u^{(j)} \in \Lambda$ for any $j \in \mathbb{N}$, $I^j \cap I^k = \emptyset$ for any $k \neq j$ and $\cup_{j=1}^{\infty} I^j = \mathbb{N}$. From Lemma 4.5 there exist disjoint subsets $\left\{J_n^j\right\}_{n=1}^{\infty} \subset I^j$ for $j \in \mathbb{N}$, such that $\sum_{k \in J_n^j} w_k = w_n$.

We will define a sequence $\{x^{(n)}\}$ by
$x^{(1)} = \sum_{s=1}^{\infty} y_s \sum_{k \in J_s^1} e_k,\ x^{(2)} = \sum_{s=1}^{\infty} y_s \sum_{k \in J_s^2} e_k,\ \ldots,\ x^{(n)} = \sum_{s=1}^{\infty} y_s \sum_{k \in J_s^n} e_k,\ \ldots$

From the construction of the sets $J_n^j$ it follows that $\widetilde{M_w}\left(x^{(n)}\right) = \widetilde{M_w}(y)$. By the construction $\mathrm{supp}(x^{(i)}) \cap \mathrm{supp}(x^{(j)}) = \emptyset$ for any $i \neq j$ and $x^{(i)} \in S_{\ell_M(w)}$. Since

$$\widetilde{M_w}\left(\frac{x^{(k_1)} \pm x^{(k_2)} \pm \cdots \pm x^{(k_m)}}{d_m - \varepsilon}\right) = m\widetilde{M_w}\left(\frac{y}{d_m - \varepsilon}\right) \ge m\widetilde{M_w}\left(\frac{y}{c_{y,m}}\right) = 1$$

we get that for any $x^{(k_1)}, x^{(k_2)}, \ldots, x^{(k_m)}$ there holds the inequality $\|x^{(k_1)} \pm x^{(k_2)} \pm \cdots \pm x^{(k_m)}\| \ge d_m - \varepsilon$, i.e. $D_m(x) \ge d_m - \varepsilon$. Consequently by the definition of $D_m$ we get that $D_m \ge D_m(x) \ge d_m - \varepsilon$. From the arbitrary choice of $\varepsilon \in (0, d_m)$ it follows that $D_m \ge d_m$.

It remains to prove that $D_m \le d_m$. We will consider the case $m$ being an odd number. The case when $m$ is an even number can be proven in a similar fashion.

Let $\varepsilon > 0$ and $x = \left\{x^{(k)}\right\}_{k=1}^{\infty} \in S_{\ell_M(w)}$. Let us denote $\varepsilon_1 = \frac{\varepsilon}{d_m + \varepsilon}$. Then for every $k \in \mathbb{N}$ there holds

$$\widetilde{M_w}\left(\frac{x^{(k)}}{d_m + \varepsilon}\right) \le \frac{d_m}{d_m + \varepsilon}\widetilde{M_w}\left(\frac{x^{(k)}}{d_m}\right) \le \frac{1 - \varepsilon_1}{m}. \tag{4.2}$$

From Lemma 4.4 it follows that there exists $\delta > 0$ such that the inequality

$$\left|\sum_{i \in I} w_i M(\alpha_i + \beta_i) - \sum_{i \in I} w_i M(\alpha_i)\right| < \frac{\varepsilon_1}{m}$$

holds whenever there hold $\sum_{i\in I} w_i M(\alpha_i) < 1$ and $\sum_{i\in I} w_i M(\beta_i) < \delta$. By the assumption $M \in \Delta_2$ it follows that there exists $c > 0$, such that $M(mu) \le cM(u)$, provided that $0 \le u \le M^{-1}(1)$.

From Lemma 4.2 it follows that any sequence $x = \left\{x^{(k)}\right\}_{k=1}^{\infty} \subset S_{\ell_M(w)}$ has a subsequence $\left\{y^{(k)}\right\}_{k=1}^{\infty}$ that satisfies

(1′) $\sum_{i=p_k+1}^{\infty} w_i M\left(y_i^{(k)}\right) < \frac{\delta}{c}$ for every $k \in \mathbb{N}$;

(2′) $\sum_{i=1}^{p_{k-1}} w_i M\left(y_i^{(n)} - y_i^{(s)}\right) < \frac{\delta}{c}$ for every $k \in \mathbb{N}$, $s, n \ge k$;

(3′) $\sum_{i=p_{k-1}+1}^{p_k} w_i M\left(y_i^{(n)}\right) < \frac{\delta}{c}$ for every $k \ge 2$, $n \ge k$.

We will prove that $\left\|y^{(1)} + \sum_{k=2}^{m}(-1)^k y^{(k)}\right\| \le d_m + \varepsilon$.

We will use the notation $\widetilde{M_w}(z|_n^m) = \sum_{i=n}^{m} w_i M(z_i)$. Since

$$\begin{aligned} S_2 &= \widetilde{M_w}\left(\frac{y^{(2)}-y^{(3)}+\cdots+y^{(m-1)}-y^{(m)}}{d_m+\varepsilon}\Big|_1^{p_1}\right) = \sum_{i=1}^{p_1} w_i M\left(\frac{\sum_{k=2}^{m}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \\ &\le \sum_{i=1}^{p_1} w_i M\left(\sum_{k=2}^{m}(-1)^k y_i^{(k)}\right) = \sum_{i=1}^{p_1} w_i M\left(\sum_{k=1}^{\frac{m-1}{2}}\left(y_i^{(2k)} - y_i^{(2k+1)}\right)\right) \\ &\le \frac{2}{m-1}\sum_{k=1}^{\frac{m-1}{2}}\left(\sum_{i=1}^{p_1} w_i M\left(\frac{m-1}{2}\left(y_i^{(2k)} - y_i^{(2k+1)}\right)\right)\right) \\ &\le \frac{2c}{m-1}\left(\sum_{k=1}^{\frac{m-1}{2}}\left(\sum_{i=1}^{p_1} w_i M\left(y_i^{(2k)} - y_i^{(2k+1)}\right)\right)\right) \le \frac{2c}{m-1}\cdot\frac{m-1}{2}\cdot\frac{\delta}{c} = \delta, \end{aligned}$$

the choice of $\delta$, recalling that $\left\{y^{(k)}\right\}_{k=1}^{\infty}$ is a subsequence of $\left\{x^{(k)}\right\}_{k=1}^{\infty}$ and (4.2) it follows that

$$\begin{aligned} S_3 &= \widetilde{M_w}\left(\frac{y^{(1)}+y^{(2)}-y^{(3)}+\cdots+y^{(m-1)}-y^{(m)}}{d_m+\varepsilon}\Big|_1^{p_1}\right) \\ &\le \sum_{i=1}^{p_1} w_i M\left(\frac{y_i^{(1)}}{d_m+\varepsilon}\right) + \frac{\varepsilon_1}{m} \le \frac{1-\varepsilon_1}{m} + \frac{\varepsilon_1}{m} = \frac{1}{m}. \end{aligned}$$

