CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
35 (2019), No. 1, 31 - 40 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Bounds for the skew Laplacian spectral radius of oriented graphs

BILAL A. CHAT[1], HILAL A. GANIE[2] and S. PIRZADA[2]

ABSTRACT. We consider the skew Laplacian matrix of a digraph $\overrightarrow{G}$ obtained by giving an arbitrary direction to the edges of a graph G having n vertices and m edges. We obtain an upper bound for the skew Laplacian spectral radius in terms of the adjacency and the signless Laplacian spectral radius of the underlying graph G. We also obtain upper bounds for the skew Laplacian spectral radius and skew spectral radius, in terms of various parameters associated with the structure of the digraph $\overrightarrow{G}$ and characterize the extremal graphs.

1. INTRODUCTION

Consider a simple graph G with n vertices and m edges and having the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $\overrightarrow{G}$ be a digraph obtained by assigning arbitrarily a direction to each of the edges of G. The digraph $\overrightarrow{G}$ is called an orientation of G or oriented graph corresponding to G. Also the graph G is called the underlying graph of $\overrightarrow{G}$. Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \ldots, n$ be respectively the out-degree, in-degree and degree of the vertices of $\overrightarrow{G}$. The out-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n\times n$ matrix $A^+ = A^+(\overrightarrow{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix of the digraph $\overrightarrow{G}$ is the $n \times n$ matrix $A^- = A^-(\overrightarrow{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. We note that $A^- = (A^+)^t$. The skew adjacency matrix of a digraph $\overrightarrow{G}$ is the $n \times n$ matrix $S = S(\overrightarrow{G}) = (s_{ij})$, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $S(\overrightarrow{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. For recent developments on the theory of skew spectrum, we refer to the papers [2, 14, 16, 23, 26, 28]. Let $D^+ = D^+(\overrightarrow{G}) = diag(d_1^+, d_2^+, \ldots, d_n^+)$, $D^- = D^-(\overrightarrow{G}) = diag(d_1^-, d_2^-, \ldots, d_n^-)$ and $D(\overrightarrow{G}) = diag(d_1, d_2, \ldots, d_n)$ be respectively the diagonal matrix of vertex out-degrees, vertex in-degrees and vertex degrees of $\overrightarrow{G}$. Further, let A^+ and A^- be respectively the out-adjacency and in-adjacency matrix of a digraph $\overrightarrow{G}$. If $S(\overrightarrow{G})$ is the skew adjacency matrix of $\overrightarrow{G}$ and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\overrightarrow{G}$, clearly $A(G) = A^+ + A^-$ and $S(\overrightarrow{G}) = A^+ - A^-$. Analogous to the definition of Laplacian matrix of a graph, Cai et al. [4] called the matrix $\widetilde{SL}(\overrightarrow{G}) = \widetilde{D}(\overrightarrow{G}) - S(\overrightarrow{G})$, where $\widetilde{D}(\overrightarrow{G}) = D^+(\overrightarrow{G}) - D^-(\overrightarrow{G})$, as the *skew Laplacian matrix* of the digraph

Received: 28.02.2018. In revised form: 03.12.2018. Accepted: 10.12.2018
2010 *Mathematics Subject Classification.* 05C50, 05C69, 05C70.
Key words and phrases. *Digraph, skew Laplacian matrix, skew Laplacian spectrum.*
Corresponding author: S. Pirzada; pirzadasd@kashmiruniversity.ac.in

$\overrightarrow{G}$. Clearly the matrix $\widetilde{SL}(\overrightarrow{G})$ is not symmetric and so its eigenvalues need not be real. The characteristic polynomial

$$P_{sl}(\overrightarrow{G}, x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

of the matrix $\widetilde{SL}(\overrightarrow{G})$ is called the *skew Laplacian characteristic polynomial* of the digraph $\overrightarrow{G}$. The zeros of the polynomial $P_{sl}(\overrightarrow{G}, x)$, that is, the eigenvalues of the matrix $\widetilde{SL}(\overrightarrow{G})$ are the skew Laplacian eigenvalues of the digraph $\overrightarrow{G}$ and are denoted by $\nu_1, \nu_2, \ldots, \nu_n$. The skew Laplacian spectrum of the digraph $\overrightarrow{G}$ is denoted by $\text{Spect}_{sl}(\overrightarrow{G})$. The sign of the even cycle $C_k = u_1 u_2 \ldots u_k u_1$, denoted by $sgn(C_k)$, is defined as $sgn(C_k) = s_{12} s_{23} \ldots s_{k-1k} s_{k1}$. An even oriented cycle C_k is called evenly-oriented (oddly-oriented) if its sign is positive (negative). If every even cycle in $\overrightarrow{G}$ is evenly-oriented, $\overrightarrow{G}$ is called evenly-oriented. An even oriented cycle C_{2k} is said to be uniformly oriented if $sgn(C_{2k}) = (-1)^k$. The following observations are immediate from the definition of $\widetilde{SL}$.

