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Some new types multivalued F-contractions on quasi metric spaces and their fixed points

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ABSTRACT. In this paper we present two new results for the existence of fixed points of multivalued mappings with closed values on quasi metric space. First we introduce the multivalued F_d-contraction on quasi metric space (X, d) and give a fixed point result related to this concept. Then taking into account the Q-function on a quasi metric space, we establish a Q-function version of this concept as multivalued F_q-contraction and hence we present a fixed point result to see the effect of Q-function to existence of fixed point of multivalued mappings on quasi metric space.

1. INTRODUCTION

Fundamentally, fixed point theory divides into three major subject which are topological, discrete and metric. Especially, it has been intensively improving on the metric case because of useful to applications. In general, metrical fixed point theory is related to contractive type mappings and it has been developed either taking into account the new type contractions or playing the structure of the space such as fuzzy metric space, quasi metric space, metric like space etc. A quasi metric space plays a crucial role in some fields of theoretical computer service, asymmetric functional analysis and approximation theory. Now, we will recall some basic concepts of quasi metric space.

In quasi metric spaces there are many different types of Cauchyness, yielding even more notions of completeness. Another difference comes from the fact that, in contrast to the metric case, in a quasi metric space a convergent sequence could not be Cauchy (see [4] for examples confirming this situation).

Let X be nonempty set and $d : X \times X \to \mathbb{R}^+$ be a function. Consider the following conditions on d, for all $x, y, z \in X$:

(qm1) $d(x, x) = 0$,
(qm2) $d(x, y) \leq d(x, z) + d(z, y)$,
(qm3) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,
(qm4) $d(x, y) = 0 \Rightarrow x = y$.

If the function d satisfies conditions (qm1) and (qm2) then d is said to be a quasi-pseudo metric on X. Further, if a quasi-pseudo metric d satisfies condition (qm3), then d is said to be a quasi metric on X, and if a quasi metric d satisfies condition (qm4), then d is said to be a T_1-quasi metric on X. In this case, the pair (X, d) is said to be a quasi-pseudo (resp. a quasi, a T_1-quasi) metric space. It is clear that every metric space is a T_1-quasi metric space, but the converse may not be true.

Let (X, d) be a quasi-pseudo metric space. Given a point $x_0 \in X$ and a real constant $\varepsilon > 0$, the set

$$B_d(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\}$$

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is called open ball with center x_0 and radius ε. Each quasi-pseudo metric d on X generates a topology τ_d on X which has a base the family of open balls $\{B_d(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. If d is a quasi metric on X, then τ_d is a T_0 topology, and if d is a T_1-quasi metric, then τ_d is a T_1 topology on X.

If d is a quasi-pseudo metric on X, then the function d^{-1} defined by

$$d^{-1}(x,y) = d(y,x)$$

is a quasi-pseudo metric on X and

$$d^s(x,y) = \max\left\{d(x,y), d^{-1}(x,y)\right\}$$

is a quasi metric. If d is a quasi metric, then d^{-1} is also a quasi metric, and d^s is a metric on X. The closure of a subset A of X with respect to τ_d, $\tau_{d^{-1}}$ and τ_{d^s} are denoted by $cl_d(A)$, $cl_{d^{-1}}(A)$ and $cl_{d^s}(A)$. It is clear that $cl_{d^s}(A) \subseteq cl_d(A)$. We will call a subset A of X as τ_d-closed (τ_d-compact) if it is closed (compact) with respect to τ_d.

Let (X,d) be a quasi metric space, A a nonempty subset of X and $x \in X$. Then

$$x \in cl_d A \Leftrightarrow d(x,A) := \inf\{d(x,a) : a \in A\} = 0.$$

Similarly,

$$x \in cl_{d^{-1}} A \Leftrightarrow d(A,x) := \inf\{d(a,x) : a \in A\} = 0.$$

It is well known that if (X,d) is a metric space and A is a compact subset of X, then for each $x \in X$, there is $a \in A$ such that $d(x,a) = d(x,A)$. However, if (X,d) is a quasi metric space (even if it is a T_1-quasi metric space), this property is not satisfied. (See [4]). Additionally, if A is a $\tau_{d^{-1}}$-compact subset of a quasi metric space (X,d), then for each $x \in X$, there is $a \in A$ such that $d(x,a) = d(x,A)$. The convergence of a sequence $\{x_n\}$ to x with respect to τ_d called d-convergence and denoted by $x_n \overset{d}{\to} x$, is defined