Continuing in the same fashion we get

$$\begin{aligned} S_4 &= \widetilde{M_w}\left(\frac{y^{(1)}-y^{(3)}+\cdots+y^{(m-1)}-y^{(m)}}{d_m+\varepsilon}\Big|_1^{p_2}\right) = \sum_{i=1}^{p_2} w_i M\left(\frac{y_i^{(1)}+\sum_{k=3}^{m}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \\ &\le \frac{1}{m-1}\left(\sum_{i=1}^{p_2} w_i M\left((m-1)\left(y_i^{(1)}\right)\right) + \sum_{k=3}^{m}\left(\sum_{i=1}^{p_2} w_i M\left((m-1)y_i^{(k)}\right)\right)\right) \\ &\le \frac{c}{m-1}\left(\sum_{i=1}^{p_2} w_i M\left(\left(y_i^{(1)}\right)\right) + \sum_{k=3}^{m}\left(\sum_{i=1}^{p_2} w_i M\left(y_i^{(k)}\right)\right)\right) \le \frac{c}{m-1}(m-1)\frac{\delta}{c} = \delta. \end{aligned}$$

From the last inequality, the choice of $\delta$, recalling that $\left\{y^{(k)}\right\}_{k=1}^{\infty}$ is a subsequence of $\left\{x^{(k)}\right\}_{k=1}^{\infty}$ and (4.2) it follows that

$$\widetilde{M_w}\left(\frac{y^{(1)}+y^{(2)}-y^{(3)}+\cdots+y^{(m-1)}-y^{(m)}}{d_m+\varepsilon}\Big|_1^{p_2}\right) \le \sum_{i=1}^{p_2} w_i M\left(\frac{y_i^{(2)}}{d_m+\varepsilon}\right) + \frac{\varepsilon_1}{m} \le \frac{1-\varepsilon_1}{m} + \frac{\varepsilon_1}{m} = \frac{1}{m}.$$

Similarly we get for $k = 3, 4, \ldots, m-1$ the inequalities

$$\widetilde{M_w}\left(\frac{y^{(1)}+y^{(2)}-y^{(3)}+\cdots+y^{(m-1)}-y^{(m)}}{d_m+\varepsilon}\Big|_1^{p_k}\right) \le \sum_{i=1}^{p_k} w_i M\left(\frac{y_i^{(k)}}{d_m+\varepsilon}\right) + \frac{\varepsilon_1}{m} \le \frac{1-\varepsilon_1}{m} + \frac{\varepsilon_1}{m} = \frac{1}{m}.$$

From the inequality

$$\widetilde{M_w}\left(\frac{y^{(1)}+y^{(2)}-y^{(3)}+\cdots+y^{(m-1)}}{d_m+\varepsilon}\Big|_{p_{m-1}+1}^{\infty}\right) = \sum_{i=p_{m-1}+1}^{\infty} w_i M\left(\frac{y_i^{(1)}+\sum_{k=2}^{m-1}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \le \delta$$

it follows that

$$\begin{aligned} S_5 &= \widetilde{M_w}\left(\frac{y^{(1)}+y^{(2)}-y^{(3)}+\dots+y^{(m-1)}}{d_m+\varepsilon}\Big|_{p_{m-1}+1}^{\infty}\right) \\ &\leq \sum_{i=p_{m-1}+1}^{\infty} w_i M\left(\frac{y_i^{(m)}}{d_m+\varepsilon}\right)+\frac{\varepsilon_1}{m} \leq \frac{1-\varepsilon_1}{m}+\frac{\varepsilon_1}{m}=\frac{1}{m}. \end{aligned}$$

Summing the above $n$ inequalities we get

$$\begin{aligned} S_6 &= \widetilde{M_w}\left(\frac{y_i^{(1)}+\sum_{k=2}^{m}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \leq \sum_{k=1}^{m-1}\sum_{i=1}^{p_k} w_i M\left(\frac{y_i^{(1)}+\sum_{k=2}^{m}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \\ &+\sum_{i=p_{m-1}+1}^{\infty} w_i M\left(\frac{y_i^{(1)}+\sum_{k=2}^{m}(-1)^k y_i^{(k)}}{d_m+\varepsilon}\right) \leq 1, \end{aligned}$$

i.e. $\left\|y^{(1)}+\sum_{k=2}^{m}(-1)^k y^{(k)}\right\| \leq d_m+\varepsilon$. Since $\{y^{(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{x^{(k)}\}_{k=1}^{\infty}$ it follows that $D_m(x) \leq d_m+\varepsilon$ and by the arbitrary choice of $\varepsilon$ we get that $D_m(x) \leq d_m$. □

Following ([6], p. 149) from Lemma 4.6 and (2.1) we get the next result.

**Theorem 4.4.** *Let $M \in \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then $K((\ell_M(w), \|\cdot\|)) = d_2$ and $\Gamma_{(\ell_M(w),\|\cdot\|)} = \frac{d_2}{2+d_2}$.*

The next proposition seem to be well known but there is no proof known to the author. It is well known that $\ell_\Phi = h_\Phi$ if and only if $\Phi$ satisfies the $\delta_2$–condition at zero [20, 23], but it is difficult to check that when $w \in \Lambda$ and $M \in \Delta_2$ then the MO function $\Phi = \{w_n M\}_{n=1}^{\infty}$ satisfies the $\delta_2$–condition at zero. It is possible for an Orlicz function $M \notin \Delta_2$ to choose a suitable wighted sequence $w$ so that the space $\ell_M(w)$ to have different properties [27, 51].

For the proof of the next proposition we will follow the technique from ([32], Proposition 4.a.4)

**Proposition 4.3.** *Let $M$ be an Orlicz function and $w \in \Lambda$. Then the following conditions are equivalent.*

*(i) $M$ satisfies the uniform $\Delta_2$–condition at zero;*
*(ii) $\ell_M(w) = h_M(w)$;*

*Proof.* If $M$ satisfies the $\Delta_2$ condition at zero. Then following the proof in ([32], Proposition 4.a.4) we will get that $\ell_M(w) = h_M(w)$.

Let us assume that $M$ does not satisfy the $\Delta_2$ condition at zero. Just for simplicity of the notations let us denote the subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, satisfying the conditions $\sum_{k=1}^{\infty} w_{n_k} = \infty$ and $\lim_{k\to\infty} w_{n_k} = 0$ by $\{w_n\}_{n=1}^{\infty}$. From the assumption that $M \notin \Delta_2$ it follows that there exists a convergent to zero sequence $\{\alpha_k\}_{k=1}^{\infty}$, such that $M(\alpha_n) \leq \frac{1}{2^n}$ and the inequalities $M(2\alpha_n) > 2^n M(\alpha_n)$ hold for every $n \in \mathbb{N}$.

We can choose two sequences of natural numbers $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$, such that $1 \leq p_1 < q_1 < p_2 < q_2 < \cdots < q_{n-1} < p_n < q_n \ldots$ and $\frac{1}{2^{n+1}M(\alpha_n)} < \sum_{i=p_n}^{q_n} w_i \leq \frac{1}{2^n M(\alpha_n)}$.

