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Hemi-slant submanifolds in metallic Riemannian manifolds

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ABSTRACT. The aim of our paper is to focus on some properties of hemi-slant submanifolds in metallic (and Golden) Riemannian manifolds. We give some characterizations of hemi-slant submanifolds in metallic or Golden Riemannian manifolds and we obtain integrability conditions for the distributions involved. Examples of hemi-slant submanifolds in metallic and Golden Riemannian manifolds are given.

1. INTRODUCTION

The geometry of slant submanifolds in complex manifolds, studied by B. Y. Chen in ([7]) in the early 1990s, was extended to semi-slant submanifold, pseudo-slant submanifold and bi-slant submanifold, respectively, in different types of differentiable manifolds. The pseudo-slant submanifolds (also called hemi-slant submanifolds) in Kenmotsu or nearly Kenmotsu manifolds ([1], [2]) or in locally decomposable Riemannian manifolds ([3]) were studied by M. Atçeken *et al.* Properties of hemi-slant submanifolds in locally product Riemannian manifolds were studied by H. M. Taştan and F. Ozdem in ([15]).

The notion of metallic structure (and, in particular, Golden structure) on a Riemannian manifold was initially studied in ([4], [5], [6], [8],[12],[13],[14]). In ([12]), the authors of the present paper studied the properties of the slant and semi-slant submanifolds in metallic or Golden Riemannian manifolds.

The purpose of the present paper is to investigate the properties of hemi-slant submanifolds in metallic (or Golden) Riemannian manifolds. Using a polynomial structure on a manifold ([9]) and the metallic numbers ([16]), we defined the metallic structure J ([14]). The name of this structure is provided by the metallic number $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ (i.e. the positive solution of the equation $x^2 - px - q = 0$) for positive integer values of p and q. If $\overline{M}$ is an m-dimensional manifold endowed with a tensor field J of type $(1,1)$ such that:

$$J^2 = pJ + qI, \tag{1.1}$$

for p, $q \in \mathbb{N}^*$, where I is the identity operator on the Lie algebra $\Gamma(T\overline{M})$, then the structure J is a *metallic structure*. In this situation, the pair $(\overline{M}, J)$ is called *metallic manifold*.

In particular, if $p = q = 1$ one obtains the *Golden structure* ([8]) determined by a $(1,1)$-tensor field J which verifies $J^2 = J + I$. In this case, $(\overline{M}, J)$ is called *Golden manifold* ([8]).

If $(\overline{M}, \overline{g})$ is a Riemannian manifold endowed with a metallic (or a Golden) structure J, such that the Riemannian metric $\overline{g}$ is J-compatible, i.e.:

$$\overline{g}(JX, Y) = \overline{g}(X, JY), \tag{1.2}$$

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for any $X, Y \in \Gamma(T\overline{M})$, then $(\overline{g}, J)$ is called a *metallic* (or a *Golden) Riemannian structure* and $(\overline{M}, \overline{g}, J)$ is a *metallic (or a Golden) Riemannian manifold* ([14]). Moreover, we have:

$$\overline{g}(JX, JY) = \overline{g}(J^2X, Y) = p\overline{g}(JX, Y) + q\overline{g}(X, Y), \tag{1.3}$$

for any $X, Y \in \Gamma(T\overline{M})$ ([14]).

Any almost product structure F on $\overline{M}$ induces two metallic structures on $\overline{M}$:

$$J = \frac{p}{2}I \pm \frac{2\sigma_{p,q} - p}{2}F, \tag{1.4}$$

where I is the identity operator on the Lie algebra $\Gamma(T\overline{M})$ ([14]).

2. Submanifolds in the metallic Riemannian manifolds

Let M be an m'-dimensional submanifold, isometrically immersed in the m-dimensional metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $m, m' \in \mathbb{N}^*$ and $m > m'$. Let T_xM be the tangent space of M in a point $x \in M$ and $T_x^{\perp}M$ the normal space of M in x. The tangent space $T_x\overline{M}$ can be decomposed into the direct sum: $T_x\overline{M} = T_xM \oplus T_x^{\perp}M$, for any $x \in M$. Let i_* be the differential of the immersion $i : M \to \overline{M}$. The induced Riemannian metric g on M is given by $g(X, Y) = \overline{g}(i_*X, i_*Y)$, for any $X, Y \in \Gamma(TM)$. For the simplification of the notations, in the rest of the paper we shall note by X the vector field i_*X, for any $X \in \Gamma(TM)$. Properties of submanifolds in metallic Riemannian manifolds was studied in ([10]) and ([11]). If we denote by TX and NX, respectively, the tangential and normal parts of JX, for any $X \in \Gamma(TM)$, then we get:

$$JX = TX + NX, \tag{2.1}$$

$T : \Gamma(TM) \to \Gamma(TM)$, $TX := (JX)^T$ and $N : \Gamma(TM) \to \Gamma(T^{\perp}M)$, $NX := (JX)^{\perp}$. For any $V \in \Gamma(T^{\perp}M)$, the tangential and normal parts of JV satisfy:

$$JV = tV + nV, \tag{2.2}$$

$t : \Gamma(T^{\perp}M) \to \Gamma(TM)$, $tV := (JV)^T$ and $n : \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M)$, $nV := (JV)^{\perp}$.