Theorem 1.1. [4]

(i) *If $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\overrightarrow{G})$, then $\sum_{i=1}^{n} \nu_i = 0$.*

(ii) *0 is an eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ with multiplicity at least p, where p is the number of components of $\overrightarrow{G}$ with all ones vector $(1, 1, \ldots, 1)$ as the corresponding eigenvector.*

(iii) *If $P_{sl}(\overrightarrow{G}, x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ is the skew Laplacian characteristic polynomial of the digraph $\overrightarrow{G}$, then $a_1 = 0,\ a_2 = m + \sum_{i<j} (d_i^+ - d_i^-)(d_j^+ - d_j^-),\ a_n = 0$.*

Evidently a good amount of research work has been done on spectral theory of skew matrices of oriented graphs, see [16], but the work on the skew Laplacian spectrum of a digraph $\overrightarrow{G}$ has been recently started and it will be of interest to develop the theory in this direction. For some recent work, see [10, 12] and the references therein. Although the skew Laplacian matrix of a digraph was so defined that it uses the structure of the digraph and at the same time enjoys the same characteristics as possessed by the Laplacian matrix of a graph, it seems that the definition of $\widetilde{SL}$ uses the structure of the digraph, but not all the properties of $L(G)$ are possessed by $\widetilde{SL}$. It is well-known that 0 is an eigenvalue of $L(G)$ with multiplicity equal to the number of components of G. In fact, the eigenvalue 0 in the spectrum of $L(G)$ decides the connectedness of the graph G. This need not be true for the matrix $\widetilde{SL}$, as is clear from the following observation, the proof of which follows from Theorem 2.1 in [27].

Theorem 1.2. *Let G be a bipartite graph and let $\overrightarrow{G}$ be the corresponding digraph of G. If $\overrightarrow{G}$ is an Eulerian digraph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$, then the multiplicity of 0 in the spectrum of $\widetilde{SL}$ is same as the multiplicity of 0 in the spectrum of $A(G)$.*

As usual, we denote the complete graph on n vertices by K_n, the complete bipartite graph on $s + t$ vertices by $K_{s,t}$, the cycle on n vertices by C_n. For other undefined notations and terminology from graphs and spectral graph theory, the readers are referred to [6, 22]. Let $K_{r,s}$ be the complete bipartite graph with both r and s even. Orient the edges of $K_{r,s}$ in such a way that in the resulting digraph $\overrightarrow{G}$ all the even cycles are oriented uniformly. Since 0 is an adjacency eigenvalue of $K_{r,s}$ of multiplicity $r + s - 2$, from Theorem 1.2, it follows that 0 is the skew Laplacian eigenvalue of $\overrightarrow{G}$ of multiplicity $r + s - 2$.

2. Skew Laplacian spectrum of an oriented graph

Let $\widetilde{SL}$ be the skew Laplacian matrix of the digraph $\overrightarrow{G}$. If we reverse the direction of all the edges of $\overrightarrow{G}$, we obtain a new digraph $\overleftarrow{G}$, which we call the *converse digraph* of $\overrightarrow{G}$. Clearly $-\widetilde{SL}$ is the skew Laplacian matrix of $\overleftarrow{G}$. Therefore, we have the following observation.

Theorem 2.3. *If $\overleftarrow{G}$ is the converse digraph of the digraph $\overrightarrow{G}$, then $Spect_{sl}(\overleftarrow{G}) = -Spect_{sl}(\overrightarrow{G})$.*

Let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the induced subgraph H of G and let $\overrightarrow{H}^* = \overrightarrow{H} \cup (n - n(H))K_1$, that is, $\overrightarrow{H}$ together with $n - n(H)$ isolated vertices. Let $\overrightarrow{G} - E(\overrightarrow{H})$ be the subdigraph obtained by removing the arcs of $\overrightarrow{H}$ in $\overrightarrow{G}$ and $\overrightarrow{G} - \overrightarrow{H}$ be the subdigraph obtained by deleting the vertices of $\overrightarrow{H}$ and the arcs incident at the vertices of $\overrightarrow{H}$. With out loss of generality, we can choose a labelling of the vertices of $\overrightarrow{G}$, so that

$$S(\overrightarrow{G}) = \begin{pmatrix} S(\overrightarrow{H}^*) & X \\ -X^t & S(\overrightarrow{G} - \overrightarrow{H}) \end{pmatrix} = S(\overrightarrow{H}^*) + S(\overrightarrow{G} - E(\overrightarrow{H})),$$
$$\text{and} \quad \tilde{D}(\overrightarrow{G}) = \tilde{D}(\overrightarrow{H}^*) + \tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})),$$

where X corresponds to the arcs connecting $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$. Therefore,

$$\widetilde{SL}(\overrightarrow{G}) = \tilde{D} - S(\overrightarrow{G}) = \Big(\tilde{D}(\overrightarrow{H}^*) - S(\overrightarrow{H}^*)\Big) + \Big(\tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})) - S(\overrightarrow{G} - E(\overrightarrow{H}))\Big).$$

Suppose both $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ are Eulerian subdigraphs of $\overrightarrow{G}$. Let $\overrightarrow{G}_1$ be the digraph obtained from $\overrightarrow{G}$ by reversing the direction of all the arcs in $\overrightarrow{H}$ and keeping the other arcs unchanged. We have

$$\begin{aligned} \widetilde{SL}(\overrightarrow{G}_1) &= \Big(\tilde{D}(\overleftarrow{H}^*) - S(\overleftarrow{H}^*)\Big) + \Big(\tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})) - S(\overrightarrow{G} - E(\overrightarrow{H})\Big) \\ &= \Big(\tilde{D}(\overrightarrow{H}^*) + S(\overrightarrow{H}^*)\Big) + \Big(\tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})) - S(\overrightarrow{G} - E(\overrightarrow{H}))\Big), \end{aligned}$$

