CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
35 (2019), No. 1, 23 - 30 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Ulam stability with respect to a directed graph for some fixed point equations

APIMUK BUAKIRD and SATIT SAEJUNG

ABSTRACT. In this paper, we introduce a new concept of Ulam stability of fixed point equation with respected to a directed graph. Two fixed point theorems of Matkowski and of Jachymski are discussed further in the sense of this stability concept. Some examples about the validity of our notion are given. Finally, we discuss the vagueness of the recent stability results of Sintunavarat [Sintunavarat, W., *A new approach to α-ψ-contractive mappings and generalized Ulam–Hyers stability, well-posedness and limit shadowing results*, Carpathian J. Math., **31** (2015), 395–401].

1. INTRODUCTION

Ulam posed the following interesting question in 1940 (see also [19]).

> Suppose that $G_1 := (G_1, *)$ and $G_2 := (G_2, \diamond)$ are two groups and $d : G_2 \times G_2 \to [0, \infty)$ is a metric. For a given $\varepsilon > 0$ does there exist a number $\delta := \delta(\varepsilon) > 0$ such that if $f : G_1 \to G_2$ satisfies
> $$d(f(x * y), f(x) \diamond f(y)) \leq \delta \text{ for all } x, y \in G_1,$$
> then one can find a homomorphism $F : G_1 \to G_2$ such that $d(f(x), F(x)) \leq \varepsilon$ for all $x \in G_1$?

Hyers [6] gave a partial answer to Ulam's question in 1941 where G_1 and G_2 are Banach spaces. In this setting, he also obtained that $\delta(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$.

There are strict connections between Ulam stability and fixed point theory and for further information we refer to the survey by Brzdęk et al. (see [2]). Ulam's question was reformulated in the context of fixed point equation as follows. For more detail, we refer to the excellent survey by Rus and Şerban [14].

> Suppose that $X := (X, d)$ is a metric space and $T : X \to X$ is given with a fixed point set $\mathrm{Fix}(T) := \{p \in X : p = Tp\}$. For a given $\varepsilon > 0$ does there exist a number $\delta := \delta(\varepsilon) > 0$ such that if $w \in X$ satisfies
> $$d(w, Tw) \leq \delta,$$
> then one can find a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \varepsilon$?

If the preceding is true for the mapping T, then we say that the fixed point equation $x = Tx$ is *Ulam stable*. For simplicity from now on, we simply say that T is *Ulam stable* if the fixed point equation $x = Tx$ is Ulam stable. If there exists a constant $c > 0$ such that $\delta(\varepsilon) \leq c\varepsilon$ for all $\varepsilon > 0$, then we say that T is *Ulam–Hyers stable*. That is, T is Ulam–Hyers stable if and only if there exists $c > 0$ such that for any pair $(w, \varepsilon) \in X \times (0, \infty)$ with $d(w, Tw) \leq \varepsilon$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq c\varepsilon$. In the literature, the following generalization of Ulam–Hyers stability was introduced.

Received: 13.07.2018. In revised form: 10.01.2019. Accepted: 15.01.2019

2010 *Mathematics Subject Classification.* 47H10, 54H25, 39B82.

Key words and phrases. *Ulam–Hyers stability, Ulam stability, fixed point equation, contraction with respect to a directed graph.*

Corresponding author: Satit Saejung; saejung@kku.ac.th

Definition 1.1. Let $X := (X, d)$ be a metric space and $T : X \to X$ be given. We say that T is *generalized Ulam–Hyers stable* [14] if there exists an increasing function $\xi : [0, \infty) \to [0, \infty)$ such that $\xi(0) = 0$, ξ is continuous at zero, and for any pair $(w, \varepsilon) \in X \times (0, \infty)$ with $d(w, Tw) \leq \varepsilon$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\varepsilon)$.