We remark that the maps T and n are $\overline{g}$-symmetric ([5]):

$$(i)\ \overline{g}(TX, Y) = \overline{g}(X, TY), \quad (ii)\ \overline{g}(nU, V) = \overline{g}(U, nV), \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Moreover, we get

$$\overline{g}(NX, U) = \overline{g}(X, tU), \tag{2.4}$$

for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$. By using (2.1), (2.2) and (1.1), we obtain:

Remark 2.1. If M is a submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then:

$$(i)\ T^2X = pTX + qX - tNX, \quad (ii)\ pNX = NTX + nNX, \tag{2.5}$$

$$(i)\ n^2V = pnV + qV - NtV, \quad (ii)\ ptV = TtV + tnV, \tag{2.6}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

For $p = q = 1$ and M is a submanifold in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, for any $X \in \Gamma(TM)$ we get $T^2X = TX + X - tNX$, $NX = NTX + nNX$ and for any $V \in \Gamma(T^{\perp}M)$ we get $n^2V = nV + V - NtV$, $tV = TtV + tnV$.

Remark 2.2. ([11]) Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an almost product structure F and let J be one of the two metallic structures induced by F on $\overline{M}$. If M is a submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ and for any $X \in$

$\Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$ we have $FX = fX + \omega X$, $FV = BV + CV$, with $fX := (FX)^T$, $\omega X := (FX)^{\perp}$, $BV := (FV)^T$ and $CV := (FV)^{\perp}$, then:

$$(i)\; TX = \frac{p}{2}X \pm \frac{2\sigma - p}{2} fX, \quad (ii)\; NX = \pm\frac{2\sigma - p}{2}\omega X \tag{2.7}$$

$$(i)\; tV = \pm\frac{2\sigma - p}{2}BV, \quad (ii)\; nV = \frac{p}{2}V \pm \frac{2\sigma - p}{2}CV. \tag{2.8}$$

In the next considerations we denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $(\overline{M}, \overline{g})$ and its submanifold (M, g), respectively. The Gauss and Weingarten formulas are given by:

$$(i)\; \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (ii)\; \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the shape operator. The second fundamental form and the shape operator verify:

$$\overline{g}(h(X, Y), V) = \overline{g}(A_V X, Y). \tag{2.10}$$

Definition 2.1. ([10]) If $(\overline{M}, \overline{g}, J)$ is a metallic (or Golden) Riemannian manifold and J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e. $\overline{\nabla} J = 0$), we say that $(\overline{M}, \overline{g}, J)$ is a *locally metallic (or locally Golden) Riemannian manifold*.

The covariant derivatives of the tangential and normal parts of JX (and JV), T and N (t and n, respectively) are given by ([10],[1]):

$$(i)\; (\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \quad (ii)\; (\overline{\nabla}_X N)Y = \nabla^{\perp}_X NY - N(\nabla_X Y), \tag{2.11}$$

$$(i)\; (\nabla_X t)V = \nabla_X tV - t(\nabla^{\perp}_X V), \quad (ii)\; (\overline{\nabla}_X n)V = \nabla^{\perp}_X nV - n(\nabla^{\perp}_X V), \tag{2.12}$$

for any X, $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. From $\overline{g}(JX, Y) = \overline{g}(X, JY)$, it follows:

$$\overline{g}((\overline{\nabla}_X J)Y, Z) = \overline{g}(Y, (\overline{\nabla}_X J)Z), \tag{2.13}$$

for any X, Y, $Z \in \Gamma(T\overline{M})$. Moreover, if M is an isometrically immersed submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then ([6]):

$$\overline{g}((\nabla_X T)Y, Z) = \overline{g}(Y, (\nabla_X T)Z), \tag{2.14}$$

for any X, Y, $Z \in \Gamma(TM)$.

Lemma 2.1. ([11]) *If M is a submanifold in a locally metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the covariant derivatives of T and N verify:*

$$(i) (\nabla_X T)Y = A_{NY} X + th(X, Y), \quad (ii)\; (\overline{\nabla}_X N)Y = nh(X, Y) - h(X, TY), \tag{2.15}$$

$$(i) (\nabla_X t)V = A_{nV} X - TA_V X, \quad (ii)\; (\overline{\nabla}_X n)V = -h(X, tV) - NA_V X, \tag{2.16}$$

for any X, $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Remark 2.3. If M is a submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then we obtain:

$$\overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}((\nabla_X t)V, Y), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. From (2.15) (ii) and (2.3) (ii) we get $\overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}(h(X, Y), nV) - \overline{g}(h(X, TY), V) = \overline{g}(A_{nV} X - TA_V X, Y)$ and using (2.16)(i) we obtain (2.17). □