as $-S(\overrightarrow{H}^*) = S(\overleftarrow{H}^*)$ and $\tilde{D}(\overleftarrow{H}^*) = \tilde{D}(\overrightarrow{H}^*)$. The last equality is due to the fact that the only non-zero contribution to the $(i,i)^{th}$ element $d_i^+ - d_i^-$ of the matrix $\tilde{D}(\overrightarrow{H}^*)$ is due to the arcs connecting the vertices in $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$. Let $\overrightarrow{G}_2$ be the digraph obtained from $\overrightarrow{G}$ by reversing the direction of all arcs in $\overrightarrow{G} - E(\overrightarrow{H})$ and keeping other arcs unchanged. Since $-S(\overrightarrow{G} - E(\overrightarrow{H})) = S(\overleftarrow{G} - E(\overleftarrow{H}))$ and the only non-zero contribution to the $(i,i)^{th}$ element $d_i^+ - d_i^-$ of the matrix $\tilde{D}(\overrightarrow{G} - E(\overrightarrow{H}))$ is due to the arcs connecting the vertices in $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$, it follows that

$$\begin{aligned} \widetilde{SL}(\overrightarrow{G}_2) &= \Big(- \tilde{D}(\overrightarrow{H}^*) - S(\overrightarrow{H}^*)\Big) + \Big(\tilde{D}(\overleftarrow{G} - E(\overleftarrow{H})) - S(\overleftarrow{G} - E(\overleftarrow{H})\Big) \\ &= \Big(- \tilde{D}(\overrightarrow{H}^*) - S(\overrightarrow{H}^*)\Big) + \Big(- \tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})) + S(\overrightarrow{G} - E(\overrightarrow{H}))\Big) \\ &= -\Big[\Big(\tilde{D}(\overrightarrow{H}^*) + S(\overrightarrow{H}^*)\Big) + \Big(\tilde{D}(\overrightarrow{G} - E(\overrightarrow{H})) - S(\overrightarrow{G} - E(\overrightarrow{H}))\Big)\Big] \\ &= -\widetilde{SL}(\overrightarrow{G}_1). \end{aligned}$$

Therefore, it follows that the skew Laplacian spectrum of $\overrightarrow{G}_2$ is negative of the skew Laplacian spectrum of $\overrightarrow{G}_1$.

Again, if both $\overrightarrow{H}$ and $\overrightarrow{G} - \overrightarrow{H}$ are Eulerian subdigraphs of $\overrightarrow{G}$, and $\overrightarrow{G}_3$ is the digraph obtained from $\overrightarrow{G}$ by reversing the direction of the arcs having one end in $\overrightarrow{H}$ and other end in $\overrightarrow{G}-\overrightarrow{H}$, then proceeding similarly as above, it can seen that $\widetilde{SL}(\overrightarrow{G}_3)=-\widetilde{SL}(\overrightarrow{G}_4)$, where $\overrightarrow{G}_4$ is the digraph obtained from $\overrightarrow{G}$ by reversing the direction of arcs in both $\overrightarrow{H}$ and $\overrightarrow{G}-\overrightarrow{H}$ and keeping other arcs unchanged. From this, it follows that the skew Laplacian spectrum of $\overrightarrow{G}_4$ is negative of the skew Laplacian spectrum of $\overrightarrow{G}_3$. Thus, we have proved the following.

Theorem 2.4. *Let $\overrightarrow{G}$ be an orientation of a graph G and let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the subgraph H of G. If the subdigraphs $\overrightarrow{H}$ and $\overrightarrow{G}-\overrightarrow{H}$ of $\overrightarrow{G}$ are Eulerian, then*

$$(i)\ Spect_{sl}(\overrightarrow{G}_1)=-Spect_{sl}(\overrightarrow{G}_2),\quad (ii)\ Spect_{sl}(\overrightarrow{G}_3)=-\,Spect_{sl}(\overrightarrow{G}_4),$$

where $\overrightarrow{G}_1,\ \overrightarrow{G}_2,\ \overrightarrow{G}_3$ and $\overrightarrow{G}_4$ are the digraphs defined above.

If $\overrightarrow{G}$ is itself an Eulerian digraph, the conclusion of Theorem 2.4 holds for all induced subdigraphs. A subset W of the vertex set $V(\overrightarrow{G})$ is said to be independent if the induced subdigraph $\langle W\rangle$ is an empty digraph. In other words, W is an independent subset of $V(\overrightarrow{G})$ if the vertices in W are mutually non-adjacent. We have the following observation.

Theorem 2.5. *Let $\overrightarrow{G}$ be an orientation of a graph G and let $\overrightarrow{H}$ be an induced subdigraph of $\overrightarrow{G}$ corresponding to the subgraph H of G. If the subdigraph $\overrightarrow{H}$ is Eulerian and the subdigraph $\overrightarrow{G}-\overrightarrow{H}$ is independent, then*

$$(i)\ Spect_{sl}(\overrightarrow{G}_1)=-Spect_{sl}(\overrightarrow{G}_2),\quad (ii)\ Spect_{sl}(\overrightarrow{G}_3)=-Spect_{sl}(\overrightarrow{G}_4),$$

where $\overrightarrow{G}_1,\ \overrightarrow{G}_2,\ \overrightarrow{G}_3$ and $\overrightarrow{G}_4$ are the digraphs defined as above.