Let us denote $x^{(n)} = \sum_{i=p_n}^{q_n} \alpha_n e_i$ and $u = \sum_{n=1}^{\infty} x^{(n)}$. Since

$$\widetilde{M_w}(u) = \sum_{n=1}^{\infty}\left(M(\alpha_n)\sum_{i=p_n}^{q_n} w_i\right) \leq \sum_{k=1}^{\infty}\frac{1}{2^n} = 1$$

it follows that $u \in \ell_M(w)$. From

$$\widetilde{M_w}(2u) = \sum_{n=1}^{\infty}\left(M(2\alpha_n)\sum_{i=p_n}^{q_n} w_i\right) \geq \sum_{n=1}^{\infty}\left(2^n M(\alpha_n)\sum_{i=p_n}^{q_n} w_i\right) \geq \sum_{k=1}^{\infty}\frac{1}{2} = \infty$$

it follows that $u \in h_M(w)$ and thus $\ell_M(w) \neq h_M(w)$. □

**Theorem 4.5.** *If $M \notin \Delta_2$ and $w \in \Lambda$, then $(\ell_M(w), \|\cdot\|)$ has a subspace isometric to $\ell_\infty$.*

*Proof.* From Proposition 4.3 it follows that the spaces $\ell_M(w)$ and $h_M(w)$ coincide if and only if $M \in \Delta_2$ at zero. The MO sequence spaces $\ell_\Phi$ and $h_\Phi$ coincide if and only if $\Phi$ has the $\delta_2$ conditon at zero [20, 26]. Thus if $M \notin \Delta_2$ and $w \in \Lambda$ it follows from Proposition 4.3 that $\ell_M(w) \neq h_M(w)$ and by [20, 26] we get that the MO function $\Phi = \{\Phi_k\}_{k=1}^\infty$, where $\Phi_k(t) = w_k M(t)$, which generates the weighted Orlicz sequence space $\ell_M(w)$ does not satisfy the $\delta_2$ condition at zero. Therefore according to ([23], Theorem 1.1) there is an isometric copy of $\ell_\infty$ in $\ell_M(w)$. □

Theorem 4.5 can be proven also by using techniques that are very similar to that in [4].

**Theorem 4.6.** *Let $M \notin \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then $K((\ell_M(w), \|\cdot\|)) = 2$ and $\Gamma_{(\ell_M(w),\|\cdot\|)} = \frac{1}{2}$.*

*Proof.* The proof follows by Theorem 4.5 and the well known fact [6] that $K(\ell_\infty) = 2$. □

For any positive measure space $(\Omega, \Sigma, \mu)$ the Orlicz function space $L_M(\mu)$ is defined as the set of all equivalence classes of $\mu$–measurable scalar functions $x$ on $\Omega$ such that for some $\lambda > 0$ there holds $\widetilde{M}(x/\lambda) = \int_\Omega M(x(t)/\lambda)d\mu(t) < \infty$. For $\Omega = \mathbb{N}$ and $w = \{w_j\}_{j=1}^\infty = \{\mu(j)\}_{j=1}^\infty$ we get the weighed Orlicz sequence space $\ell_M(w)$. Therefore when investigating the sequence space $\ell_M(w)$ we can use the known results about the corresponding Orlicz function spaces. As far as we are considering weighted Orlicz sequence spaces $\ell_M(w)$ generated by a weight sequence $w \in \Lambda$ then only the behavior of the Orlicz function on small arguments matters [15].

Deep results about the geometry of $p$–Amemiya Orlicz spaces $(L_M(\mu), |||\cdot|||_{M,p})$ are obtained in [8].

To any Orlicz function the following function $N(v) = \sup\{u|v| - M(u) : u \geq 0\}$ is associated and is called complementary function to $M$. Following [8] let us denote $\alpha_p(x) = \left(\widetilde{M}\right)^{p-1} \widetilde{N}(p_+(|x|)) - 1$, for $p \in [1, +\infty)$, where $N$ is the complementary function to $M$, and $p_+$ is the right derivative of $M$. Let us denote $k_p^* = \inf\{k \geq 0 : \alpha_p(kx) \geq 0\}$ and $k_p^{**} = \sup\{k \geq 0 : \alpha_p(kx) \leq 0\}$. There holds the inequality $k_p^* \leq k_p^{**}$. Let us denote $c_M = \sup\{u \geq 0 : M(u) < \infty\}$. If $c_M = \infty$, then according to [8] $k_p^* < \infty$ if and only if $M$ is not linear for $p \in (1, +\infty)$ and $M$ does not have an asymptote at $\infty$ for $p = 1$. When considering the Orlicz sequence spaces $\ell_M$ or the MO sequence space $\ell_M(w)$, $w \in \Lambda$ only the behavior of $M$ at zero is significant. Thus we get that if $c_M = \infty$, then according to [8] $k_p^* < \infty$ in $\ell_M(w)$ if and only if $M$ is not linear. If $M$ is linear and $w \in \Lambda$ then by the equivalence of the $p$–Amemiya norm and the Luxemburg norm it follows that there is an isomrophic copy of $\ell_1$ in $(\ell_M(w), |||\cdot|||_{M,p})$. According to [19] $X$ containes an almost isometric copy $\ell_1$ and thus $K(\ell_M(w), |||\cdot|||_{M,p}) = K(\ell_1) = 2$ and $\Gamma_{(\ell_M(w),|||\cdot|||_{M,p})} = \frac{1}{2}$.

If $M \in \Delta_2$ and $w \in \Lambda$ then for any $k > 1$ and any $p \in [1, \infty)$ there is a unique $d^{x,k,p} > 0$, so that $\left(\widetilde{M_w}\left(\frac{kx}{d^{x,k,p}}\right)\right)^p = \frac{k^p-1}{2^p}$. Let us denote $d^{x,p} = \inf\{d^{x,k,p} : k > 1\}$ and $d^p = \sup\{d^{x,p} : x \in S_{(\ell_M(w),|||\cdot|||_{M,p})}\}$.

If $M$ is not linear, $M \in \Delta_2$ and $w \in \Lambda$, then for every $k \in [k_p^*(x), k_p^{**}]$ there holds $|||x|||_{M,p} = \frac{1}{k}\left(1 + \left(\widetilde{M_w}(kx)\right)^p\right)^{1/p}$. It is easy to observe that for every $x \in S_{(\ell_M(w),|||\cdot|||_{M,p})}$ and any $k \in [k_p^*(x), k_p^{**}]$ there hold the inequality $\left(\widetilde{M_w}\left(\frac{kx}{2}\right)\right)^p < \frac{1}{2^p}\left(\widetilde{M_w}(kx)\right)^p = \frac{k^p-1}{2^p}$ and therefore $d^{x,p} < 2$. From the definition of $|||\cdot|||_{M,p}$ we get that the inequality $\widetilde{M_w}(kx) \geq (k^p-1)^{\frac{1}{p}}$ holds for every $k > 0$ and hence $d^{x,p} > 1$. Combining these observation it follows that $d^p \in (1, 2]$.

If $M$ is linear and $w \in \Lambda$, then $k_p^*(x) = \infty$. Let $x \in S_{(\ell_M(w),|||\cdot|||_{M,p})}$. Then for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ so that for every $k > N_\varepsilon$ there holds $1 + \varepsilon \geq \frac{1}{k}\left(1 + \left(\widetilde{M_w}(kx)\right)^p\right)^{1/p}$. Therefore the inequality $\left(\widetilde{M_w}\left(\frac{kx}{2}\right)\right)^p = \frac{1}{2^p}\left(\widetilde{M_w}(kx)\right)^p \leq \frac{k^p(1+\varepsilon)^p - 1}{2^p}$ holds for every $k > N_\varepsilon$. From continuity arguments it follows that there exists $\delta_\varepsilon > 0$ so that $d^{x,p} \leq 2 + \delta_\varepsilon$ and $\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$. By the arbitrary choice of $\varepsilon > 0$ it follows that and therefore $d^{x,p} \leq 2$.