Theorem 2.1. *Let M be a submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then $(\overline{\nabla}_X N)Y = 0$ and $(\nabla_X t)V = 0$, for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$ if and only if the shape operator A verifies:*

$$A_{nV}X = TA_V X = A_V TX. \tag{2.18}$$

Proof. From (2.3)(ii) we get $\overline{g}(nh(X,Y),V) = \overline{g}(h(X,Y),nV)$, for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. Thus, we obtain:

$$\overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}(h(X,Y),nV) - \overline{g}(h(X,TY),V) = \overline{g}(A_{nV}X,Y) - \overline{g}(A_V X, TY),$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. From (2.15)(ii) and (2.10) we have

$$\overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}(A_{nV}X - TA_V X, Y) = \overline{g}(A_{nV}Y - A_V TY, X), \tag{2.19}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. Thus, from (2.19) and (2.17) we obtain the conclusion. □

Theorem 2.2. ([11]) *If M is a submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then:*

$$T([X,Y]) = \nabla_X TY - \nabla_Y TX - A_{NY}X + A_{NX}Y \tag{2.20}$$

$$N([X,Y]) = h(X,TY) - h(TX,Y) + \nabla^{\perp}_X NY - \nabla^{\perp}_Y NX, \tag{2.21}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on $\Gamma(TM)$.

3. Hemi-slant submanifolds in metallic Riemannian manifolds

In this section we recall the definition of a slant distribution and of a bi-slant submanifold in a metallic (or Golden) Riemannian manifold. Then, we define the hemi-slant submanifold and find some properties regarding the distributions involved in this type of submanifold, using a similar definition as for Riemannian product manifold ([15]).

Definition 3.2. ([11]) Let M be an immersed submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. A differentiable distribution D on M is called a *slant distribution* if the angle θ_D between JX_x and the vector subspace D_x is constant, for any $x \in M$ and any nonzero vector field $X_x \in \Gamma(D_x)$. The constant angle θ_D is called the *slant angle* of the distribution D.

Theorem 3.3. ([11]) *Let D be a differentiable distribution on a submanifold M of a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. The distribution D is a slant distribution if and only if there exists a constant $\lambda \in [0,1]$ such that:*

$$(P_D T)^2 X = \lambda(pP_D TX + qX), \tag{3.1}$$

for any $X \in \Gamma(D)$, where P_D is the orthogonal projection on D. Moreover, if θ_D is the slant angle of D, then it satisfies $\lambda = \cos^2\theta_D$.

Definition 3.3. ([11]) Let M be an immersed submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. We say that M is a *bi-slant submanifold* of $\overline{M}$ if there exist two orthogonal differentiable distribution D_1 and D_2 on M such that $TM = D_1 \oplus D_2$, and D_1, D_2 are slant distributions with the slant angles θ_1 and θ_2, respectively. Moreover, M is a *proper bi-slant submanifold* of $\overline{M}$ if $dim(D_1) \cdot dim(D_2) \neq 0$.

Definition 3.4. An immersed submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ is a *hemi-slant submanifold* if there exist two orthogonal distributions D^{θ} and $D^{\perp}$ on M such that:

(1) TM admits the orthogonal direct decomposition $TM = D^{\theta} \oplus D^{\perp}$;

(2) The distribution D^{θ} is slant with angle $\theta \in [0, \frac{\pi}{2}]$;

(3) The distribution $D^{\perp}$ is anti-invariant distribution (i.e. $J(D^{\perp}) \subseteq \Gamma(T^{\perp}M)$).

Moreover, if $dim(D^{\theta}) \cdot dim(D^{\perp}) \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper hemi-slant submanifold.

Remark 3.4. If M is a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with $TM = D^{\theta} \oplus D^{\perp}$, for particular cases we get:

(1) if $\theta = 0$ and $dim(D^{\perp}) = 0$, then M is an invariant submanifold;
(2) if $dim(D^{\theta}) = 0$ or $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold;
(3) if $dim(D^{\perp}) = 0$ and $\theta \neq 0$, then M is a slant submanifold;
(4) if $dim(D^{\theta}) \cdot dim(D^{\perp}) \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.

Remark 3.5. If M is a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with $TM = D^{\theta} \oplus D^{\perp}$, then we get that M is an anti-invariant submanifold if $\theta = \frac{\pi}{2}$ and $g(JX, Y) = 0$, for any $X \in \Gamma(D^{\theta})$ and $X \in \Gamma(D^{\perp})$.

Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with $TM = D^{\theta} \oplus D^{\perp}$ and let P_1 and P_2 be the orthogonal projections on D^{θ} and $D^{\perp}$, respectively. Thus, for any $X \in \Gamma(TM)$, we can consider the decomposition of $X = P_1X + P_2X$, where $P_1X \in \Gamma(D^{\theta})$ and $P_2X \in \Gamma(D^{\perp})$. From $J(D^{\perp}) \subseteq \Gamma(T^{\perp}M)$ we obtain:

Lemma 3.2. *If M is a hemi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, for any $X \in \Gamma(TM)$ we have:*

$$JX = TP_1X + NP_1X + NP_2X = TP_1X + NX \tag{3.2}$$

$$(i) JP_2X = NP_2X, \ (ii) TP_2X = 0, \ (iii) TP_1X \in \Gamma(D^{\theta}). \tag{3.3}$$

Remark 3.6. If M is a hemi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then:

$$T^{\perp}M = N(D^{\theta}) \oplus N(D^{\perp}) \oplus \mu, \tag{3.4}$$

where μ is an invariant subbundle of $T^{\perp}M$.