The skew Laplacian spectral radius of the digraph $\overrightarrow{G}$ is denoted by $\rho_{sl}(\overrightarrow{G})$ and is defined as

$$\rho_{sl}(\overrightarrow{G})=\max_i\{|\nu_i|:\ i=1,2,\dots,n\}.$$

The singular values of a square matrix X of order n are defined as the positive square roots of the eigenvalues of the matrix X^*X. If $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_n$ are the singular values and $|\lambda_1|\geq|\lambda_2|\geq\cdots\geq|\lambda_n|$ are the absolute values of the eigenvalues of X, then it is well known [15] that $|\lambda_1|\leq\sigma_1$, with equality if and only if X is a normal matrix (a matrix X is said to be normal if $XX^*=X^*X$). This observation implies that any upper bound for the largest singular value σ_1 gives an upper bound for the spectral radius.

In this paper, we obtain upper bounds for the skew Laplacian spectral radius of $\overrightarrow{G}$ in terms of various parameters associated with the structure of the digraph $\overrightarrow{G}$ and the underlying graph G. The spectral radius of the matrices has been discussed in general for all matrices real or complex. But when restricted to a special kind of a matrix associated to a graph or a digraph, it is always interesting to find estimates for the spectral radius in terms of the structure of the graph or the digraph. Also, when restricted to a particular class of graphs or digraphs, it is of interest to characterize the graphs or digraphs which attain the extremal values in that class. A reasonable amount of work has been done in these directions and various research articles can be found in the literature regarding the spectral radius of a graph with respect to different matrices, like adjacency matrix $A(G)$, Laplacian matrix $L(G)$, signless Laplacian matrix $Q(G)$, distance matrix, etc, for example, see the recent articles [3, 8, 9, 11, 21]. Similarly, various papers can be found in the literature regarding the spectral radius of a digraph with respect to different matrices like adjacency matrix $A(\overrightarrow{G})$, Laplacian matrix $L(\overrightarrow{G})$, skew matrix $S(\overrightarrow{G})$, etc, associated to the digraph $\overrightarrow{G}$. Here, we consider spectral radius of the digraph $\overrightarrow{G}$ with respect to its skew Laplacian matrix $\widetilde{SL}(\overrightarrow{G})$. The following observation is due to Perron and Frobenius [15].

Lemma 2.1. *Let $S = (s_{ij})$ be a complex matrix and X be an irreducible matrix of the same order. Let $|S|$ denote the matrix whose (i, j)-entry is $|s_{ij}|$. If $|S| \leq X$ and S has t as an eigenvalue, then $|t| \leq \lambda_1(X)$. If the equality holds, then $|S| = X$, and there is a diagonal matrix E with diagonal entries of absolute value 1 and a constant c of absolute value 1, such that $S = cEXE^{-1}$.*

Now, we obtain an upper bound for $\rho_{sl}(\overrightarrow{G})$ in terms of the adjacency spectral radius and signless Laplacian spectral radius of the underlying graph G.

Theorem 2.6. *Let $\overrightarrow{G}$ be an orientation of a connected graph G of order n and let $\rho_{sl}(\overrightarrow{G})$ be the skew Laplacian spectral radius of $\overrightarrow{G}$. Let λ_1 and q_1 be respectively the largest adjacency eigenvalue and the largest signless Laplacian eigenvalue of the graph G. Then*

$$\rho_{sl}(\overrightarrow{G}) \leq \begin{cases} \lambda_1, & \text{if } \overrightarrow{G} \text{ is Eulerian} \\ q_1, & \text{if } \overrightarrow{G} \text{ is non-Eulerian.} \end{cases}$$

If $\overrightarrow{G}$ is Eulerian, equality occurs if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

Proof. Let G be a connected graph of order n and let $\overrightarrow{G}$ be an orientation of G. Further, let $Q(G) = L(G) + A(G)$ be the signless Laplacian matrix of the graph G and let $\widetilde{SL}(\overrightarrow{G})$ be the skew Laplacian matrix of the digraph $\overrightarrow{G}$. With out loss of generality, suppose that the vertices of $\overrightarrow{G}$ and G are labelled in the same order. If $\overrightarrow{G}$ is Eulerian, then $\widetilde{SL}(\overrightarrow{G}) = -S(\overrightarrow{G})$ and so $|\widetilde{SL}(\overrightarrow{G})| = |-S(G)| = A(G)$. Since the matrix $A(G)$ is irreducible, by Lemma 2.1, it follows that $\rho_{sl}(\overrightarrow{G}) \leq \lambda_1$. On the other hand, if $\overrightarrow{G}$ is non-Eulerian, then for any orientation $\overrightarrow{G}$ of G, we always have $|\widetilde{SL}(\overrightarrow{G})| \leq Q(G)$. Since the matrix $Q(G)$ is irreducible, by Lemma 2.1, it follows that $\rho_{sl}(\overrightarrow{G}) \leq q_1$.