From the definition of $|||\cdot|||_{M,p}$ we get that the inequality $\widetilde{M_w}(kx) \geq (k^p - 1)^{\frac{1}{p}}$ holds for every $k > 0$ and hence $d^{x,p} > 1$.

Combining these observations it follows that $d^p \in (1, 2]$, provided that $M \in \Delta_2$ and $w \in \Lambda$.

**Theorem 4.7.** *Let $M \in \Delta_2$ be an Orlicz function, $w \in \Lambda$ be a weight sequence and $1 \leq p < \infty$. Then there holds $K((\ell_M(w), |||\cdot|||_{M,p})) = d^p$ and $\Gamma_{(\ell_M(w),|||\cdot|||_{M,p})} = \frac{d^p}{2+d^p}$.*

*Proof.* First we will show that $K((\ell_M(w), |||\cdot|||_{M,p})) \geq d^p$. For any $\varepsilon > 0$ there exists $y \in S_{(\ell_M(w),|||\cdot|||_{M,p})}$, such that $d^{y,p} > d^p - \varepsilon$ and of course $d^{y,k,p} > d^p - \varepsilon$ holds for all $k > 1$.

It is easy to observe that by using a diagonal argument any sequence $w = \{w_i\}_{i=1}^\infty \in \Lambda$ can be split into countably many sequences $u^{(j)} = \{w_k\}_{k \in I^j}$, so that $u^{(j)} \in \Lambda$ for any $j \in \mathbb{N}$, $I^j \cap I^k = \emptyset$ for any $j \neq k$ and $\cup_{j=1}^\infty I^j = \mathbb{N}$. By Lemma 4.5 there exist disjoint subsets $\left\{J_n^j\right\}_{n=1}^\infty \subset I^j$ for $j \in \mathbb{N}$, such that $\sum_{k \in J_n^j} w_k = w_n$.

We will define a sequence $\{x^{(n)}\}$ by
$x^{(1)} = \sum_{s=1}^\infty y_s \sum_{k \in J_s^1} e_k,\ x^{(2)} = \sum_{s=1}^\infty y_s \sum_{k \in J_s^2} e_k,\ \ldots,\ x^{(n)} = \sum_{s=1}^\infty y_s \sum_{k \in J_s^n} e_k,\ \ldots$

From the construction of the sets $J_n^j$ it follows that $\widetilde{M_w}\left(x^{(n)}\right) = \widetilde{M_w}(y)$ and consequently $|||x^{(n)}|||_{M,p} = 1$ for all $n \in \mathbb{N}$. By the construction $\operatorname{supp}(x^{(i)}) \cap \operatorname{supp}(x^{(j)}) = \emptyset$ for any $i \neq j$. Let us consider the sequence $x = \left\{x^{(n)}\right\}_{n=1}^\infty$. Then

$$\begin{aligned} S_9 &= \frac{1}{k}\left(1 + \widetilde{M_w}^p\left(k\frac{x^{(n)} - x^{(m)}}{d^p - \varepsilon}\right)\right)^{1/p} = \frac{1}{k}\left(1 + 2^p\widetilde{M_w}^p\left(k\frac{x^{(n)}}{d^p - \varepsilon}\right)\right)^{1/p} \\ &> \frac{1}{k}\left(1 + 2^p\widetilde{M_w}^p\left(k\frac{x^{(n)}}{d^{y,k,p}}\right)\right)^{1/p} = \frac{(1 + k^p - 1)^{1/p}}{k} = 1. \end{aligned}$$

Consequently $|||x^{(n)} - x^{(m)}|||_{M,p} > d^p - \varepsilon$ and by the arbitrary choice of $\varepsilon > 0$ it follows that $K((\ell_M(w), |||\cdot|||_{M,p})) \geq d^p$.

For the proof that $K((\ell_M(w), |||\cdot|||_{M,p})) \leq d^p$ we start by choosing a sequence $\left\{x^{(n)}\right\}_n \in S_{(\ell_M(w),|||\cdot|||_{M,p})}$. For any $\varepsilon > 0$ there exists $k_n > 1$, such that $b_n < d^p + \varepsilon$, where $b_n$ satisfies $\widetilde{M_w}^p\left(\frac{k_n x^{(n)}}{b_n}\right) = \frac{k_n^p - 1}{2^p}$.

Case I) Let $k_n$ be an unbounded sequence. WLOG we may assume that $\lim_{n\to\infty} k_n = \infty$. Let us assume that $d^p + \varepsilon < 2$ then

$$\begin{aligned} S_{10} &= \frac{k_n^p - 1}{2^p} = \widetilde{M_w}^p\left(\frac{k_n x^{(n)}}{b_n}\right) \geq \widetilde{M_w}^p\left(\frac{k_n x^{(n)}}{d^p + \varepsilon}\right) > \left(\frac{2}{d^p + \varepsilon}\right)^p \widetilde{M_w}^p\left(\frac{k_n x^{(n)}}{2}\right) \\ &\geq \left(\frac{2}{d^p + \varepsilon}\right)^p \left(\left|\left|\left|\frac{k_n x^{(n)}}{2}\right|\right|\right|_{M,p}^p - 1\right) = \left(\frac{2}{d^p + \varepsilon}\right)^p \left(\left(\frac{k_n}{2}\right)^p - 1\right). \end{aligned}$$

Letting $n \to \infty$ we get $1 < \left(\frac{2}{d^p + \varepsilon}\right)^p \leq \lim_{n\to\infty} \frac{k_n^p - 1}{k_n^p - 2^p} = 1$, which is a contradiction.

Consequently $d^p + \varepsilon \geq 2$. Using the inequality $K(X) \leq 2$, which holds for any Banach space $X$ it follows that $K((\ell_M(w), |||\cdot|||_{M,p})) \leq 2 \leq d^p + \varepsilon$.

Case II) Let us assume that $\{k_n\}_n$ is a bounded sequence. WLOG we may assume that $\lim_{s\to\infty} k_s = k \geq 1$. Since $M \in \Delta_2$ there exists $c > 1$ such that the inequality

$$M\left(\frac{k}{d^p+\varepsilon}u\right) \leq cM(u) \tag{4.3}$$

holds for every $u \in [0,1]$. From Lemma 4.4 it follows that there exists $\delta \in (0,\varepsilon)$ so that the inequality

$$\widetilde{M_w}(x+y) \leq \widetilde{M_w}(x) + \varepsilon, \tag{4.4}$$

holds whenever $\widetilde{M_w}(x) \leq c$ and $\widetilde{M_w}(y) \leq \delta$. From Lemma 4.2 it follows that there is a subsequence $\left\{x^{(n_k)}\right\}_k$ of $\left\{x^{(n)}\right\}_n$, which just for simplicity of the notations we will denote again by $\left\{x^{(n)}\right\}_n$, so that

$$\sum_{i=p_k+1}^{\infty} w_i M\left(x_i^{(k)}\right) < \delta/c \text{ for every } k \in \mathbb{N}; \tag{4.5}$$

$$\sum_{i=1}^{p_{k-1}} w_i M\left(x_i^{(n)} - x_i^{(m)}\right) < \delta/c \text{ for every } 2 \leq k \leq n,\ m \geq k; \tag{4.6}$$

$$\sum_{i=p_{k-1}+1}^{p_k} w_i M\left(x_i^{(n)}\right) < \delta/c \text{ for every } k \geq 2, n \geq k. \tag{4.7}$$