Remark 1.1. If T is generalized Ulam–Hyers stable, then it is Ulam stable. To see this, let $\varepsilon > 0$ be given. Since T is generalized Ulam–Hyers stable, there exists an increasing function $\xi : [0, \infty) \to [0, \infty)$ such that $\xi(0) = 0$, ξ is continuous at zero, and for any pair $(w, \eta) \in X \times (0, \infty)$ with $d(w, Tw) \leq \eta$ there exists a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\eta)$. We choose $\eta := \xi^{-1}(\varepsilon) > 0$. It follows that if $w \in X$ satisfies

$$d(w, Tw) \leq \eta,$$

then one can find a fixed point $p \in \mathrm{Fix}(T)$ such that $d(p, w) \leq \xi(\eta) = \varepsilon$. In particular, we have the following implications.

Ulam–Hyers stability	$\Longrightarrow$	generalized Ulam–Hyers stability	$\Longrightarrow$	Ulam stability

There are several other types of stability, for more detail we refer to [10] and [4]. It is easy to see that every Banach contraction defined on a complete metric space is Ulam–Hyers stable. Recall that a mapping $T : X \to X$ is a *Banach's contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

In the literature, there are many generalizations of a Banach's contraction. We are mainly interested in the following two types of generalizations due to Matkowski [9] and to Jachymski [7], respectively.

Matkowski's contractions.

Definition 1.2. Let $X := (X, d)$ be a metric space and $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. A mapping $T : X \to X$ is a *Matkowski's contraction* or *ψ-contraction* if

$$d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in X.$$

Remark 1.2. *If $\psi(t) = \alpha t$ where $\alpha \in (0, 1)$, then a ψ-contraction becomes a Banach's contraction.*

Jachymski's contractions. Recently, Jachymski introduced a class of mappings including all Banach's contractions and proved a fixed point theorem for mappings in this class. From now on, by saying that X is a metric space with a directed graph G on X, we mean that the vertex set $V(G)$ of G is X and the edge set $E(G)$ of G is a subset of the Cartesian product $X \times X$ such that $(x, x) \in E(G)$ for all $x \in X$.

Definition 1.3. Let $X := (X, d)$ be a metric space with a directed graph G on X. A mapping $T : X \to X$ is a *Banach G-contraction* if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ the following two conditions hold:

- $(Tx, Ty) \in E(G)$;
- $d(Tx, Ty) \leq \alpha d(x, y)$.

Remark 1.3. It is clear that if $E(G) = X \times X$, then a Banach G-contraction becomes a Banach's contraction.

Remark 1.4. We note that not every Banach G-contraction is Ulam stable. See Example 2.1.

In this paper, we introduce a new type of Ulam stability to explain the stability of Banach G-contractions. In fact, our result includes a wider class of mappings whose contractiveness in the sense of Matkowski is given with respect to a directed graph. We also discuss some vague result proved by Sintunavarat [17] concerning the generalized Ulam–Hyers stability. As pointed out by the referee, the results of the paper are related to some simplified versions of outcomes in several other papers (which can be found in the references of the survey [2]) such as Corollary 3.2 in [3].

2. MAIN RESULTS

In this paper we introduce the following concept.

Definition 2.4. Let (X, d) be a metric space with a directed graph G on X and $T : X \to X$ be a mapping such that $\mathrm{Fix}(T) \neq \varnothing$. We say that a mapping $T : X \to X$ is *Ulam stable with respect to G* if for each $\varepsilon > 0$ there is a $\delta := \delta(\varepsilon) > 0$ such the following implication holds:

$$d(w, Tw) \leq \delta \text{ and } (w, Tw) \in E(G) \implies \text{there exists } p \in \mathrm{Fix}(T) \text{ such that } d(p, w) \leq \varepsilon.$$

In the preceding notion, if there exists a constant $c > 0$ such that $\delta(\varepsilon) \leq c\varepsilon$ for all $\varepsilon > 0$, then we say that T is *Ulam–Hyers stable with respect to G*.

Remark 2.5. In particular, if $E(G) := X \times X$, then the Ulam stability with respect to G (Ulam–Hyers stability with respect to G, respectively) becomes the Ulam stability (Ulam–Hyers stability, respectively).

Inspired by the works of Matkowski [9] and of Jachymski [7], we introduce the following mappings.