If $\overrightarrow{G}$ is Eulerian, then $\widetilde{SL}(\overrightarrow{G}) = -S(\overrightarrow{G})$ and so by Theorem 2.1 of [27] equality occurs if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$. On the other hand if $\overrightarrow{G}$ is non-Eulerian, then G is either a bipartite graph or a non-bipartite graph. If G is a non-bipartite graph, the equality can never occur. This is due to the fact that for non-bipartite graphs the smallest eigenvalue for the matrix $Q(G)$ and so for the matrix $cEQ(G)E^{-1}$ is positive while as the smallest eigenvalue of $\widetilde{SL}(\overrightarrow{G})$ is always zero. So, assume that G is a bipartite graph and $\overrightarrow{G}$ is a non-Eulerian graph. Let (V_1, V_2) be the bipartition of the vertex set $V(G)$ of the graph G. If we choose an orientation $\overrightarrow{G}$ such that all the arcs are directed from V_1 to V_2, then with out loss of generality we can label the vertices of $\overrightarrow{G}$ so that its skew Laplacian matrix is $\widetilde{SL}(\overrightarrow{G}) = \begin{pmatrix} D_1 & -X \\ X^t & -D_2 \end{pmatrix}$, where $\tilde{D} = \begin{pmatrix} D_1 & 0 \\ 0 & -D_2 \end{pmatrix}$ and $S(G) = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}$. Clearly $|\widetilde{SL}| = Q(G)$, but equality can not occur in this case. If $E = diag(\alpha_1, \alpha_2, \ldots, \alpha_n),\ |\alpha_i| = 1$, $Q(G) = (q_{ij}),\ \widetilde{SL}(\overrightarrow{G}) = (s_{ij})$, it can be seen that $cEQ(G)E^{-1} = \widetilde{SL}(\overrightarrow{G})$ gives $c\alpha_i\alpha_j^{-1}a_{ij} = s_{ij}$, for all $i, j = 1, 2, \ldots, n$. Since $a_{ii} \neq 0$ and $s_{ii} \neq 0$, there is no such $c,\ |c| = 1$, for which this equality occurs. This completes the proof. □

Remark 2.1. From Theorem 2.6, we observe that if the orientation $\overrightarrow{G}$ is an Eulerian digraph, then any upper bound for $\lambda_1(G)$ gives an upper bound for $\rho_{sl}(\overrightarrow{G})$ and if the orientation $\overrightarrow{G}$ is a non-Eulerian digraph, then any upper bound for $q_1(G)$ gives an upper bound for $\rho_{sl}(\overrightarrow{G})$. For example, if the digraph $\overrightarrow{G}$ of the graph G is Eulerian, then $\rho_{sl}(\overrightarrow{G}) \leq \Delta = \max_i\{d_i^+ + d_i^-\}$, this is because $\lambda_1(G) \leq \Delta$. If $\overrightarrow{G}$ is a non-Eulerian digraph, then $\rho_{sl}(\overrightarrow{G}) \leq \max_{v_i \sim v_j}\{d_i + d_j\}$, because $q_1(G) \leq \max_{v_i \sim v_j}\{d_i + d_j\}$.

The following observation is immediate from Theorem 2.6.

Corollary 2.1. *Among all the Eulerian digraphs, the bipartite Eulerian digraphs with each even cycle of G oriented uniformly in $\overrightarrow{G}$ has the maximum skew Laplacian spectral radius.*

For Eulerian digraphs $\overrightarrow{G}$, since the skew Laplacian spectral radius $\rho_{sl}(\overrightarrow{G})$ and skew spectral radius $\rho_s(\overrightarrow{G})$ are same, we have the following observation, the proof of which follows from Theorem 2.1 of [27] and Corollary 2.1.

Corollary 2.2. *Among all the bipartite Eulerian digraphs, the complete bipartite digraph with partite sets of even cardinality and with each even cycle oriented uniformly has the maximum skew Laplacian spectral radius.*

It will be interesting to determine the oriented graphs among the oriented trees, oriented unicyclic graphs, oriented bicyclic graphs, oriented non-Eulerian graphs, etc, which attain the extremal values for the skew Laplacian spectral radius. Now, we obtain an upper bound for $\rho_{sl}(\overrightarrow{G})$ in terms of the out-degree d_i^+ and in-degree d_i^- of the vertex v_i of the digraph $\overrightarrow{G}$.

Theorem 2.7. *Let $\rho_{sl}(\overrightarrow{G})$ be the skew Laplacian spectral radius of the digraph $\overrightarrow{G}$. If d_i^+ and d_i^- are the out and in-degrees of the vertices of $\overrightarrow{G}$, then*

$$\rho_{sl}(\overrightarrow{G}) \leq \max_i\{|d_i^+ - d_i^-| + d_i\}.$$

If $\overrightarrow{G}$ is Eulerian, equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

Proof. It is well known [15] that the spectral radius $\rho(X)$ of an $n \times n$ matrix $X = (x_{ij})$ always satisfies

$$\rho(X) \leq \min_i\{R_i,\ C_i\},$$

where

$$R_i = \max_i\{\sum_{k=1}^{n} |x_{ik}| : 1 \leq i \leq n\}$$

and

$$C_i = \max_i\{\sum_{k=1}^{n} |x_{ki}| : 1 \leq i \leq n\}.$$

Taking $X = \widetilde{SL}(\overrightarrow{G})$ and using the fact that $R_i = C_i = \max_i\{|d_i^+ - d_i^-| + d_i\}$, the result follows. By Theorem 2.6, $\rho_{sl}(\overrightarrow{G}) = \lambda_1$, where λ_1 is the adjacency spectral radius of G, if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$. The result now follows by using the fact that the adjacency matrix of G is irreducible and non-negative. $\square$

If $\rho_s(\overrightarrow{G})$ is the skew spectral radius of the digraph $\overrightarrow{G}$, we have the following observation, the proof of which is similar to that of Theorem 2.7.