$$x_n \overset{d}{\to} x \Leftrightarrow d(x,x_n) \to 0.$$

Similarly, the convergence of a sequence $\{x_n\}$ to x with respect to $\tau_{d^{-1}}$ called d^{-1}-convergence and denoted by $x_n \overset{d^{-1}}{\to} x$, is defined

$$x_n \overset{d^{-1}}{\to} x \Leftrightarrow d(x_n,x) \to 0.$$

Finally, the convergence of a sequence $\{x_n\}$ to x with respect to τ_{d^s} called d^s-convergence and denoted by $x_n \overset{d^s}{\to} x$, is defined

$$x_n \overset{d^s}{\to} x \Leftrightarrow d^s(x_n,x) \to 0.$$

It is clear that $x_n \overset{d^s}{\to} x \Leftrightarrow x_n \overset{d}{\to} x$ and $x_n \overset{d^{-1}}{\to} x$. More and detailed information about some important properties of quasi metric spaces and their topological structures can be found in [12, 16, 17, 18].

Definition 1.1 ([23]). Let (X,d) be a quasi metric space. A sequence $\{x_n\}$ in X is called

- left K-Cauchy (or forward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$\forall n,k,\ n \geq k \geq n_0,\ d(x_k,x_n) < \varepsilon,$$
- right K-Cauchy (or backward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$\forall n,k,\ n \geq k \geq n_0,\ d(x_n,x_k) < \varepsilon,$$
- d^s-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$\forall n,k \geq n_0,\ d(x_n,x_k) < \varepsilon.$$

If a sequence is left K-Cauchy with respect to d, then it is right K-Cauchy with respect to d^{-1}. A sequence is d^s-Cauchy if and only if it is both left K-Cauchy and right K-Cauchy. Let $\{x_n\}$ be a sequence in a quasi metric space (X, d) such that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty,$$

then it is left K-Cauchy sequence (see [12]).

It is well known that a metric space is said to be complete if every Cauchy sequence is convergent. The completeness of a quasi metric space, however, can not be uniquely defined. Taking into account the convergence and the Cauchyness of sequences in a quasi metric space, one obtains several notions of completeness, most of them being already available in the literature (see [1, 11, 12, 17, 23]) with different notations. It can be found a detailed classification, some important properties and relations for completeness of quasi metric spaces in [4]

Definition 1.2. Let (X, d) be a quasi metric space. Then (X, d) is said to be

- left (right) K-complete if every left (right) K-Cauchy sequence is d-convergent,
- left (right) M-complete if every left (right) K-Cauchy sequence is d^{-1}-convergent,
- left (right) Smyth complete if every left (right) K-Cauchy sequence is d^s-convergent.

Remark 1.1. It is clear that a quasi metric space (X, d) is left M-complete if and only if (X, d^{-1}) is right K-complete. Also, a quasi metric space (X, d) is right M-complete if and only if (X, d^{-1}) is left K-complete.

In [2, 19, 20], considering some contractive conditions with respect to q-function, which is introduced by Al-Hamidan et al. [2], the authors proved some fixed point results on quasi metric space. As understood from the recent papers [2, 19, 20, 21] it is more suitable using the w-distance or Q-function (a slight generalization of w-distance) instead of the quasi metric d in contractive condition.

A Q-function on a quasi metric space (X, d) is a function $q : X \times X \to [0, \infty)$ satisfying the following conditions:

(Q_1) $q(x, z) \le q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(Q_2) if $x \in X$, $M > 0$ and $\{y_n\}$ is a sequence in X such that d^{-1}-converges to a point $y \in X$ and satisfies $q(x, y_n) \le M$ for all $n \in \mathbb{N}$, then $q(x, y) \le M$,

(Q_3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \le \delta$ and $q(x, z) \le \delta$ imply $d(y, z) \le \varepsilon$ (and so $d^s(y, z) \le \varepsilon$).

Note that, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$. It is clear that if (X, d) is a metric space then d is a Q-function on (X, d). However, as it can be seen in [2], if d is a quasi metric, then d may not be a Q-function on (X, d).