Proof. For any $X \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$ we get $\overline{g}(NX, NZ) = \overline{g}(JX, JZ) = p\overline{g}(X, TZ) + q\overline{g}(X, Z) = 0$. Thus, the distributions $N(D^{\theta})$ and $N(D^{\perp})$ are mutually perpendicular in $T^{\perp}M$. If we denote by μ the orthogonal complementary subbundle of $J(TM)$ in $T^{\perp}M$, then we obtain (3.4). □

Remark 3.7. If M is a hemi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then: $\overline{g}(JP_1X, TP_1X) = \cos\theta(X)\|TP_1X\| \cdot \|JP_1X\|$ and the cosine of the slant angle $\theta(X) =: \theta$ of the distribution D^{θ} is constant, for any nonzero $X \in \Gamma(TM)$. Thus, for any nonzero $X \in \Gamma(TM)$, we get:

$$\cos\theta = \frac{\overline{g}(JP_1X, TP_1X)}{\|TP_1X\| \cdot \|JP_1X\|} = \frac{\|TP_1X\|}{\|JP_1X\|}. \tag{3.5}$$

Theorem 3.4. *If M is a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, for any $X, Y \in \Gamma(TM)$, we have:*

$$\overline{g}(TP_1X, TP_1Y) = \cos^2\theta[p\overline{g}(TP_1X, P_1Y) + q\overline{g}(P_1X, P_1Y)] \tag{3.6}$$

$$\overline{g}(NX, NY) = \sin^2\theta[p\overline{g}(TP_1X, P_1Y) + q\overline{g}(P_1X, P_1Y)]. \tag{3.7}$$

Proof. Taking $X+Y$ in (3.5) then, for any $X, Y \in \Gamma(TM)$ we have $\overline{g}(TP_1X, TP_1Y) = \cos^2\theta\overline{g}(JP_1X, JP_1Y) = \cos^2\theta[p\overline{g}(JP_1X, P_1Y) + q\overline{g}(P_1X, P_1Y)]$, and using (3.3)(iii) we get (3.6). Thus, from (3.2) we get, for any $X, Y \in \Gamma(TM)$:
$\overline{g}(TP_1X, TP_1Y) = \overline{g}(JP_1X, JP_1Y) - \overline{g}(NX, NY)$ and (3.7) holds. □

Remark 3.8. A hemi-slant submanifold M in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D^θ verifies (3.6) and (3.7) with $p = q = 1$.

Theorem 3.5. *Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D^θ. Then:*

$$(TP_1)^2 = \cos^2\theta(pTP_1 + qI), \tag{3.8}$$

where I is the identity on $\Gamma(D^\theta)$ and

$$\nabla((TP_1)^2) = p\cos^2\theta\nabla(TP_1). \tag{3.9}$$

Remark 3.9. Let M be a hemi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, with $TM = D^\theta \oplus D^\perp$. Then $T(D^\theta) = D^\theta$ and $T(D^\perp) = 0$.

Proof. By using (2.3)(i), we get $\overline{g}(TX, Z) = \overline{g}(X, TZ) = 0$, for any $X \in \Gamma(D^\theta)$, $Z \in \Gamma(D^\perp)$. Thus, $T(D^\theta) \perp D^\perp$. Since $T(D^\theta) \subset \Gamma(TM)$ we obtain that $T(D^\theta) \subseteq D^\theta$. Moreover, from (3.8) we obtain $X = \frac{1}{q}T(TX - p\cos^2\theta X)$, for any $X \in \Gamma(D^\theta)$ (i.e. $P_1X = X$), where $(\overline{M}, \overline{g}, J)$ is a metallic Riemannian manifold. If $(\overline{M}, \overline{g}, J)$ is a Golden Riemannian manifold, then $X = T(TX - \cos^2\theta X)$, for any $X \in \Gamma(D^\theta)$. Thus, $D^\theta \subseteq T(D^\theta)$. Since $T(D^\theta) \subseteq D^\theta$, we get $T(D^\theta) = D^\theta$. By using (3.3)(ii) we obtain that $D^\perp$ is anti-invariant with respect to J and $T(D^\perp) = 0$. □

Theorem 3.6. *Let M be an immersed submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then M is a hemi-slant submanifold in $\overline{M}$ if and only if there exists a constant $\lambda \in [0,1]$ such that $D = \{X \in \Gamma(TM) | T^2X = \lambda(pTX + qX)\}$ is a distribution and $TY = 0$, for any Y orthogonal to D, $Y \in \Gamma(TM)$, where $p, q \in \mathbb{N}^*$.*