Theorem 2.8. *Let $\rho_s(\overrightarrow{G})$ be the skew spectral of the digraph $\overrightarrow{G}$. If d_i^+ and d_i^-, are the out and in-degrees of the vertices of $\overrightarrow{G}$, then*

$$\rho_s(\overrightarrow{G}) \leq \max_i\{d_i\}.$$

Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

For each vertex $v_i \in V(\overrightarrow{G})$, let $\alpha_i = d_i^+ - d_i^-$ be its oriented degree. Let t_i be the sum of the absolute values of the oriented degrees of the vertices which are adjacent to v_i, that is, $t_i = \sum_{v_j, v_iv_j\in E(G)} |\alpha_j|$. We call t_i the 2-oriented degree and $m(v_i) = \frac{t_i}{|\alpha_i|}$ as the average oriented degree of the vertex v_i. Let $m^+(v_i) = \frac{t_i^+}{d_i^+}$, where $t_i^+ = \sum_{v_j, v_iv_j\in E(G)} d_j^+$, be the average positive degree of the vertex v_i. The next result gives an upper bound for $\rho_{sl}(\overrightarrow{G})$ in terms of oriented degrees α_i and the average oriented degrees $m(v_i)$ of the vertices of the digraph $\overrightarrow{G}$.

Theorem 2.9. *Let $\overrightarrow{G}$ be an orientation of a connected graph G. If α_i is the oriented degree and $m(v_i)$ is the average oriented degree of the vertex v_i, then*

$$\rho_{sl}(\overrightarrow{G}) \leq \begin{cases} \max_i\{|\alpha_i| + m(v_i)\}, & \text{if } \alpha_i \neq 0, \text{ for all } i \\ \max_i\{\delta_i^k\}, & \text{if } \alpha_i = 0, \text{ for } 1 \leq i \leq k \\ \max_i\{\delta_i\}, & \text{if } \alpha_i = 0, \text{ for } 1 \leq i \leq n, \end{cases}$$

where for $1 \leq i \leq k$, we have

$$\delta_i^k = \sum_{\substack{v_j,\ v_iv_j\in E(G)\\ j\leq k}} \beta_i^{-1}\beta_j + \sum_{\substack{v_j,\ v_iv_j\in E(G)\\ j\geq k+1}} \beta_i^{-1}|\alpha_j|,$$

and for $k+1 \leq i \leq n$, we have

$$\delta_i^k = \sum_{\substack{v_j,\ v_iv_j\in E(G)\\ j\leq k}} \beta_j|\alpha_i^{-1}| + \sum_{\substack{v_j,\ v_iv_j\in E(G)\\ j\geq k+1}} |\alpha_i^{-1}\alpha_j| + |\alpha_i|$$

and

$$\delta_i = \sum_{v_j, v_iv_j\in E(G)} \beta_i^{-1}\beta_j.$$

If $\overrightarrow{G}$ is Eulerian, equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

Proof. Let G be a connected graph of order n having m edges and let $\overrightarrow{G}$ be an orientation of G. First suppose that the oriented degrees α_i of each of the vertices v_i of the orientation $\overrightarrow{G}$ are non-zero. If $\tilde{D} = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $\tilde{D}^{-1} = diag(\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1})$ exists. Consider the matrix $\tilde{D}^{-1}\widetilde{SL}\tilde{D}$. It is easy to see that the $(i,i)^{th}$ entry of this matrix is $\alpha_i = d_i^+ - d_i^-$ and the $(i,j)^{th}$ entry is $(d_i^+ - d_i^-)^{-1}(d_j^+ - d_j^-)s_{ij}$, where $s_{ij} = -1$ or 0 or -1. Using the fact that the matrices $\widetilde{SL}$ and $\tilde{D}^{-1}\widetilde{SL}\tilde{D}$ are similar, it follows that $\rho_{sl}(\overrightarrow{G}) = \rho(\tilde{D}^{-1}\widetilde{SL}\tilde{D})$. Now, proceeding similarly as in Theorem 2.7, and by taking $X = \tilde{D}^{-1}\widetilde{SL}\tilde{D}$, the result follows in this case.

Let some k, $1 \leq k \leq n-1$, vertices of $\overrightarrow{G}$ have oriented degree zero. Without loss of generality, suppose these vertices are $v_1, v_2, \ldots, v_k$. Label the vertices of $\overrightarrow{G}$ in such a way that the first k rows and columns of the matrix $\widetilde{SL}$ correspond to the vertices $v_1, v_2, \ldots, v_k$. Let $D_1 = diag(\beta_1, \beta_2, \ldots, \beta_k, \alpha_{k+1}, \ldots, \alpha_n)$, where $\beta_i = \min_i\{d_i^+, d_i^-\}$. Clearly, the matrix $\tilde{D}^{-1} = diag(\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_k^{-1}, \alpha_{k+1}^{-1}, \ldots, \alpha_n^{-1})$ exists. Consider the matrix $D_1^{-1}\widetilde{SL}D_1$. It is easy to see that the $(i,i)^{th}$ entry of this matrix is equal to 0, for $1 \leq i \leq k$, and equal to $\alpha_i = d_i^+ - d_i^-$, for $k+1 \leq i \leq n$. Also, for $1 \leq i \leq k$, its $(i,j)^{th}$ entry is equal to $\beta_i^{-1}\beta_j$, for $1 \leq j \leq k$ and equal to $\beta_i^{-1}(d_j^+ - d_j^-)s_{ij}$, for $k+1 \leq j \leq n$; and for $k+1 \leq i \leq n$ its $(i,j)^{th}$ entry is equal to $\beta_i(d_j^+ - d_j^-)^{-1}s_{ij}$, for $1 \leq j \leq k$ and equal to $(d_i^+ - d_i^-)^{-1}(d_j^+ - d_j^-)s_{ij}$, for $k+1 \leq j \leq n$, where $s_{ij} = -1$ or 0 or 1. Since the matrices $\widetilde{SL}$ and $D_1^{-1}\widetilde{SL}D_1$ are similar, the result follows by taking $X = \tilde{D}^{-1}\widetilde{SL}\tilde{D}$, in Theorem 2.7.