There are $m, n \in \mathbb{N}$ such that there hold

$$|k_n - k| < \delta,\ |k_m - k| < \delta. \tag{4.8}$$

By (4.3) and (4.6) it follows that

$$\sum_{i=1}^{p_{n-1}} w_i M\left(\frac{k}{d^p+\varepsilon}\left(x_i^{(n)} - x_i^{(m)}\right)\right) < c\sum_{i=1}^{p_{n-1}} w_i M\left(x_i^{(n)} - x_i^{(m)}\right) < \delta < \varepsilon. \tag{4.9}$$

By (4.3) and (4.7) we have $\sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k}{d^p+\varepsilon}x_i^{(m)}\right) < c\sum_{i=p_{n-1}+1}^{p_n} w_i M\left(x_i^{(m)}\right) < \delta$. Thus from (4.4) we get

$$\sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k\left(x_i^{(n)} - x_i^{(m)}\right)}{d^p+\varepsilon}\right) < \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{kx_i^{(n)}}{d^p+\varepsilon}\right) + \varepsilon. \tag{4.10}$$

From (4.8) we obtain the chain of inequalities

$$\sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{(k-k_n)x_i^{(n)}}{d^p+\varepsilon}\right) \leq |k_n - k| \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{x_i^{(n)}}{d^p+\varepsilon}\right) \leq \delta\widetilde{M_w}(x^{(n)}) \leq \delta$$

and consequently using (4.4) we get

$$\sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k}{d^p+\varepsilon}x_i^{(n)}\right) \leq \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k_n}{d^p+\varepsilon}x_i^{(n)}\right) + \varepsilon.$$

Combining the last inequality with (4.10) and $b_n < d^p + \varepsilon$ we obtain

$$\begin{aligned} S_{11} &= \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k}{d^p+\varepsilon}\left(x_i^{(n)} - x_i^{(m)}\right)\right) \\ &\leq \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k_n}{d^p+\varepsilon}x_i^{(n)}\right) + 2\varepsilon \leq \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k_n}{b_n}x_i^{(n)}\right) + 2\varepsilon \\ &\leq \frac{k_n-1}{2} + 2\varepsilon = \frac{k-1}{2} + \frac{k_n-k}{2} + 2\varepsilon \leq \frac{k-1}{2} + \frac{5\varepsilon}{2}. \end{aligned} \tag{4.11}$$

Finally by (4.3), (4.7) and (4.4) we get

$$(4.12)\qquad \sum_{i=p_n+1}^{\infty} w_i M\left(\frac{k}{d^p+\varepsilon}\left(x_i^{(n)}-x_i^{(m)}\right)\right) \le \sum_{i=p_n+1}^{\infty} w_i M\left(\frac{k}{d^p+\varepsilon}x_i^{(m)}\right)+\varepsilon.$$

Thus we have

$$(4.13)\qquad \textstyle\sum_{i=p_n+1}^{\infty} w_i M\left(\frac{k}{d^p+\varepsilon}\left(x_i^{(n)}-x_i^{(m)}\right)\right) \le \frac{k-1}{2}+\frac{5\varepsilon}{2}.$$

From (4.9), (4.11) and (4.12) we obtain

$$\begin{aligned} S_{12} &= \left|\left|\left|\tfrac{x^{(n)}-x^{(m)}}{d^p+\varepsilon}\right|\right|\right|_{M,p} \le \tfrac{1}{k}\left(1+\widetilde{M_w}^p\left(\tfrac{k}{d^p+\varepsilon}\left(x^{(n)}-x^{(m)}\right)\right)\right)^{1/p} \\ &= \tfrac{1}{k}\left(1+[W_1+W_2+W_3]^p\right)^{1/p} \le \frac{\left(1+\left(\frac{(k^p-1)^{1/p}}{2}+\frac{5\varepsilon}{2}+\frac{(k^p-1)^{1/p}}{2}+\frac{5\varepsilon}{2}+\varepsilon\right)^p\right)^{1/p}}{k} \\ &= \frac{\left(1+\left((k^p-1)^{1/p}+6\varepsilon\right)^p\right)^{1/p}}{k}, \end{aligned}$$

where $W_1 = \sum_{i=1}^{p_{n-1}} w_i M\left(\frac{k\left(x_i^{(n)}-x_i^{(m)}\right)}{d^p+\varepsilon}\right)$, $W_2 = \sum_{i=p_{n-1}+1}^{p_n} w_i M\left(\frac{k\left(x_i^{(n)}-x_i^{(m)}\right)}{d^p+\varepsilon}\right)$ and $W_3 = \sum_{i=p_n+1}^{\infty} w_i M\left(\frac{k\left(x_i^{(n)}-x_i^{(m)}\right)}{d^p+\varepsilon}\right)$.

We will prove that there exists $\varepsilon_0 > 0$ such that the inequality

$$\left(1+\left((k^p-1)^{1/p}+6\varepsilon\right)^p\right)^{1/p} < k+12\varepsilon$$

holds for any $k,p \in [1,\infty)$ and every $\varepsilon \in (0,\varepsilon_0)$. Let us consider the function $F(\varepsilon) = \left(1+\left((k^p-1)^{1/p}+6\varepsilon\right)^p\right)^{1/p} - k - 12\varepsilon$. Then $F(0) = \left(1+\left((k^p-1)^{1/p}\right)^p\right)^{1/p} - k = 0$. After a differentiation we get $F'(\varepsilon) = 6\left(1+\left((k^p-1)^{1/p}+6\varepsilon\right)^p\right)^{\frac{1-p}{p}}\left((k^p-1)^{1/p}+6\varepsilon\right)^{p-1} - 12$. From the inequality

$$F'(0) = 6\left(1+\left((k^p-1)^{1/p}\right)^p\right)^{\frac{1-p}{p}}\left((k^p-1)^{1/p}\right)^{p-1} - 12 = 6\left(1-\frac{1}{k^p}\right)^{\frac{p-1}{p}} - 12 < 0$$

and the continuity of the function it follows that existence of $\varepsilon_0 > 0$ such that $F'(\varepsilon) < 0$ for every $k,p \in [1,\infty)$ and every $\varepsilon \in (0,\varepsilon_0)$. Therefore $F$ is a decreasing function in the interval $(0,\varepsilon_0)$ and consequently $F(\varepsilon) \le F(\varepsilon_0) = 0$ for every $k,p \in [1,\infty)$ and every $\varepsilon \in (0,\varepsilon_0)$.

Thus we get $S_{15} \le 1 + \frac{12\varepsilon}{k}$ whenever we have chosen $\varepsilon \in (0,\varepsilon_0)$. Consequently there holds the inequality $\inf\{|||x^{(n)}-x^{(m)}|||_{M,p} : n \ne m\} \le (1+12\varepsilon)(d^p+\varepsilon)$. Since $\{x^{(n)}\}_{n=1}^{\infty} \subset S_{(\ell_M(w),|||\cdot|||_{M,p})}$ and $\varepsilon \in (0,\varepsilon_0)$ be arbitrary it follows that $K((\ell_M(w),|||\cdot|||_{M,p})) \le d^p$.