Definition 2.5. Let (X, d) be a metric space with a directed graph G on X. Define $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function. We say that $T : X \to X$ is

(i) a (ψ, G)-*contraction of type I* if the following conditions hold
- $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;
- $(Tx, T^2x) \in E(G)$ whenever $(x, Tx) \in E(G)$;
- $d(Tx, Ty) \leq \psi(d(x, y))$ whenever $(x, y) \in E(G)$;

(ii) a (ψ, G)-*contraction of type II* if the following conditions hold
- $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;
- $(Tx, T^2x) \in E(G)$ whenever $(x, Tx) \in E(G)$;
- $d(Tx, Ty) \leq \psi(d(x, y))$ whenever $(x, y) \in E(G)$.

Our stability results rely on the following two additional assumptions (see [7]).

(J1) If $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$.

(J2) For each $x, y \in X$ if $T^{n_k}x \to y$ and $(T^{n_k}x, T^{n_k+1}x) \in E(G)$ for all $k \geq 1$, then $T(T^{n_k}x) \to Ty$.

Remark 2.6.
- Since we will mention some fixed point theorems proved under a bit stronger assumption than the condition (J1), we refer to this assumption as (J1*). More precisely, it is defined as follows.
  (J1*) If $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$, then $(x_n, p) \in E(G)$ for all $n \geq 1$.
- The condition (J2) is sometimes referred as the *orbital G-continuity of T* [7].

2.1. (ψ, G)-contractions of type I.

2.1.1. *Fixed point theorem.*

Theorem 2.1. *Let (X,d) be a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type I and suppose that either the condition (J1) or (J2) holds. If there exists an element $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then $T^n x_0 \to p$ for some $p \in \mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) \neq \varnothing$.*

Proof. Set $x_1 := x_0$ and $x_{n+1} := Tx_n$ for all $n \geq 1$. If $d(x_1, x_2) = 0$, then $x_1 = Tx_1$ and we are done. We now assume that $d(x_1, x_2) > 0$. Note that $(x_n, x_{n+1}) \in E(G)$ and $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for all $n \geq 1$. In particular,

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_1, x_2))$$

for all $n \geq 1$. Note that $\sum_{n=1}^{\infty} \psi^n(d(x_1, x_2)) < \infty$, and hence $\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) < \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is an element $p \in X$ such that $x_n \to p$. We now show that $p = Tp$. The proof is divided into 2 cases.

Case 1: We assume the condition (J1). Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$. We consider the following

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_k+1}) + d(x_{n_k+1}, Tp) \\ &= d(p, x_{n_k+1}) + d(Tx_{n_k}, Tp) \\ &\leq d(p, x_{n_k+1}) + \psi(d(x_{n_k}, p)) \end{aligned}$$

for all $k \geq 1$. Letting $k \to \infty$ gives $p = Tp$, that is, p is a fixed point of T.

Case 2: We assume the condition (J2). In this case, we have

$$p = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T^n x_1 = \lim_{n\to\infty} T(T^n x_1) = Tp.$$

Then p is a fixed point of T.

This completes the proof. □

2.1.2. *Ulam stability with respect to G.*

Lemma 2.1. *Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\sum_{k=1}^{\infty} \psi^k(t) < \infty$ for all $t > 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Then $\sum_{k=1}^{\infty} \psi^k(\varepsilon) < \infty$. In particular, there exists a positive integer N such that $\sum_{k=N}^{\infty} \psi^k(\varepsilon) \leq \varepsilon$. We now choose $\delta := \psi^N(\varepsilon)$. Hence, $\delta + \sum_{k=1}^{\infty} \psi^k(\delta) = \sum_{k=N}^{\infty} \psi^k(\varepsilon) \leq \varepsilon$ as desired. □

We present two Ulam stability results with respect to G for (ψ, G)-contractions of type I.