Lastly, we suppose that the oriented degrees of all the vertices of $\overrightarrow{G}$ are zero. Let $D_2 = diag(\beta_1, \beta_2, \ldots, \beta_n)$. Clearly, the matrix $\tilde{D}^{-1} = diag(\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_n^{-1})$ exists. Now, consider the matrix $D_2^{-1}\widetilde{SL}D_2$ and proceeding similarly as above, it can be seen that the result follows in this case as well. If λ_1 is the adjacency spectral radius of G, then as shown in Theorem 2.6, $\rho_{sl}(\overrightarrow{G}) = \lambda_1$, if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$. Therefore, if G is a regular graph, equality follows by using the fact that the adjacency matrix of G is irreducible and non-negative. This completes the proof. □

Proceeding similarly as in Theorem 2.9, we obtain the following upper bound for the skew spectral radius $\rho_s(\overrightarrow{G})$ of the digraph $\overrightarrow{G}$.

Theorem 2.10. *Let $\overrightarrow{G}$ be an orientation of a connected graph G. Then*

$$\rho_s(\overrightarrow{G}) \leq \max_i\{\sum_{v_j, v_iv_j \in E(G)} \beta_i^{-1}\beta_j\},$$

where $\beta_i = \min_i\{d_i^+, d_i^-\}$, if $d_i^+d_i^- \neq 0$ and $\beta_i = \max_i\{d_i^+, d_i^-\}$, if $d_i^+d_i^- = 0$. Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in $\overrightarrow{G}$.

The following observation can be found in [15].

Lemma 2.2. *Let A and B be square complex matrices of order n with singular values*

$$\sigma_1(A) \geq \cdots \geq \sigma_n(A)$$

and

$$\sigma_1(B) \geq \cdots \geq \sigma_n(B).$$

Then

$$\sigma_{i+j-1}(A+B) \leq \sigma_i(A) + \sigma_j(B), \tag{2.1}$$

where

$$\sigma_1(A+B) \geq \cdots \geq \sigma_n(A+B)$$

are the singular values of $A+B$.

The following result gives a relation between $\rho_{sl}(\overrightarrow{G})$ and $\rho_s(\overrightarrow{G})$ for a digraph $\overrightarrow{G}$.

Theorem 2.11. *Let $\rho_{sl}(\overrightarrow{G})$ and $\rho_s(\overrightarrow{G})$ be respectively, the skew Laplacian spectral radius and the skew spectral radius of the digraph $\overrightarrow{G}$. Then*

$$\rho_{sl}(\overrightarrow{G}) \leq \max_i\{|d_i^+ - d_i^-|\} + \rho_s(\overrightarrow{G}),$$

with equality if $\overrightarrow{G}$ is Eulerian.

Proof. Let G be a connected graph of order n having m edges and let $\overrightarrow{G}$ be an orientation of G. Since $\widetilde{SL}(\overrightarrow{G}) = \tilde{D} - S(\overrightarrow{G})$, from Lemma 2.2, it follows that

$$\sigma_1(\widetilde{SL}) \leq \sigma_1(\tilde{D}) + \sigma_1(S(\overrightarrow{G})). \tag{2.2}$$

For any complex matrix X, it is well known [15], that the spectral radius $\rho(X)$ satisfies

$$\rho(X) \leq \sigma_1(X),$$

with equality if and only if X is a normal matrix. Since the matrices $\tilde{D}$ and $S(\overrightarrow{G})$ are normal, from (1), it follows that

$$\rho(\widetilde{SL}) \leq \sigma_1(\widetilde{SL}) \leq \rho(\tilde{D}) + \rho(S(\overrightarrow{G})),$$

which implies that

$$\rho_{sl}(\overrightarrow{G}) \leq \max_i\{|d_i^+ - d_i^-|\} + \rho_s(\overrightarrow{G}).$$

If $\overrightarrow{G}$ is an Eulerian digraph, then $\tilde{D} = 0$ and so

$$\widetilde{SL}(\overrightarrow{G}) = -S(\overrightarrow{G}).$$

Therefore equality occurs in this case. □

Remark 2.2. If $\overrightarrow{G}$ is an Eulerian digraph, the upper bound given by Theorem 2.11, becomes $\rho_{sl}(\overrightarrow{G}) \leq \rho_s(\overrightarrow{G})$. Since $\rho_s(\overrightarrow{G}) \leq \lambda_1$, for Eulerian digraphs, it follows that the upper bound given by Theorem 2.11 is better than the upper bound given by Theorem 2.6.

Acknowledgements. The authors thank the anonymous referees for their valuable comments and suggestions which improved the presentation of the paper. The research of S. Pirzada is supported by the SERB-DST research project number EMR/2015/001047 and MTR/2017/000084.