On the other hand, the following family of functions, introduced by Wardowski [26], has been thought recently to give more general contractive condition for fixed point theory on metric spaces:

Let $\mathcal{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

(F2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if

$$\lim_{n\to\infty} F(a_n) = -\infty$$

(F3) There exists $k \in (0, 1)$ such that

$$\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0.$$

Many authors have extended fixed point results on metric space by considering the family $\mathcal{F}$. For instance, by inspiration the recent papers as [3, 10], some fixed point results for multivalued mappings which are compact set valued on metric space have been obtained in [6, 7, 8, 9, 13, 24, 25]. Furthermore, in the same papers some fixed point results for multivalued mappings with closed values defined on a metric space have been obtained by adding the following condition:

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by $\mathcal{F}_*$ the set of all functions F satisfying (F1)-(F4).

Dağ et al. [14] proved the quasi metric versions of Theorem 5 and Theorem 6 of [22] and their results also includes the quasi metric version of Feng-Liu's [15] fixed point theorem.

For the sake of completeness we recall the following: Let (X, d) be a quasi metric space. $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X, $\mathcal{C}_d(X)$ denotes the family of all nonempty, τ_d-closed subsets of X and $\mathcal{K}_d(X)$ denotes the family of all nonempty τ_d-compact subsets of X. We will say that a nonempty subset A of X is d-proximinal set if for all $x \in X$ there exists $a \in A$ such that $d(x, A) = d(x, a)$. We indicate the family of all d-proximinal subsets of X by $\mathcal{A}_d(X)$.

If (X, d) is a metric space, then it is clear that $\mathcal{K}_d(X) \subseteq \mathcal{A}_d(X) \subseteq \mathcal{C}_d(X)$. However, if (X, d) is a quasi metric space, then each one of these classes is independent from each other. However, although there is no connection between these classes on quasi metric space, if (X, d) is a T_1-quasi metric space, then $\mathcal{A}_d(X) \subseteq \mathcal{C}_d(X)$ (see [5, 14] for more details). Let $T : X \to \mathcal{P}(X)$ be a multivalued mapping, $F \in \mathcal{F}$ and $\sigma \geq 0$. For $x \in X$ with $d(x, Tx) > 0$, define the set $F_\sigma^x \subseteq X$ as

$$F_\sigma^x = \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\}.$$

Theorem 1.1 ([14]). *Let (X, d) be a left K-complete quasi metric space, $T : X \to \mathcal{C}_d(X)$ be a multivalued mapping and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y, Ty)) \leq F(d(x, y)),$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous with respect to τ_d.

Theorem 1.2 ([14]). *Let (X, d) be a left M-complete quasi metric space, $T : X \to \mathcal{C}_d(X)$ be a multivalued mapping and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y, Ty)) \leq F(d(x, y)),$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous with respect to $\tau_{d^{-1}}$.

In the same study, taking into account the class of $\mathcal{A}_d(X)$ instead of $\mathcal{C}_d(X)$, Dağ et al. [14] removed the condition (F4) on F and presented two fixed point results. However, they need the space to be a T_1-quasi metric space.

In this paper, we introduce two new type multivalued contractions, called multivalued F_d-contraction and multivalued F_q-contraction, on quasi metric space. Then taking into account multivalued F_d-contraction, we present a fixed point result different from Theorem 1.1 for multivalued mappings with closed values by omitting the condition (F4). However, we still need the distance d to be a T_1-quasi metric. To overcome this situation, we consider the multivalued F_q-contraction and provide a new result for multivalued mappings with closed values on quasi metric space.

2. Main results

First we introduce the new contractions, which are mentioned above, for multivalued mappings on quasi metric space.