Proof. If M is a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with $D^\theta := D$ and $TM = D^\theta \oplus D^\perp$ then, from (3.8) and $\theta(X) \neq 0$ we have $\lambda = \cos^2\theta \in [0,1]$. Conversely, if there exists a real number $\lambda \in [0,1]$ such that $T^2X = \lambda(pTX + qX)$, for any $X \in \Gamma(D)$, it follows that $\cos^2\theta(X) = \lambda$ which implies that $\theta(X) = \arccos(\sqrt{\lambda})$ does not depend on X. If we consider the orthogonal direct sum $TM = D \oplus D^\perp$, since $T(D) \subseteq D$ and $TY = 0$, for any Y orthogonal to D, $Y \in \Gamma(TM)$, we obtain that M is a hemi-slant submanifold in $\overline{M}$ with $D^\theta := D$. □

Example 3.1. Let $\mathbb{R}^4$ be the Euclidean space endowed with the usual Euclidean metric $< \cdot, \cdot >$. Let $f : M \to \mathbb{R}^4$ be the immersion given by: $f(u,v) = (u\cos t, u\sin t, v, \frac{\sigma}{\sqrt{q}}v)$, where $M := \{(u,v) \mid u > 0, t \in (0, \frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number $\overline{\sigma} = p - \sigma$ ($p, q \in N^*$). We can find a local orthonormal frame on TM given by: $Z_1 = \cos t\frac{\partial}{\partial x_1} + \sin t\frac{\partial}{\partial x_2}$, and $Z_2 = \frac{\partial}{\partial x_3} + \frac{\sigma}{\sqrt{q}}\frac{\partial}{\partial x_4}$. We define the metallic structure $J : \mathbb{R}^4 \to \mathbb{R}^4$ by:
$J(X_1, X_2, X_3, X_4) = (\sigma X_1, \overline{\sigma}X_2, \sigma X_3, \overline{\sigma}X_4)$, and we can easily verify that $J^2X = pJ + qI$ and $< JX, Y > = < X, JY >$, for any $X := (X_1, X_2, X_3, X_4)$, $Y := (Y_1, Y_2, Y_3, Y_4) \in \mathbb{R}^4$. We remark that $JZ_2 \perp span\{Z_1, Z_2\}$ and $\cos\theta = \frac{<JZ_1, Z_1>}{\|Z_1\| \cdot \|JZ_1\|} = \frac{\sigma\cos^2 t + \overline{\sigma}\sin^2 t}{\sqrt{\sigma^2\cos^2 t + \overline{\sigma}^2\sin^2 t}}$.

We define the distributions $D^\perp = span\{Z_2\}$ $(J(D^\perp) \subset \Gamma(T^\perp M))$ and $D^\theta = span\{Z_1\}$ is a slant distribution with the slant angle θ. The Riemannian metric tensor of $D^\theta \oplus D^\perp$

is given by $g = du^2 + \frac{p\sigma+2q}{q}dv^2$. Thus, M is a hemi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^4, <\cdot,\cdot>, J)$, with $TM = D^\theta \oplus D^\perp$.

Example 3.2. If we consider $p = q = 1$ in the example 3.1 and $\phi := \sigma_{1,1}$ is the Golden number ($\overline{\phi} := 1-\phi$), for M given in the example 3.1 we define the immersion $f : M \to \mathbb{R}^4$ by $f(u,v) = (u\cos t, u\sin t, v, \phi v)$. The Golden structure $J : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $J(X_1,X_2,X_3,X_4) = (\phi X_1, \overline{\phi}X_2, \phi X_3, \overline{\phi}X_4)$. The distribution $D^\theta = span\{Z_1\}$ has the slant angle $\theta = \arccos \frac{\phi\cos^2 t+\overline{\phi}\sin^2 t}{\sqrt{(\phi\cos^2 t+\overline{\phi}\sin^2 t)+1}}$ and $D^\perp = span\{Z_2\}$. The Riemannian metric tensor of $D^\theta \oplus D^\perp$ is given by $g = du^2 + (\phi+2)dv^2$. Thus, M is a hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^4, <\cdot,\cdot>, J)$.

Example 3.3. If M and f are the same as in the example 3.1, we define the metallic structure $\overline{J} : \mathbb{R}^4 \to \mathbb{R}^4$ given by $\overline{J}(X_1,X_2,X_3,X_4) = (\sigma X_1, \sigma X_2, \sigma X_3, \overline{\sigma}X_4)$. We obtain: $\overline{J}Z_1 = \sigma Z_1$, the distributions $D^\perp = span\{Z_2\}$ and $D^\theta = span\{Z_1\}$ has the slant angle $\theta = 0$. Thus, $TM = D^\theta \oplus D^\perp$ and M is a semi-invariant submanifold in the metallic Riemannian manifold $(\mathbb{R}^4, <\cdot,\cdot>, \overline{J})$. Similarly, for $p = q = 1$ we obtain that M is a semi-invariant submanifold in the Golden Riemannian manifold $(\mathbb{R}^4, <\cdot,\cdot>, \overline{J})$.