REFERENCES

[1] Adiga, C., Balakrishnan, R. and So, W., *The skew energy of a digraph*, Linear Algebra Appl., **432** (2010), 1825–1835

[2] Berinde, Z.-M., *Comparing the molecular graph degeneracy of Wiener, Harary, Balaban, Randić and ZEP topological indices*, Creat. Math. Inform., **23** (2014), No. 2, 165–174

[3] Bravo, D., Cubría, F. and Rada, J., *Energy of matrices*, Appl. Math. Comput., **312** (2017), 149–157

[4] Cai, Q., Li, X. and Song, J., *New skew Laplacian energy of simple digraphs*, Trans. Combin., **2** (2013), No. 1, 27–37.

[5] Carvalho, P. and Rama, P., *The modified Schultz index of graph operations*, Creat. Math. Inform., **26** (2017), No. 1, 53–60

[6] Cvetkovic, D., Doob, M. and Sachs, H., *Spectra of Graphs-Theory and Application*, Academic Press, New York, 1980

[7] Ebrahimi Vishki, M. R., Mirzavaziri, K. and Mirzavaziri, M., *k-combinations of an unlabelled graph*, Creat. Math. Inform., **24** (2015), No. 1, 89–95

[8] Ganie, H. A. and Pirzada, S., *On the bounds for signless Laplacian energy of a graph*, Discrete Appl. Math., **228** (2017), 3–13

[9] Ganie, H. A., Alghamdi, A. M. and Pirzada, S., *On the sum of the Laplacian eigenvalues of a graph and Brouwers conjecture*, Linear Algebra Appl., **501** (2016), 376–389

[10] Ganie, H. A. and Chat, B. A., *Skew Laplacian energy of digraphs*, Afr. Mat., **29** (2018), No. 3-4, 499–507

[11] Ganie, H. A., Chat, B. A. and Pirzada, S., *Signless Laplacian energy of a graph and energy of a line graph*, Linear Algebra Appl., **544** (2018) 306–324

[12] Ganie, H. A., Chat, B. A. and Pirzada, S., *On skew Laplacian spectra and skew Laplacian energy of digraphs*, Kragujevac J. Math., **43** (2019), No. 1, 87–98

[13] Ghorbani, M., *Computing the Wiener index of graphs on triples*, Creat. Math. Inform., **24** (2015), No. 1, 49–52

[14] Hou, Y. P. and Fang, A. X., *Unicyclic graphs with reciprocal skew eigenvalues property*, Acta Math. Sinica (Chinese Series), **57** (2014), No. 4, 657–664
[15] Horn, R. and Johnson, C., *Topics in Matrix Analysis*, Cambridge University Press, 1991
[16] Li, X. and Lian, H., *A survey on the skew energy of oriented graphs*, arXiv:1304.5707v6 [math.CO], 18 May 2015
[17] Milovanović, I. Z., Milovanović, E. I., Popović, M. R. and Stanković, R. M., *Remark on the Laplacian-energy-like and Laplacian incidence energy invariants of graphs*, Creat. Math. Inform., **24** (2015), No. 2, 181–185
[18] Milovanović, I. Z., Bekakos, P. M., Bekakos, M. P. and Milovanović, E. I., *Remark on upper bounds of Randić index of a graph*, Creat. Math. Inform., **25** (2016), No. 1, 71–75
[19] Mohammadyari, R. and Darafsheh, M. R., *Topological indices of the double odd graph* $2O_k$, Creat. Math. Inform., **20** (2011), No. 2, 163–170
[20] Mohammadyari, R. and Darafsheh, M. R., *The PI index of polyomino chains of* $2k$*-cycles*, Creat. Math. Inform., **22** (2013), No. 1, 81–86
[21] Pirzada, S, and Ganie, H. A., *On the Laplacian eigenvalues of a graph and Laplacian energy*, Linear Algebra Appl., **486** (2015), 454–468
[22] Pirzada, S., *An Introduction to Graph Theory*, Universities Press, Orient BlackSwan, India, 2012
[23] Shader, B. and So, W., *Skew spectra of oriented graphs*, Electron. J. Combin., **16** (2009), No. 1, Note 32, 6 pp.
[24] Shang, Y., *The natural connectivity of colored random graphs*. Creat. Math. Inform., **20** (2011), no. 2, 197–202
[25] Shekarriz, M. H. and Mirzavaziri, M., *Strong twins of ordinary star-like self-contained graphs*, Creat. Math. Inform., **25** (2016), No. 1, 93–98
[26] Wang, Y. and Zhou, B., *A note on skew spectrum of graphs*, Ars Combin., **110** (2013) 481–485
[27] Xu, G., *Some inequalities on the skew-spectral radii of oriented graphs*, J. Inequal. Appl., (2012), 2012:211, 13 pp.
[28] Xu, G. and Gong, S., *On oriented graphs whose skew spectral radii do not exceed 2*, Linear Algebra Appl., **439** (2013), 2878–2887

[1] Department of Mathematical Sciences
Islamic University of Science and Technology
Awantipora, Kashmir, India
E-mail address: bchat1118@gmail.com

[2] Department of Mathematics
University of Kashmir
Hazratbal, Srinagar, Kashmir, India
E-mail address: hilahmad1119kt@gmail.com
E-mail address: pirzadasd@kashmiruniversity.ac.in