Theorem 2.2. *Let (X,d) be a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type I. Suppose that either the condition (J1) or (J2) holds. If $\mathrm{Fix}(T) \neq \varnothing$, then T is Ulam stable with respect to G.*

Proof. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type I. Let $\varepsilon > 0$. By Lemma 2.1, there exists a $\delta > 0$ such that $\delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon$. Let w be an element in X such that $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta$. (Note that such an element w exists because $\{(x,x) : x \in X\} \subset E(G)$ and $\mathrm{Fix}(T) \neq \varnothing$.) Set $x_1 := w$ and define $x_{n+1} := Tx_n$ for all

$n \geq 1$. It follows from Theorem 2.1 that $x_n \to p$ for some $p \in \text{Fix}(T)$. Furthermore, since $d(x_k, x_{k+1}) \leq \psi^k(d(x_1, x_2)) \leq \psi^k(\delta)$, we also have

$$d(w,p) = \lim_{n\to\infty} d(x_1, x_{n+1}) \leq \lim_{n\to\infty} \sum_{k=1}^{n} d(x_k, x_{k+1}) \leq \delta + \lim_{n\to\infty} \sum_{k=1}^{n} \psi^k(\delta) = \delta + \sum_{k=1}^{\infty} \psi^k(\delta) \leq \varepsilon.$$

Thus, T is Ulam stable with respect to G. □

The following example shows that our concept of Ulam stability with respect to G is more suitable for (ψ, G)-contractions than the classical Ulam stability.

Example 2.1. Let $X := [0,1]$ be a metric space with a usual metric d. Define $T : X \to X$ by

$$Tx := \begin{cases} x/2 & \text{if} \quad x \in [0,1/2] \cup \{1\} \\ 1 & \text{if} \quad x \in (1/2, 1). \end{cases}$$

Then the following statements are true.

(a) T is a (ψ, G)-contraction of type I where $\psi(t) = t/2$ for all $t \geq 0$ and $E(G) := [0,1/2]^2 \cup \{(x,x) : x \in (1/2, 1]\}$.
(b) T is not Ulam stable.
(c) T is Ulam stable with respect to G.

Proof. (a) It is clear that ψ is nondecreasing and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Moreover, if $(x,y) \in E(G)$, then $(Tx, Ty) \in E(G)$ and $d(Tx, Ty) \leq \psi(d(x,y))$. Hence T is a (ψ, G)-contraction of type I.

(b) To see this, we choose $\varepsilon := 1/2$ and set $x_n := (n+1)/(n+2)$ for all $n \geq 1$. It follows that $d(x_n, Tx_n) = d((n+1)/(n+2), 1) \to 0$. Note that 0 is the only one fixed point of T and $d(x_n, 0) \geq 1/2$ for all $n \geq 1$. Hence T is not Ulam stable.

(c) We conclude the result by using Theorem 2.2. In fact, we show that the condition (J1) is satisfied. Suppose that $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$. The result follows easily if $p \neq 1/2$. We now consider the case $p = 1/2$. If there exists an integer N such that $x_n > p$ for all $n \geq N$, then $x_n = x_N$ for all $n \geq N$ which is impossible. Hence there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq p$ for all $k \geq 1$. In particular, $(x_{n_k}, p) \in E(G)$ for all $k \geq 1$. So T is Ulam stable with respect to G. □

2.2. (ψ, G)-contractions of type II.

2.2.1. *Fixed point theorem.*

Lemma 2.2 ([1]). *Suppose that $\{x_n\}$ is a sequence in a metric space (X,d). If $\{x_n\}$ is not a Cauchy sequence, then there exist a constant $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that the following two conditions hold: for all $k \geq 1$ one has*

$$n_k < m_k \quad \textit{and} \quad d(x_{n_k}, x_{m_k - 1}) < \varepsilon \leq d(x_{n_k}, x_{m_k}).$$

Theorem 2.3. *Let (X,d) be a complete metric space with a directed graph G on X. Suppose that G is transitive, that is, $(x,z) \in E(G)$ whenever $(x,y) \in E(G)$ and $(y,z) \in E(G)$. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type II and suppose that either the condition (J1) or (J2) holds. If there exists an element $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then $T^n x_0 \to p$ for some $p \in \text{Fix}(T)$, that is, $\text{Fix}(T) \neq \varnothing$.*