The equality $\Gamma_{(\ell_M(w),|||\cdot|||_{M,p})} = \frac{d^p}{2+d^p}$ follows from (2.1). □

We say that an MO function $\Phi$ satisfies the uniform $\Delta_2$–condition at zero if there exists a constant $K < \infty$ and an integer $N \in \mathbb{N}$ and a real $t_0 > 0$ such that the inequality $\frac{\Phi_n(2t)}{\Phi_n(t)} \le K$ holds for every $n \ge N$ and every $t \in (0,t_0]$. We say that an MO function $\Phi$ satisfies the uniform $\Delta_2^*$–condition at zero if there exists a constant $k > 0$ and an integer $N \in \mathbb{N}$ and a real $t_0 > 0$ such that the inequality $\frac{\Phi_n(2t)}{\Phi_n(t)} \ge k$ holds for every $n \ge N$ and every $t \in (0,t_0]$. Recall that given MO functions $\Phi$ and $\Psi$ the spaces $\ell_\Phi$ and $\ell_\Psi$ coincide with equivalence of norms [34] if and only if $\Phi$ is equivalent to $\Psi$, that is there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \text{ and } \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$. According to [47] $\ell_\Phi = h_\Phi$ if and only if $\Phi$ is equivalent to a function $\Psi$, which satisfies the uniform $\Delta_2$ condition. Let us recall [32] that an MO sequence space $\ell_\Phi$ is reflexive if and only if the MO function $\Phi$ is equivalent to an MO function $\Psi$, for which uniform $\Delta_2$ and $\Delta_2^*$–conditions hold.

**Theorem 4.8.** *Let $M \notin \Delta_2$ be an Orlicz function, $w \in \Lambda$ be a weight sequence and $1 \leq p < \infty$. Then $\Gamma_{(\ell_M(w), |||\cdot|||_{M,p})} = \frac{1}{2}$ and $K((\ell_M(w), |||\cdot|||_{M,p})) = 2$.*

*Proof.* If $M \notin \Delta_2$, then $\ell_M(w) \neq h_M(w)$. If we suppose that $\ell_M(w)$ is a reflexive space then there exists a MO function $\Psi$, which is equivalent to $\{w_i M\}_{i=1}^{\infty}$ and $\Psi$ has the uniform $\Delta_2$ and $\Delta_2^*$–conditions. Consequently form $\Psi$ satisfying the uniform $\Delta_2$ condition it follows that $\ell_\Psi = h_\Psi$ and therefore $\ell_M(w) = h_M(w)$, which is a contradiction. Thus $\ell_M(w)$ is not a reflexive Banach space and according to [17] we get the equalities $\Gamma_{(\ell_M(w), |||\cdot|||_{M,p})} = \frac{1}{2}$ and $K(\ell_M(w), |||\cdot|||_{M,p}) = 2$. □

## 5. REISZ ANGLE IN KÖTHE SEQUENCE SPACES

Let us mention that the Orlicz sequence spaces $\ell_M$, $\ell_M(w)$, $w \in \Lambda$, endowed with the Luxemburg norm, provided that $M \in \Delta_2$ are order continuous Köthe sequence spaces with the Fatou property and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ is unconditional and boundedly complete [32]. From the equivalence of the Luxemburg and $p$–Amemiya norms it follows that $\ell_M$, $\ell_M(w)$, $w \in \Lambda$, endowed with $p$–Amemiya norms, provided that $M \in \Delta_2$ are order continuous Köthe sequence spaces and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ is unconditional and boundedly complete.

If $M$ does not satisfy the $\Delta_2$–condition at zero, then the Orlicz sequence spaces $\ell_M$, $\ell_M(w)$, $w \in \Lambda$, endowed with the Luxemburg or $p$–Amemiya norm are not order continuous Köthe sequence spaces and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ is not boundedly complete. Therefore we could not apply Theorem 3.2 for calculating of the Reisz angle in this case.

**Definition 5.10.** ([33], p.1) A partially ordered Banach space $X$ over the reals is called a Banach lattice provided

(i) $x \leq x$ implies $x + z \leq y + z$ for every $x, y, x \in X$
(ii) $ax \geq 0$ for every $x \geq 0$ in $X$ and every nonnegative real $a$
(iii) for all $x, y \in X$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$
(iv) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

The sequence spaces $\ell_M$, $\ell_M(w)$, endowed with Luxemburg or $p$–Amemiya norms are Banach lattices [32].

We will need the next lemma, which is similar to that proven in [17].

Let us first recall that we say that the Banach space $X$ containes an almost isometric copy of $Y$ if for every $\varepsilon > 0$ there exists a linear operator $P : Y \to X$ such that the inequality $\|y\|_Y \leq \|Py\|_X \leq (1+\varepsilon)\|y\|_Y$ holds for every $y \in Y$.

**Lemma 5.7.** *Let $X$ and $Y$ be two Köthe sequence spaces and let $X$ contains an almost isometric copy of $Y$. Then $\alpha(X) \geq \alpha(Y)$.*

*Proof.* Let $\varepsilon > 0$ be arbitrary chosen. There exist $x, y \in S_Y$ such that $\alpha(Y) - \varepsilon < \|(|x| \vee |y|)\|$ and $|x| \wedge |y| = 0$. Let $P : Y \to X$ be a linear operator that satisfies the inequality $\|y\|_Y \leq$

$\|Py\|_X \le (1+\varepsilon)\|y\|_Y$. Define $x = \frac{Px}{\|Px\|_X}$ and $y = \frac{Py}{\|Py\|_X}$. We will need the fact, that if $|x| \wedge |y| = 0$, then $|Px| \wedge |Py| = 0$. Indeed if we assume that $|Px| \wedge |Py| \neq 0$, then there exist $\alpha_i, \beta_i$, $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} \alpha_i^2 > 0$, $\sum_{i=1}^{\infty} \beta_i^2 > 0$ such that $P\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) + P\left(\sum_{i=1}^{\infty} \beta_i y_i\right) = 0$. Consequently there holds the inequality

$$\begin{aligned} 0 &= \|P\left(\textstyle\sum_{i=1}^{\infty} \alpha_i x_i + \beta_i y_i)\right)\|_X \ge \|\textstyle\sum_{i=1}^{\infty}(\alpha_i x_i + \beta_i y_i)\|_Y \\ &= \|(|\textstyle\sum_{i=1}^{\infty} \alpha_i x_i| \vee |\textstyle\sum_{i=1}^{\infty} \beta_i y_i|)\|_X > 0, \end{aligned}$$

which is a contradiction and thus $|Px| \wedge |Py| = 0$.

Using the inequality $\left|\frac{1}{\|Px\|_X} - \frac{1}{\|Py\|_X}\right| \le 1 - \frac{1}{1+\varepsilon} < \varepsilon$ we get the chain of inequalities

$$\begin{aligned} \tfrac{1}{1+\varepsilon}\|x-y\|_Y - \varepsilon &\le \left\|\tfrac{x-y}{\|P(x)\|_X}\right\|_Y - \left|\tfrac{1}{\|Px\|_X} - \tfrac{1}{\|Py\|_X}\right| \\ &\le \left\|\tfrac{x}{\|Px\|_X} - \tfrac{y}{\|Py\|_X}\right\|_Y \le \left\|P\left(\tfrac{x}{\|Px\|_X}\right) - P\left(\tfrac{y}{\|Py\|_X}\right)\right\|_X. \end{aligned}$$

Therefore for any two $x, y \in S_Y$ such that $\alpha(Y) - \varepsilon < \|(|x| \vee |y|)\|$ and $|x| \wedge |y| = 0$ there exists $u = \frac{Px}{\|Px\|_X} \in S_X$ and $v = \frac{Py}{\|Py\|_X} \in S_X$, such that $|u| \wedge |v| = 0$ and $\|(|u| \vee |v|)\| > \frac{1}{1+\varepsilon}\|(|x| \vee |y|)\| - \varepsilon \ge \frac{\alpha(Y)-\varepsilon}{1+\varepsilon} - \varepsilon$. From the arbitrary choice of $\varepsilon > 0$ it follows that $\alpha(X) \ge \alpha(Y)$. □

**Theorem 5.9.** *Any nonreflexive Banach lattice $X$ has a Riesz angle equal to 2.*

*Proof.* By assumption $X$ contains an isomorphic copy of $c_0$ or $\ell_1$ [14]. According to [19] $X$ contains an almost isometric copy of $c_0$ or $\ell_1$, respectively. From Theorem 5.7 it follows that $\alpha(X) \ge \alpha(c_0) = 2$ or $\alpha(X) \ge \alpha(\ell_1) = 2$ [2]. □

The MO sequence spaces, equipped with a $p$–Amemiya norm, $p \in [1, +\infty]$ are Banach lattices.