Definition 2.3. Let (X,d) be a quasi metric space, $T : X \to \mathcal{P}(X)$ and $F \in \mathcal{F}$. Then T is said to be multivalued F_d-contraction if there exists $\tau > 0$ such that for each $x, y \in X$ with $d(x,y) > 0$ and for each $u \in Tx$, there exists $v \in Ty$ satisfying either $d(u,v) = 0$ or $d(u,v) > 0$ such that

$$\tau + F(d(u,v)) \le F(d(x,y)).$$

Definition 2.4. Let q be a Q-function on quasi metric space (X,d), $T : X \to \mathcal{P}(X)$ and $F \in \mathcal{F}$. Then T is said to be multivalued F_q-contraction if for all $x, y \in X$ the following conditions hold:

(i) $q(x,y) = 0$ implies $q_T(x,y) = 0$, where

$$q_T(x,y) = \inf\{q(x,u) : u \in Ty\},$$

(ii) $q(x,y) > 0$ implies there exists $\tau > 0$ such that for each $u \in Tx$, there exists $v \in Ty$ satisfying either $q(u,v) = 0$ or $q(u,v) > 0$ such that

$$\tau + F(q(u,v)) \le F(q(x,y)).$$

Now we present some examples to discuss these concepts.

Example 2.1. Let $X = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $d(x,y) = \max\{y - x, 0\}$, then (X,d) is a quasi metric space. Define $T : X \to \mathcal{P}(X)$ by

$$Tx = \begin{cases} \left\{\frac{1}{2^{n+1}}, 1\right\} & , \quad x = \frac{1}{2^n} \\ \left\{0, \frac{1}{2}\right\} & , \quad x = 0 \end{cases}.$$

Then T is a multivalued F_d-contraction with $F(\alpha) = \ln \alpha$ and $\tau = \ln 2$. Indeed, if $d(x,y) > 0$, then $y > x$ and so there are two cases:

Case 1. $x = 0$ and $y = \frac{1}{2^n}$, then $Tx = \left\{0, \frac{1}{2}\right\}$ and $Ty = \left\{\frac{1}{2^{n+1}}, 1\right\}$. For $u = 0$, by choosing $v = \frac{1}{2^{n+1}} \in Ty$, we have

$$\begin{aligned} \tau + F(d(u,v)) &= \ln 2 + \ln(d(0, \frac{1}{2^{n+1}})) \\ &= \ln \frac{1}{2^n} = \ln(d(0, \frac{1}{2^n})) \\ &= F(d(x,y)). \end{aligned}$$

For $u = \frac{1}{2}$, by choosing $v = \frac{1}{2^{n+1}} \in Ty$, we have $d(u,v) = 0$.

Case 2. $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}$ with $m < n$, then $Tx = \{\frac{1}{2^{n+1}}, 1\}$ and $Ty = \left\{\frac{1}{2^{m+1}}, 1\right\}$. Now for $u = \frac{1}{2^{n+1}}$, by choosing $v = \frac{1}{2^{m+1}} \in Ty$, we have

$$\begin{aligned}\tau + F(d(u,v)) &= \ln 2 + \ln(d(\frac{1}{2^{n+1}}, \frac{1}{2^{m+1}})) \\ &= \ln d(\frac{1}{2^n}, \frac{1}{2^m}) \\ &= F(d(x,y)).\end{aligned}$$

For $u = 1$, by choosing $v = 1 \in Ty$, we have $d(u,v) = 0$.

Example 2.2. Let $X = [0,1]$, $d(x,y) = |x-y|$ and $q(x,y) = y$, then (X,d) is a (quasi) metric space and q is a Q-function on X. Define $T : X \to \mathcal{P}(X)$ by $Tx = [0, x^2]$. Then T is not multivalued F_d-contraction but it is multivalued F_q-contraction with for all $F \in \mathcal{F}$ and for all $\tau > 0$. Indeed, consider $x = 1$ and $y = 0$, then we have $T1 = [0,1]$ and $T0 = \{0\}$. Thus, for $u = 1$ we get $d(u,v) > 0$ and

$$\tau + F(d(u,v)) = \tau + F(1) > F(1) = F(d(x,y))$$

for all $\tau > 0$ and for all $F \in \mathcal{F}$. That is, T is not multivalued F_d-contraction. Now we show T is a multivalued F_q-contraction.

(i) Since $0 \in Ty$ we have $q_T(x,y) = 0$ for all $x, y \in X$. Therefore condition (i) of Definition 2.4 holds.

(ii) It is obvious that $q(u,v) = 0$ for all $x, y \in X$ and for all $u \in Tx$ with $v = 0 \in Ty$. Therefore condition (ii) of Definition 2.4 holds.

Now we give our main results.