Proof. Set $x_1 := x_0$ and $x_{n+1} := Tx_n$ for all $n \geq 1$. If $d(x_1, x_2) = 0$, then $x_1 = Tx_1$ and we are done. We now assume that $d(x_1, x_2) > 0$. Note that $(x_n, x_{n+1}) \in E(G)$ and $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for all $n \geq 1$. In particular,

$$d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_1, x_2))$$

for all $n \geq 1$. In particular, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. There exist an $\eta > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of $\{n\}$ such that $k \leq n_k \leq m_k$ and $d(x_{n_k}, x_{m_k-1}) < \eta \leq d(x_{n_k}, x_{m_k})$ for all $k \geq 1$. We note from the transitivity of G that $(x_{n_k}, x_{m_k-1}) \in E(G)$ and we obtain the following

$$\begin{aligned}\eta - d(x_{n_k}, x_{n_k+1}) &\leq d(x_{n_k}, x_{m_k}) - d(x_{n_k}, x_{n_k+1}) \\ &\leq d(x_{n_k+1}, x_{m_k}) \\ &\leq \psi(d(x_{n_k}, x_{m_k-1})) \\ &\leq \psi(\eta).\end{aligned}$$

Letting $k \to \infty$ gives $\eta \leq \psi(\eta)$, that is, $\eta = 0$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is an element p in X such that $x_n \to p$. The proof that p is a fixed point of T follows exactly as the same as the proof of Theorem 2.1 so it is left for the reader to verify. □

2.2.2. *Ulam stability with respect to G.*

Lemma 2.3 ([15, 18]). *If $\psi : [0,\infty) \to [0,\infty)$ is nonexpansive, that is, $|\psi(s) - \psi(t)| \leq |s - t|$ for all $s, t \geq 0$, then $I - \psi$ is nondecreasing where I is the identity mapping.*

We present the Ulam stability result with respect to G for (ψ, G)-contractions of type II.

Theorem 2.4. *Let (X, d) be a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a (ψ, G)-contraction of type II where G is transitive and ψ is nonexpansive. Suppose that either the condition (J1) or (J2) holds. If $\mathrm{Fix}(T) \neq \varnothing$, then T is Ulam stable with respect to G.*

Proof. Suppose that $T : X \to X$ is a (ψ, G)-contraction mapping of type II where ψ is a nonexpansive mapping and G is transitive. Let $\varepsilon > 0$. We choose $\delta := (\varepsilon - \psi(\varepsilon))/2$. Let w be an element in X such that $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta$. Set $x_1 := w$ and define $x_{n+1} := Tx_n$ for all $n \geq 1$. It follows from Theorem 2.3 that $x_n \to p$ where $p \in \mathrm{Fix}(T)$. We consider

$$\begin{aligned}d(w, p) = d(x_1, p) &\leq d(x_1, x_2) + d(x_2, p) \\ &= d(x_1, Tx_1) + d(Tx_1, Tp) \\ &\leq d(x_1, Tx_1) + \psi(d(x_1, p)) \\ &\leq \delta + \psi(d(w, p)).\end{aligned}$$

In particular, $(I - \psi)(d(w, p)) \leq \delta$. Suppose that $\varepsilon < d(w, p)$. Then $(I - \psi)(\varepsilon) \leq (I - \psi)(d(w, p)) \leq \delta = (\varepsilon - \psi(\varepsilon))/2$ which is a contradiction. Hence, $d(w, p) \leq \varepsilon$. Thus, T is Ulam stable with respect to G. □

There exists a nondecreasing and nonexpansive function $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\sum_{n=1}^{\infty} \psi^n(t) = \infty$ for all $t > 0$. In particular, this reveals the importance of Theorem 2.4.

Example 2.2. Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t/(1 + t)$ for all $t \geq 0$. Then ψ is nonexpansive, $\lim_{n\to\infty} \psi^n(t) = 0$, and $\sum_{n=1}^{\infty} \psi^n(t) = \infty$ for all $t > 0$. In fact, for each $n \geq 1$, we note that $\psi^n(t) = \frac{t}{1+nt}$ for all $t \geq 0$.

3. Deduced results and some remarks

3.1. **Ulam–Hyers stability with respect to G of Banach G-contractions.** By Theorem 2.2, we obtain the following result which supplements the result of Jachymski [7].