Example 3.4. Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric $<\cdot,\cdot>$. Let $f : M \to \mathbb{R}^7$ be the immersion given by:

$$f(u,v,w) = \left(\frac{1}{\sqrt{3}}u\cos t, \frac{1}{\sqrt{3}}u\sin t, v, \frac{\sigma}{\sqrt{q}}v, \frac{\sqrt{q}}{\sigma}w, w, \frac{\sqrt{2}}{\sqrt{3}}u\right),$$

where $M := \{(u,v,w) \mid u > 0, t \in (0,\frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q}$ is the metallic number ($p, q \in N^*$). We can find a local orthonormal frame on TM given by: $Z_1 = \frac{1}{\sqrt{3}}\cos t\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{3}}\sin t\frac{\partial}{\partial x_2} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\partial}{\partial x_7}$, $Z_2 = \frac{\partial}{\partial x_3} + \frac{\sigma}{\sqrt{q}}\frac{\partial}{\partial x_4}$, and $Z_3 = \frac{\sqrt{q}}{\sigma}\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}$. We define the metallic structure $J : \mathbb{R}^7 \to \mathbb{R}^7$ by: $J(X_1,X_2,X_3,X_4,X_5,X_6,X_7) = (\sigma X_1, \overline{\sigma}X_2, \sigma X_3, \overline{\sigma}X_4, \sigma X_5, \overline{\sigma}X_6, \sigma X_7)$ and we can easily verify that $J^2X = pJ + qI$ and $< JX, Y >=< X, JY >$, for any $X := (X_1,X_2,X_3,X_4,X_5,X_6,X_7)$, $Y := (Y_1,Y_2,Y_3,Y_4,Y_5,Y_6,Y_7) \in \mathbb{R}^7$. We find that $JZ_2 \perp span\{Z_1,Z_2,Z_3\}$ and $JZ_3 \perp span\{Z_1,Z_2,Z_3\}$. Thus, we get $\cos\theta = \frac{\sigma(\cos^2 t+2)+\overline{\sigma}\sin^2 t}{\sqrt{3[\sigma^2(\cos^2 t+2)+\overline{\sigma}^2\sin^2 t]}}$.

We define the distributions $D^\perp = span\{Z_2, Z_3\}$ ($J(D^\perp) \subset \Gamma(T^\perp M)$) and $D^\theta = span\{Z_1\}$ is a slant distribution, with the slant angle θ. The Riemannian metric tensor of $D^\theta \oplus D^\perp$ is given by $g = du^2 + \frac{p\sigma+2q}{q}dv^2 + \frac{p\sigma+2q}{p\sigma+q}dw^2$. Thus, $TM = D^\theta \oplus D^\perp$ and M is a hemi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^7, <\cdot,\cdot>, J)$.

Example 3.5. We consider $p = q = 1$ in the example 3.4 and $\phi := \sigma_{1,1}$ is the Golden number ($\overline{\phi} := 1-\phi$). We define, for M given in the example 3.1, the immersion $f : M \to \mathbb{R}^7$ by

$$f(u,v,w) = (\frac{1}{\sqrt{3}}u\cos t, \frac{1}{\sqrt{3}}u\sin t, v, \phi v, \overline{\phi}w, w, \frac{\sqrt{2}}{\sqrt{3}}u),$$

and the Golden structure $J : \mathbb{R}^7 \to \mathbb{R}^7$ by

$$J(X_1,X_2,X_3,X_4,X_5,X_6,X_7) = (\phi X_1, \overline{\phi}X_2, \phi X_3, \overline{\phi}X_4, \phi X_5, \overline{\phi}X_6, \phi X_7).$$

The distributions $D^\perp = span\{Z_2, Z_3\}$ verifies $J(D^\perp) \subset \Gamma(T^\perp M)$ and the slant distribution $D^\theta = span\{Z_1\}$ has the slant angle $\theta = \arccos\frac{\phi(\cos^2 t+2)+\overline{\phi}\sin^2 t}{\sqrt{3[\phi^2(\cos^2 t+2)+\overline{\phi}^2\sin^2]}}$. The Riemannian metric tensor of $D^\theta \oplus D^\perp$ is given by $g = du^2 + (\phi+2)dv^2 + \frac{\phi+2}{\phi+1}dw^2$. Thus, M is a hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^7, <\cdot,\cdot>, J)$.