From Theorem 5.7 and the fact that if an Orlicz function $M \notin \Delta_2$ or a MO function $\Phi \notin \delta_2$, then $\ell_M$, $\ell_M(w)$ for $w \in \Lambda$ and $\ell_\Phi$ are not reflexive spaces for any $p$–Amemiya norm, $p \in [1, +\infty]$ we get the next results.

**Theorem 5.10.** *Let $M \notin \Delta_2$ be an Orlicz function and $w \in \Lambda$ be a weight sequence. Then $a((\ell_M(w), |||\cdot|||_{M,p}) = 2$.*

**Theorem 5.11.** *Let $\Phi \notin \delta_2$ be an MO function. Then $a((\ell_\Phi, |||\cdot|||_{M,p}) = 2$.*

Thus from the equality $\|\cdot\|_\Phi = |||\cdot|||_{\Phi,\infty}$ [8] it follows that $a((\ell_M, \|\cdot\|) = 2$, $a((\ell_M, |||\cdot|||) = 2$, $a((\ell_\Phi, \|\cdot\|) = 2$, $a((\ell_\Phi, |||\cdot|||) = 2$.

As corollaries of Theorem 3.2 and [6] we get the results from [49].

**Theorem 5.12.** ([49]) *Let $M \in \Delta_2$ be an Orlicz function. Then the Riesz angle of $(\ell_M, \|\cdot\|)$ can be expressed as:*

$$a((\ell_M, \|\cdot\|)) = \sup\left\{k_x : \widetilde{M}\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_{\ell_M}\right\}.$$

**Theorem 5.13.** ([49]) *Let $M \in \Delta_2$ be an Orlicz function. Then the Riesz angle of $(\ell_M, |||\cdot|||)$ can be expressed as:*

$$a((\ell_M, |||\cdot|||)) = \sup_{|||x|||=1} \inf_{k>1}\left\{d_{x,k} : \widetilde{M}\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}\right\}.$$

As corollaries of Theorem 3.2 and [16] we get:

**Theorem 5.14.** *Let $M \in \Delta_2$ be an Orlicz function and $1 \leq p < \infty$. Then the Riesz angle of $(\ell_M, |||\cdot|||_{M,p})$ can be expressed as:*

$$a((\ell_M, |||\cdot|||_{M,p})) = \sup_{|||x|||_{M,p}=1} \inf_{k>1} \left\{ d_{x,k} : \widetilde{M}^p \left( \frac{kx}{d_{x,k}} \right) = \frac{k^p - 1}{2^p} \right\}.$$

As corollaries of Theorem 3.2 and Theorem 4.4 we get the results from [53].

**Theorem 5.15.** ([53]) *Let $M \in \Delta_2$ be an Orlicz function and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ be a weight sequence. Then the Riesz angle of $(\ell_M(w), \|\cdot\|)$ can be expressed as:*

$$a((\ell_M(w), \|\cdot\|)) = \sup \left\{ k_x : \widetilde{M}_w \left( \frac{x}{k_x} \right) = \frac{1}{2}, x \in S_{\ell_M(w)} \right\}.$$

As a corollary of Theorem 3.2 and Theorem 4.7 we get an expression of the Reisz angle in $(\ell_M(w), |||\cdot|||_{M,p})$.

**Theorem 5.16.** *Let $M \in \Delta_2$ be an Orlicz function, $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ be a weight sequence and $1 \leq p < \infty$. Then the Riesz angle of $(\ell_M(w), |||\cdot|||_{M,p})$ can be expressed as:*

$$a((\ell_M(w), |||\cdot|||)) = \sup_{|||x|||=1} \inf_{k>1} \left\{ d_{x,k} : \widetilde{M_w}^p \left( \frac{kx}{d_{x,k}} \right) = \frac{k^p - 1}{2^p} \right\}.$$

As a corollary of Theorem 5.16 for $p = 1$ we improve the result from [53].

As a corollary of Theorem 3.2 and [18] we get an expression of the Reisz angle in $\ell_\Phi$.

**Theorem 5.17.** *Let $\Phi$ be an MO function with the $\delta_2$–condition at zero and condition $(+)$. Then the Riesz angle of $(\ell_\Phi, \|\cdot\|)$ can be expressed as:*

$$a((\ell_\Phi, \|\cdot\|)) = d_\Phi.$$

Following [1] let $1 \leq p_i$, $i \in \mathbb{N}$ be a sequence of reals. The MO sequence space $\ell_\Phi$, where $\Phi = \{t^{p_i}\}_{i=1}^{\infty}$ is called a Nakano sequence space and is denoted by $\ell_{\{p_i\}}$.

As a corollary of Theorem 3.2 and [18] we get an expression of the Reisz angle in $\ell_{\{p_i\}}$, when $\limsup_{i\to\infty} p_i < +\infty$. As a corollary of Theorem 5.10 we get an expression of the Reisz angle in $\ell_{\{p_i\}}$, when $\limsup_{i\to\infty} p_i = +\infty$.

**Theorem 5.18.** *Let $\ell_{\{p_i\}}$ be a Nakano sequence space, where $1 \leq p_i$, $i \in \mathbb{N}$. Then the Riesz angle of $(\ell_{\{p_i\}}, \|\cdot\|)$ can be expressed as:*

$$a((\ell_{\{p_i\}}, \|\cdot\|)) = 2 \text{ when } \limsup_{i\to\infty} p_i = +\infty$$

*and*

$$a((\ell_{\{p_i\}}, \|\cdot\|)) = 2^{\frac{1}{p}} \text{ when } \limsup_{i\to\infty} p_i = p < +\infty.$$

The concept of Orlicz-Lorentz space was first introduced by A.Kaminska in [25]. Let $M$ be an Orlicz function and $w = \{w_n\}_{n=1}^{\infty}$ be a non–increasing sequence of positive scalars so that $\lim_{n\to\infty} w_n = 0$. We denote by $d(w, M)$ the Orlicz–Lorentz sequence space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ for which

$$\|x\| = \sup \left\{ \sum_{n=1}^{\infty} w_n M\left(x_{\pi(n)}\right) \right\} < \infty, \tag{5.14}$$

where the supremum is taken over all permutations $\pi$ of the $\mathbb{N}$ [50]. From (5.14) we deduce that there exists a sequence rearrangement of the natural numbers $\{\pi(n)\}_{n=1}^{\infty}$ such that $\|x\| = \left\{\sum_{n=1}^{\infty} w_n |x_{\pi(n)}|^p\right\}^{1/p}$. The space $d(w, M)$ is a Banach space and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ is a boundedly complete unconditional basis (see [32] p. 175).