Corollary 3.1. *Suppose that (X, d) is a complete metric space with a directed graph G on X. Suppose that $T : X \to X$ is a Banach G-contraction with a fixed point. If either the condition (J1) or (J2) holds, then T is Ulam–Hyers stable with respect to G.*

Proof. Suppose that there exists a constant $\alpha \in (0, 1)$ such that $(Tx, Ty) \in E(G)$ and $d(Tx, Ty) \leq \alpha d(x, y)$ for all $(x, y) \in E(G)$. Put $\psi(t) := \alpha t$ for all $t \geq 0$. Let $\varepsilon > 0$ be given. Note that if $\delta = (1-\alpha)\varepsilon$, then $\delta + \sum_{k=1}^{\infty} \psi^k(\delta) = \delta + \sum_{k=1}^{\infty} \alpha^k \delta = \varepsilon$. It follows from the proof of Theorem 2.2 that for each $w \in X$ with $(w, Tw) \in E(G)$ and $d(w, Tw) \leq \delta = (1 - \alpha)\varepsilon$ there exists a fixed point p of T such that $d(p, w) \leq \varepsilon$. This completes the proof. □

3.2. **Remarks on Sintunavarat's results.** We will discuss some vague statements in the recent result of Sintunavarat [17]. His results are established in a different context but it will be seen later that it is equivalent to the setting with a directed graph (see Remark 3.7(1)). We first recall some concepts.

Definition 3.6. Let (X, d) be a metric space. Suppose that $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Suppose that $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$.

- T is *weakly α-admissible* if $\alpha(Tx, T^2x) \geq 1$ whenever $\alpha(x, Tx) \geq 1$.
- T is an *(α, ψ)-contraction* if $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.
- X is *α-regular* if whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to p$ for some $p \in X$ it follows that $\alpha(x_n, p) \geq 1$ for all $n \geq 1$.

Definition 3.7. Let (X, d) be a metric space and $\varepsilon > 0$. A point $p \in X$ is an *ε-fixed point of a mapping* $T : X \to X$ if $d(p, Tp) \leq \varepsilon$.

We carefully restate the following result from [17, Theorems 2.1, 2.2, 2.3, and 3.4].

Theorem 3.5. *Suppose that (X, d) is a complete metric space. Suppose that $T : X \to X$ is an (α, ψ)-contraction and it is weakly α-admissible with $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Suppose in addition that either T is continuous or X is α-regular. Then the following statements are true.*

(a) $\mathrm{Fix}(T) \neq \varnothing$.
(b) *If $\alpha(p, q) \geq 1$ for all $p, q \in \mathrm{Fix}(T)$, then $\mathrm{Fix}(T)$ is a singleton.*
(c) *Suppose that $I - \psi$ is strictly increasing and onto. If $\alpha(p', q') \geq 1$ for all ε-fixed points p' and q' of T, then T is generalized Ulam–Hyers stable.*

The remarks for the preceding theorem are as follows.

Remark 3.7. (1) Suppose that $T : X \to X$ is an (α, ψ)-contraction and it is weakly α-admissible. We define a directed graph G on X by letting $E(G) := \{(x, y) : \alpha(x, y) \geq 1\}$. It follows that T is a (ψ, G)-contraction of type I. In fact, if $(x, y) \in E(G)$, then $\alpha(x, y) \geq 1$ and hence

$$d(Tx, Ty) \leq \alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

The continuity of T can be replaced by the G-orbital continuity of T, that is, the condition (J2). The α-regularity of X becomes the condition (J1*) which is a stronger assumption than the condition (J1). On the other hand, suppose that T is a (ψ, G)-contraction of type I. Now, we define $\alpha(x, y) := 1$ if $(x, y) \in E(G)$ and $\alpha(x, y) := 0$ if $(x, y) \notin E(G)$. It follows that T is an (α, ψ)-contraction and it is weakly α-admissible.