Example 3.6. If M and f are the same as in the example 3.4 and the metallic structure $\overline{J}$: $\mathbb{R}^7 \to \mathbb{R}^7$ is defined by $\overline{J}(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = (\sigma X_1, \sigma X_2, \sigma X_3, \overline{\sigma} X_4, \sigma X_5, \overline{\sigma} X_6, \sigma X_7)$. then we get: $\overline{J} Z_1 = \sigma Z_1$. We obtain the distributions $D^\perp = span\{Z_2, Z_3\}$ and $D^\theta = span\{Z_1\}$ with the slant angle $\theta = 0$. Thus, $TM = D^\theta \oplus D^\perp$ and M is a semi-invariant submanifold in the metallic Riemannian manifold $(\mathbb{R}^7, < \cdot, \cdot >, \overline{J})$. Similarly, for $p = q = 1$ we obtain that M is a semi-invariant submanifold in the Golden Riemannian manifold $(\mathbb{R}^7, < \cdot, \cdot >, \overline{J})$.

4. On the integrability of the distributions of a hemi-slant submanifold

In this section we investigate the conditions for the integrability of the distributions of a hemi-slant submanifold in a metallic (or Golden) Riemannian manifold.

Theorem 4.7. *If M is a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then*

$$\nabla_X TY - \nabla_Y TX - A_{NY} X + A_{NX} Y \in \Gamma(D^\theta), \tag{4.1}$$

for any $X, Y \in \Gamma(D^\theta)$.

Proof. By using (2.3)(i), we obtain: $\overline{g}(T([X,Y]), Z) = \overline{g}([X,Y], TZ) = 0$, for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$ (i.e. $TZ = 0$). Thus, $T([X,Y]) \in \Gamma(D^\theta)$ and from (2.20) we get (4.1). □

Theorem 4.8. *If M is a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the distribution D^θ is integrable.*

Proof. By using (1.3), we have $\overline{g}(\overline{\nabla}_X Y, Z) = \frac{1}{q}[\overline{g}(J\overline{\nabla}_X Y, JZ) - p\overline{g}(\overline{\nabla}_X Y, JZ)]$, for any $X, Y \in \Gamma(D^\theta)$, $Z \in \Gamma(D^\perp)$. From $\overline{\nabla} J = 0$ we get $J\overline{\nabla}_X Y = \overline{\nabla}_X JY$ and using $JZ = NZ$, for any $Z \in \Gamma(D^\perp)$, we obtain $q\overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(\overline{\nabla}_X JY, NZ) - p\overline{g}(\overline{\nabla}_X Y, NZ)$. From (2.9) and (2.10) we get $q\overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(h(X, TY), NZ) + \overline{g}(\nabla^\perp_X NY, NZ) - p\overline{g}(h(X,Y), NZ)$. From (2.11)(ii) and (2.15)(ii) we obtain $\nabla^\perp_X NY = nh(X,Y) - h(X, TY) + N\nabla_X Y$, for any $X, Y \in \Gamma(D^\theta)$. From $q\overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(nh(X,Y), NZ) + \overline{g}(N\nabla_X Y, NZ) - p\overline{g}(h(X,Y), NZ)$, we get $q\overline{g}([X,Y], Z) = \overline{g}(N\nabla_X Y, NZ) - \overline{g}(N\nabla_Y X, NZ) = \overline{g}(N[X,Y], NZ)$, for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. Thus, from (3.7) and (2.3)(i) we have

$$q\overline{g}([X,Y], Z) = \sin^2\theta[p\overline{g}(P1[X,Y], TP_1 Z) + q\overline{g}(P1[X,Y], P_1 Z)].$$

By using $P_1 Z = 0$ for any $Z \in \Gamma(D^\perp)$ (where $P_1 Z$ is the projection of Z on $\Gamma(D^\theta)$), we obtain $\overline{g}([X,Y], Z) = 0$, for any $X, Y \in \Gamma(D^\theta)$, $Z \in \Gamma(D^\perp)$ which implies that $[X,Y] \in \Gamma(D^\theta)$. □

Theorem 4.9. *Let M be a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then the distribution $D^\perp$ is integrable if and only if, for any $Z, W \in \Gamma(D^\perp)$ we have*

$$A_{NZ} W = 0. \tag{4.2}$$

Proof. If M is a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, for any $Z, W \in \Gamma(D^\perp)$ we have $TZ = TW = 0$ which implies $\nabla_Z TW = \nabla_W TZ = 0$. By using (3.3)(ii) and (2.20) we get $T([Z,W]) = 0$ if and only if $A_{NZ} W = A_{NW} Z$ holds, for any $Z, W \in \Gamma(D^\perp)$. From (2.15)(i), for any $X \in \Gamma(TM)$ and $Z, W \in \Gamma(D^\perp)$, we get $\overline{g}(A_{NZ} X, W) + \overline{g}(th(X,Z), W) = \overline{g}((\nabla_X T)Z, W) = -\overline{g}(\nabla_X Z, TW) = 0$, which implies $\overline{g}(A_{NZ} X, W) = -\overline{g}(th(X,Z), W)$. From

$$\overline{g}(A_{NZ} X, W) = \overline{g}(A_{NZ} W, X) = \overline{g}(A_{NW} Z, X) = \overline{g}(h(X,Z), NW) = \overline{g}(th(X,Z), W),$$