Theorem 2.3. *Let (X,d) be a left K-complete T_1-quasi metric space, $T : X \to \mathcal{C}_d(X)$ be a multivalued mapping and $F \in \mathcal{F}$. If T is multivalued F_d-contraction ,then T has a fixed point in X provided that the function $f(x) = d(x, Tx)$ is lower-semicontinuous with respect to τ_d.*

Proof. Fix $x_0 \in X$ and let $x_1 \in Tx_0$. If $d(x_0, x_1) = 0$, then x_0 is a fixed point of T. Now assume that $d(x_0, x_1) > 0$. Since T is multivalued F_d-contraction, then there exists $x_2 \in Tx_1$ satisfying either $d(x_1, x_2) = 0$ or $d(x_1, x_2) > 0$ such that

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

Now, if $d(x_1, x_2) = 0$, then x_1 is a fixed point of T. Assume $d(x_1, x_2) > 0$, then there exists $x_3 \in Tx_2$ satisfying either $d(x_2, x_3) = 0$ or $d(x_2, x_3) > 0$ such that

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

Continuining this process we can construct a sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$ satisfying either

(A) there exists $n_0 \in \mathbb{N}$ with $x_n = x_{n_0}$ for $n \geq n_0$,

or

(B) $d(x_n, x_{n+1}) > 0$ such that

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) \tag{2.1}$$

for all $n \in \mathbb{N}$.

If (A) holds, then it is clear that x_{n_0} is a fixed point of T and also $\{x_n\}$ τ_d-converges to x_{n_0}.

Now assume (B) holds. We will verify that $\{x_n\}$ is left K-Cauchy sequence. From (2.1) inequality, we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. From (2.2) we get

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$$

and so from $(F2)$ we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

From $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) = 0.$$

Then, by (2.2), the following holds for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1})^k [F(d(x_n, x_{n+1})) - F(d(x_0, x_1))] \leq -d(x_n, x_{n+1})^k n\tau \tag{2.3}$$

Letting $n \to \infty$ in (2.3), we obtain that

$$\lim_{n\to\infty} nd(x_n, x_{n+1})^k = 0. \tag{2.4}$$

From (2.4), there exits $n_1 \in \mathbb{N}$ such that $nd(x_n, x_{n+1})^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}. \tag{2.5}$$

In order to show that $\{x_n\}$ is a left K-Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality of d, from (2.5), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Taking into account the convergence of the series

$$\sum_{i=1}^{\infty} \frac{1}{i^{1/k}},$$

we get $d(x_n, x_m) \to 0$ as $n \to \infty$. So $\{x_n\}$ is a left K-Cauchy sequence in the left K-complete quasi metric space (X, d), so there exists $z \in X$ such that $\{x_n\}$ is d-convergent to z, that is, $d(z, x_n) \to 0$ as $n \to \infty$.

On the other hand, since $x_{n+1} \in Tx_n$, we get

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1})$$

and so we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since f is lower semi-continuous with respect to τ_d, then

$$d(z, Tz) = f(z) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Therefore $z \in cl_d(Tz) = Tz$. □

In the following theorem we will consider the concept of multivalued F_q-contraction.

Theorem 2.4. *Let (X, d) be a left M-complete quasi metric space, q be a Q-function on X, $T : X \to \mathcal{C}_d(X)$ be a multivalued mapping and $F \in \mathcal{F}$. If T is multivalued F_q-contraction then, T has a fixed point in X.*

Proof. Fix $x_0 \in X$ and let $x_1 \in Tx_0$.

If $q(x_0, x_1) = 0$, then from (i) of Definition 2.4, we get

$$q_T(x_0, x_1) = \inf\{q(x_0, u) : u \in Tx_1\} = 0.$$

Hence there exists a sequence $\{u_n\}$ in Tx_1 such that $q(x_0, u_n) \to 0$ as $n \to \infty$. Therefore, from $(Q3)$, we have $d(x_1, u_n) \to 0$ as $n \to \infty$ and so $x_1 \in cl_d(Tx_1) = Tx_1$. This shows that x_1 is a fixed point of T.