If $\sum_{n=1}^{\infty} w_n < \infty$ [25, 50] or $M \notin \Delta_2$ and $\sum_{n=1}^{\infty} w_n = \infty$ [30, 50] then $d(w, M)$ is not a reflexive space. If $M(t) = |t|^p$ we get the Lorentz sequence space $d(w, p)$.

As a corollary of Theorem 3.2 and [50] we get the results.

**Theorem 5.19.** *Let $d(w, p)$ be a Lorentz sequence space. Then $a(d(w, p)) = 2^{\frac{1}{p}}$.*

**Theorem 5.20.** *Let $d(w, M)$ be an Orlicz–Lorentz sequence space, such that $M \in \Delta_2$, $\sum_{i=1}^{\infty} w_i = \infty$ and the function $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is increasing on the interval $(0, M^{-1}(1/2w_1)]$. Then $a(d(w, M)) = \lim_{u \to 0} G_M(u)$.*

From Theorem 5.7 we get the next result.

**Theorem 5.21.** *Let $M$ be an Orlicz function. If $\sum_{n=1}^{\infty} w_n < \infty$ or $M \notin \Delta_2$ and $\sum_{n=1}^{\infty} w_n = \infty$. Then $\alpha(d(w, M)) = 2$.*

Let $p \in [1, +\infty)$. Following [10] we denote by $ces_p$ the Cesaro sequence space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ for which

$$ces_p = \left\{ x \in \ell^0 : \left( \sum_{n=1}^{\infty} \left( \frac{\sum_{i=1}^{n} |x_i|}{n} \right)^p \right)^{1/p} < \infty \right\}.$$

The Cesaro sequence space is a Banach space if endowed with the norm

$$\|x\| = \left( \sum_{n=1}^{\infty} \left( \frac{\sum_{i=1}^{n} |x_i|}{n} \right)^p \right)^{1/p}$$

and it is an order continuous Köthe sequence space with the Fatou property [10].

As a corollary of Theorem 3.2 and [10] we get the result.

**Theorem 5.22.** *Let $ces_p$ be a Cesaro sequence space. Then $\alpha(d(w, p)) = 2^{\frac{1}{p}}$.*

## 6. Conclusion

There are many articles where upper or lower estimates are found for the value of Kottman's constant. As a corollary of Theorem 3.2 and these results we can get estimates of the Reisz angle.

Following [13] let us define a modulus of asymptotic uniform convexity

$$\overline{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\substack{Z \subset X \\ \mathrm{co}-\dim Z < \infty}} \inf_{\substack{z \in Z \\ \|x\| \geq t}} \{\|x + z\| - 1\}$$

and a modulus of asymptotic uniform smoothness

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\substack{Z \subset X \\ \mathrm{co}-\dim Z < \infty}} \sup_{\substack{z \in Z \\ \|x\| \geq t}} \{\|x + z\| - 1\}.$$

As a corollary of Theorem 3.2 and [13] we get the result.

**Theorem 6.23.** *Let $M \in \Delta_2$ be an Orlicz function. Then $1 + \overline{\delta}_{(\ell_M, \|\cdot\|)}(1) \leq a((\ell_M, \|\cdot\|)) \leq 1 + \overline{\rho}_{(\ell_M, \|\cdot\|)}(1)$.*

Following [7] for $\varepsilon \in [0, 2]$ we call a modulus of convexity of $X$ the function

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \varepsilon \right\}.$$

A space is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$. We call [31] a modulus of smoothness of $X$ the function

$$\rho_X(\tau) = \inf\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 : x \in S_X, \|y\| = \tau\right\}.$$

For some properties of the two moduli just defined we refer to [33]. A space is called uniformly smooth if $\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} = 0$ for all $\varepsilon > 0$. Some generalizations of the moduli of convexity and smoothness can be found in [5, 52].

As a corollary of Theorem 3.2 and [35] we get the results.

**Theorem 6.24.** *Let $(X, \|\cdot\|)$ be a uniformly convex Köthe sequence space. Let there hold one of the following*

*(a) $X$ is order continuous with the Fatou property;*

*(b) the unit vector basis $\{e_n\}_{n=1}^\infty$ of $X$ is unconditional and boundedly complete.*

*Then there hold the inequalities*

$$\max\left\{1 + \frac{1}{2}\delta_X(2/3), \frac{1}{1-\delta_X(1)}, \frac{1}{1-\delta_X(\sqrt{2})}, \frac{1}{1-\delta_X\left(\frac{2}{\alpha(X)}\right)}\right\} \le a(X)$$

*and*

$$a(X) \le \min\{2 - 2\delta_X(1), 1 + 2\rho_X(1)\}.$$

Let us denote $\alpha_M^0 = \liminf_{u\to 0}\frac{M^{-1}(u)}{M^{-1}(2u)}$, $\beta_M^0 = \limsup_{u\to 0}\frac{M^{-1}(u)}{M^{-1}(2u)}$, where $M$ is an Orlicz function. Let $w$ be a weight sequences. Let us denote $\alpha'_{M,w} = \inf_{k\ge 1}\left\{\frac{M^{-1}\left(\frac{1}{\sum_{i=1}^{2k} w_i}\right)}{M^{-1}\left(\frac{1}{\sum_{i=1}^{k} w_i}\right)}\right\}$, $\widetilde{\alpha}_{M,w} = \inf_{u\in(0,1/2w_1]}\frac{M^{-1}(u)}{M^{-1}(2u)}$.

As a corollary of Theorem 3.2 and [50] we get the result.

**Theorem 6.25.** *Let $d(w, M)$ be an Orlicz–Lorentz sequence space, such that $M \in \Delta_2$ and $\sum_{i=1}^\infty w_i = \infty$. Then $\max\left\{\frac{1}{\alpha_M^0}, \frac{1}{\alpha'_{M,w}}\right\} \le a(d(w, M)) \le \frac{1}{\widetilde{\alpha}_{M,w}}$.*

As a corollary of Theorem 3.2 and [42] we get the result.

**Theorem 6.26.** *Let $\ell_M$ be an Orlicz sequence space, such that $M \in \Delta_2$ and $N \in \Delta_2$ be its complementary function. Then*

$$\max\left\{\frac{1}{\alpha_M^0}, \frac{1}{\alpha'_M}\right\} \le a((\ell_M, \|\cdot\|)) \le \frac{1}{\widetilde{\alpha}_M}$$

*and*

$$\max\{2\beta^0_{N^*}, 2\beta'_N\} \le a((\ell_M, |||\cdot|||)) \le \frac{1}{\alpha^*_M},$$

*where* $\widetilde{\alpha}_M = \inf\left\{\frac{M^{-1}(u)}{M^{-1}(2u)} : 0 < u \le \frac{1}{2}\right\}$, $\alpha^*_M = \inf\left\{\frac{M^{-1}(u)}{M^{-1}(2u)} : 0 < u \le \frac{1}{2}(k_1^{**} - 1)\right\}$ *and* $\alpha'_M = \inf\left\{\frac{M^{-1}\left(\frac{1}{2k}\right)}{M^{-1}\left(\frac{1}{k}\right)} : k \in \mathbb{N}\right\}$.

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