we obtain $\overline{g}(A_{NZ}W, X) = 0$ for any $X \in \Gamma(TM)$ and $Z, W \in \Gamma(D^{\perp})$. Thus, (4.2) holds. Conversely, if $A_{NZ}W = 0$, for any $Z, W \in \Gamma(D^{\perp})$ then $\overline{g}(th(X, Z), W) = \overline{g}(A_{NW}Z, X) = 0$ and from (2.15)(i) we get $0 = \overline{g}((\nabla_Z T)W, X) = \overline{g}(T\nabla_Z W, X) = \overline{g}(\nabla_Z W, TX)$, for any $Z, W \in \Gamma(D^{\perp})$, $X \in \Gamma(D^{\theta})$. From $T(D^{\theta}) = D^{\theta}$, we obtain $\nabla_Z W \in \Gamma(D^{\perp})$ which implies $[Z, W] \in \Gamma(D^{\perp})$. □

Theorem 4.10. *Let M be a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, the anti-invariant distribution $D^{\perp}$ is integrable if and only if, for any $Z, W \in \Gamma(D^{\perp})$ we have*

$$(\nabla_Z T)W = (\nabla_W T)Z. \tag{4.3}$$

Proof. By using (2.13) we get $(\nabla_Z T)W - (\nabla_W T)Z = A_{NW}Z - A_{NZ}W$, for any $Z, W \in \Gamma(D^{\perp})$ and using (4.2) we obtain the conclusion. □

Remark 4.10. Let M be a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $(\nabla_Z T)W = 0$, for any $Z, W \in \Gamma(D^{\perp})$, then $D^{\perp}$ is integrable.

Theorem 4.11. *Let M be a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $(\overline{\nabla}_X N)Y = 0$, for any $X, Y \in \Gamma(D^{\theta})$ then, either M is a D^{θ} geodesic submanifold (i.e $h(X, Y) = 0$) or $h(X, Y)$ is an eigenvector of n, with eigenvalues*

$$\lambda_1 = \frac{p\cos^2\theta + \cos\theta\sqrt{p^2\cos^2\theta + 4q}}{2}, \quad \lambda_2 = \frac{p\cos^2\theta - \cos\theta\sqrt{p^2\cos^2\theta + 4q}}{2}. \tag{4.4}$$

Proof. By using $(\overline{\nabla}_X N)Y = 0$ for any $X, Y \in \Gamma(D^{\theta})$ and (2.15)(ii) we obtain $nh(X, Y) = h(X, TY)$. From (3.8) we get $n^2h(X, Y) = h(X, T^2Y) = p\cos^2\theta nh(X, Y) + q\cos^2\theta h(X, Y)$, for any $X, Y \in \Gamma(D^{\theta})$. Thus, we obtain either M is a D^{θ} geodesic submanifold or $h(X, Y)$ is an eigenvector of n with eigenvalue λ, which verifies $\lambda^2 - p\cos^2\theta\lambda - q\cos^2\theta = 0$ and (4.1) holds. □

5. Mixed totally geodesic hemi-slant submanifolds

We consider hemi-slant submanifolds in a locally metallic (or locally Golden) Riemannian manifold and we find some conditions for these submanifolds to be $D^{\theta} - D^{\perp}$ mixed totally geodesic (i.e. $h(X, Y) = 0$, for any $X \in \Gamma(D^{\theta})$ and $Y \in \Gamma(D^{\perp})$).

Theorem 5.12. *If M is a hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then M is a $D^{\theta} - D^{\perp}$ mixed totally geodesic submanifold if and only if $A_V X \in \Gamma(D^{\theta})$ and $A_V Y \in \Gamma(D^{\perp})$, for any $X \in \Gamma(D^{\theta})$, $Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$.*

Proof. From $\overline{g}(A_V X, Y) = \overline{g}(A_V Y, X) = \overline{g}(h(X, Y), V)$, for any $X \in \Gamma(D^{\theta}), Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$ we obtain that M is a $D^{\theta} - D^{\perp}$ mixed totally geodesic submanifold in the locally metallic (or locally Golden) Riemannian manifold if and only if $A_V X \in \Gamma(D^{\theta})$ and $A_V Y \in \Gamma(D^{\perp})$, for any $X \in \Gamma(D^{\theta})$, $Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$. □

Theorem 5.13. *Let M be a proper hemi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $(\overline{\nabla}_X N)Z = 0$, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$, then M is a $D^{\theta} - D^{\perp}$ mixed totally geodesic submanifold in $\overline{M}$.*

Proof. If $X \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$ then, from $(\overline{\nabla}_X N)Z = 0$, (2.15)(ii) and $TZ = 0$ we get $h(Z, TX) = nh(X, Z) = h(X, TZ) = 0$. From $n^2h(Z, X) = h(Z, T^2X) = 0$ and (3.8) we get $p\cos^2\theta nh(Z, TX) + q\cos^2\theta h(Z, X) = 0$. From $nh(Z, TX) = 0$ and $\theta \neq \frac{\pi}{2}$ and $q \neq 0$, we obtain $h(X, Z) = 0$, for any $X \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$. □

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