Assume $q(x_0, x_1) > 0$. Then from (i) of Definition 2.4, there exists $x_2 \in Tx_1$ satisfying either $q(x_1, x_2) = 0$ or $q(x_1, x_2) > 0$ such that

$$\tau + F(q(x_1, x_2)) \leq F(q(x_0, x_1)).$$

Again, if $q(x_1, x_2) = 0$, then from (i) of Definition 2.4, x_2 is a fixed point of T. Assume $q(x_1, x_2) > 0$. Then there exists $x_3 \in Tx_2$ satisfying either $q(x_2, x_3) = 0$ or $q(x_2, x_3) > 0$ such that

$$\tau + F(q(x_2, x_3)) \leq F(q(x_1, x_2)).$$

Continuining this process we can construct a sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$ satisfying either

(C) there exists $n_0 \in \mathbb{N}$ with $x_n = x_{n_0}$ for $n \geq n_0$,

or

(D) $q(x_n, x_{n+1}) > 0$ such that

$$\tau + F(q(x_n, x_{n+1})) \leq F(q(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$.

If (C) holds, then it is clear that x_{n_0} is a fixed point of T and also $\{x_n\}$ τ_d-converges to x_{n_0}.

If (D) holds, as in the proof of Theorem 2.3, then we can obtain

$$\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < \infty.$$

Now let $\varepsilon > 0$ and $0 < \delta < \varepsilon$ for which condition $(Q3)$ is satisfied. Thus there exists $n(\delta) \in \mathbb{N}$ such that

$$\sum_{n=n(\delta)}^{\infty} q(x_n, x_{n+1}) < \delta.$$

Now, for all $n \geq n(\delta)$, we get, from $(Q1)$

$$\begin{aligned} q(x_{n(\delta)}, x_n) &\leq q(x_{n(\delta)}, x_{n(\delta)+1}) + q(x_{n(\delta)+1}, x_{n(\delta)+2}) + \cdots + q(x_{n-1}, x_n) \\ &\leq \sum_{n=n(\delta)}^{\infty} q(x_n, x_{n+1}) < \delta. \end{aligned}$$

Therefore, for all $m, n \geq n(\delta)$, we get $q(x_{n(\delta)}, x_n) < \delta$ and $q(x_{n(\delta)}, x_m) < \delta$. From $(Q3)$, we get $d^s(x_n, x_m) \leq \varepsilon$.

Consequently, $\{x_n\}$ is a d^s-Cauchy and so it is left K-Cauchy sequence in the quasi metric space (X, d). Since (X, d) left $\mathcal{M}$-complete, there exists $z \in X$ such that $\{x_n\}$ is d^{-1}-convergent to z, that is, $d(x_n, z) \to 0$ as $n \to \infty$. On the other hand, for $m > n \geq n(\delta)$ we can get $q(x_n, x_m) < \delta$.

Therefore by$(Q2)$, we get $q(x_n, z) \leq \delta < \varepsilon$ and so $q(x_n, z) \to 0$ as $n \to \infty$. Now we consider the following cases:

Case 1. If there exists $n_0 \in \mathbb{N}$ such that $q(x_{n_0}, z) = 0$, then from (i) of Definition 2.4, we have

$$q_T(x_{n_0}, z) = \inf\{q(x_{n_0}, y) : y \in Tz\} = 0.$$

Therefore there exists a sequence $\{y_n\}$ in Tz such that $q(x_{n_0}, y_n) \to 0$ as $n \to \infty$. Thus from (Q_3), we have $d(z, y_n) \to 0$ as $n \to \infty$. Hence $z \in cl_d(Tz) = Tz$.

Case 2. Assume $q(x_n, z) > 0$ for all $n \in \mathbb{N}$. Then from (ii) of Definition 2.4, for all $n \in \mathbb{N}$, there exists $v_n \in Tz$ such that either $q(z, v_n) = 0$ or $q(z, v_n) > 0$ satisfying

$$\tau + F(q(z, v_n)) \leq F(q(x_n, z)).$$

Thus by taking into account (F1), we have $q(z, v_n) \to 0$ as $n \to \infty$. Therefore, from (Q_1) we get $q(x_n, v_n) \leq q(x_n, z) + q(z, v_n) \to 0$ as $n \to \infty$. From (Q_3), we have $d(z, v_n) \to 0$ and so $z \in cl_d(Tz) = Tz$. □

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