(2) Our result for (ψ, G)-contractions of type II also provides a new information which is beyond the scope of the work of [17].
(3) No quantifier about ε is given in the statement (c) of Theorem 3.5. (The same patterns of vague statements are in [8, 11, 5, 16, 12, 13].) Moreover, in the proof of [17, Theorem 3.4] (page 400 line 7), the given $\varepsilon > 0$ is not arbitrary as required in the definition of the generalized Ulam–Hyers stability. Finally, we discuss the validity of the assumption: $\alpha(p', q') \geq 1$ for all ε-fixed points p' and q' of T. Note that if we set $X_\varepsilon := \{x : d(x, Tx) \leq \varepsilon\}$, then it follows from the continuity of T or the α-regularity X that the subset X_ε is closed and hence complete. It is clear that $T : X_\varepsilon \to X_\varepsilon$ is a ψ-contraction. From this point, the function α plays no role in the study.

Acknowledgements. The authors would like to thank the two referees for their comments and suggestions which enhance the presentation of this paper. The corresponding author's research was supported by the Thailand Research Fund and Khon Kaen University under grant RSA5980006.

REFERENCES

[1] Boyd, D. W. and Wong, J. S .W., *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458–464
[2] Brzdęk, J., Cădariu, L. and Ciepliński, K., *Fixed point theory and the Ulam stability*, J. Funct. Spaces, 2014, Art. ID 829419, 16 pp.
[3] Brzdęk, J., Karapnar, E. and Petruşel, A., *A fixed point theorem and the Ulam stability in generalized dq-metric spaces*, J. Math. Anal. Appl., **467** (2018), 501–520
[4] Brzdęk, J., Popa, D., Raşa, I. and Xu, B., *Ulam stability of operators. (Mathematical Analysis and Its Applications)*, Academic Press, London, 2018
[5] Felhi, A., Sahmim, S. and Aydi, H., *Ulam–Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces*, Fixed Point Theory Appl. 2016, Paper No. 1, 20 pp.
[6] Hyers, D. H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224
[7] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373
[8] Kutbi, M. A. and Sintunavarat, W., *Ulam–Hyers stability and well-posedness of fixed point problems for α-ψ-contraction mapping in metric spaces*, Abstr. Appl. Anal., Volume **2104** (2014), Article ID 268230, 6 pp.
[9] Matkowski, J., *Integrable solutions of functional equations*, Dissertationes Math. (Rozprawy Mat.), **127** (1975), 1–68
[10] Moszner, Z., *Stability has many names*, Aequationes Math., **90** (2016), 983–999
[11] Nashine, H. K. and Kadelburg, Z., *Existence of solutions of cantilever beam problem via $(\alpha$-β-$FG)$-contractions in b-metric-like spaces*, Filomat, **31** (2017), 3057–3074
[12] Phiangsungnoen, S. and Kumam, P., *Generalized Ulam–Hyers stability and well-posedness for fixed point equation via α-admissibility*, J. Inequal. Appl. 2014, 2014:418, 12 pp.
[13] Phiangsungnoen, S., Sintunavarat, W. and Kumam, P., *Fixed point results, generalized Ulam–Hyers stability and well-posedness via α-admissible mappings in b-metric spaces*, Fixed Point Theory Appl., 2014, 2014:188, 17 pp.
[14] Rus, I. A. and Şerban, M. A., *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., **29** (2013), 239–258
[15] Saejung, S., *Fixed Point Algorithms and Related Topics*, Yokohama Publishers, Yokohama, 2017
[16] Sintunavarat, W., *Generalized Ulam–Hyers stability, well-posedness and limit shadowing of fixed point problems for α-β-contraction mapping in metric spaces*, The Scientific World Journal, Volume **2014** (2014), Article ID 569174, 7 pp.
[17] Sintunavarat, W., *A new approach to α-ψ-contractive mappings and generalized Ulam–Hyers stability, well-posedness and limit shadowing results*, Carpathian J. Math., **31** (2015), 395–401
[18] Takahashi, W., *Introduction to nonlinear and convex analysis*, Yokohama Publishers, Yokohama, 2009
[19] Ulam, S. M., *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, No. 8 Interscience Publishers, New York-London 1960

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
KHON KAEN UNIVERSITY, KHON KAEN, 40002, THAILAND

RESEARCH CENTER FOR ENVIRONMENTAL AND HAZARDOUS SUBSTANCE MANAGEMENT
KHON KAEN UNIVERSITY KHON KAEN, 40002, THAILAND
E-mail address: saejung@kku.ac.th