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Bounds for some entropies and special functions

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ABSTRACT. We consider a family of probability distributions depending on a real parameter and including the binomial, Poisson and negative binomial distributions. The corresponding index of coincidence satisfies a Heun differential equation and is a logarithmically convex function. Combining these facts we get bounds for the index of coincidence, and consequently for Rényi and Tsallis entropies of order 2.

1. INTRODUCTION

For $c \in \mathbb{R}$, let $I_c := \left[0, -\frac{1}{c}\right]$ if $c < 0$, and $I_c := [0, +\infty)$ if $c \geq 0$.

Let $a \in \mathbb{R}$ and $k \in \mathbb{N}_0$; the binomial coefficients are defined as usual by

$$\binom{a}{k} := \frac{a(a-1)\dots(a-k+1)}{k!}$$

if $k \in \mathbb{N}$, and $\binom{a}{0} := 1$.

Consider also a real number $n > 0$ such that $n > c$ if $c \geq 0$, and $n = -cl$ for some $l \in \mathbb{N}$ if $c < 0$.

For $k \in \mathbb{N}_0$ and $x \in I_c$ define

$$p_{n,k}^{[c]}(x) := \binom{-\frac{n}{c}}{k}(-cx)^k(1+cx)^{-\frac{n}{c}-k}, \text{ if } c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!}e^{-nx}.$$

These functions were intensively used in Approximation Theory: see [3], [8], [22] and the references therein.

In particular,

$$\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1,$$

so that $\left(p_{n,k}^{[c]}(x)\right)_{k \geq 0}$ is a parameterized probability distribution.

Its index of coincidence (see [7]) is

$$S_{n,c}(x) := \sum_{k=0}^{\infty} \left(p_{n,k}^{[c]}(x)\right)^2, \quad x \in I_c.$$

The Rényi entropy of order 2 and the Tsallis entropy of order 2 are given, respectively, by (see [18], [20])

$$R_{n,c}(x) := -\log S_{n,c}(x); \quad T_{n,c}(x) := 1 - S_{n,c}(x),$$

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while the associated Shannon entropy is

$$H_{n,c}(x) := -\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x), \quad x \in I_c.$$

The cases $c = -1$, $c = 0$, $c = 1$ correspond, respectively, to the binomial, Poisson, and negative binomial distributions; see also [13], [14].

It was proved in [12], [15] that the index of coincidence $S_{n,c}$ satisfies the Heun differential equation

$$\begin{aligned} x(1+cx)(1+2cx)S''_{n,c}(x) + (4(n+c)x(1+cx)+1)\, S'_{n,c}(x) + \\ +2n(1+2cx)S_{n,c}(x) = 0, \quad x \in I_c. \end{aligned} \tag{1.1}$$

It was conjectured in [11] and proved in several papers (for details see [1], [4], [10], [12], [15], [16] and the references given there) that $S_{n,c}$ is a convex function, i.e.,

$$S''_{n,c}(x) \geq 0, \quad x \in I_c. \tag{1.2}$$

It is easy to combine (1.1) and (1.2) in order to get

$$S_{n,c}(x) \leq (4(n+c)x(1+cx)+1)^{-\frac{n}{2(n+c)}}, \quad x \in I_c. \tag{1.3}$$

Particular cases and related results can be found in [12], [15]. Let us remark also that $S_{n,c}(0) = 1$.

The upper bound for $S_{n,c}$, given by (1.3), leads obviously to lower bounds for the Rényi entropy $R_{n,c}$ and the Tsallis entropy $T_{n,c}$.

The following conjecture was formulated in [12] and [15]:

Conjecture 1.1. *For $c \in \mathbb{R}$, $S_{n,c}$ is a logarithmically convex function, i.e., $\log S_{n,c}$ is convex.*

For $c \geq 0$, U. Abel, W. Gawronski and Th. Neuschel obtained a stronger result:

Theorem 1.1. ([1]) *For $c \geq 0$ the function $S_{n,c}$ is completely monotonic, i.e.,*

$$(-1)^j S_{n,c}^{(j)}(x) > 0, \quad x \geq 0, \quad j \geq 0.$$

Consequently, for $c \geq 0$, $S_{n,c}$ is logarithmically convex.

The following corollary can be found in [16]:

Corollary 1.1. ([16])

i) Let $c \geq 0$. Then $R_{n,c}$ is increasing and concave, while $T'_{n,c}$ is completely monotonic on $[0, +\infty)$.
ii) $T_{n,c}$ is concave for all $c \in \mathbb{R}$.

Let us remark that the complete monotonicity for the Shannon entropy $H_{n,c}$ was investigated in [16], and for other entropies in [23].

In Sections 2 and 3 we shall use (1.1) in connection with the log-convexity of $S_{n,c}$, $c \geq 0$, in order to obtain upper-bounds for $S_{n,c}$, sharper than (1.3); they can be immediately converted into sharp lower-bounds for the Rényi entropy and the Tsallis entropy.

Theorem 3.3 provides an upper bound for the modified Bessel function of first kind of order 0.

Section 4 is devoted to the case $c < 0$. In this case Conjecture 1.1 was proved in [17], so that it is again possible to obtain upper-bounds for $S_{n,c}$, sharper than (1.3).

On the other hand (see [10]), $S_{n,-1}$ is related to the Legendre polynomials P_n; using results from [10] we obtain bounds for $S_{n,-1}$ and P_n.

Sharp bounds on other entropies can be found in [2], [7], [19], [21] and the references therein.

2. THE CASE $c > 0$

According to Theorem 1.1, $\log S_{n,c}(x)$, $x \in [0, +\infty)$, is a convex function, i.e.,

$$S''_{n,c}(x) \geq \left(S'_{n,c}(x)\right)^2 / S_{n,c}(x), \quad x \in [0, +\infty). \tag{2.4}$$

Denote $X := x(1 + cx)$, and therefore $X' = 1 + 2cx$. Then (1.1) becomes

$$XX'S''_{n,c}(x) + (4(n + c)X + 1)\, S'_{n,c}(x) + 2nX'S_{n,c}(x) = 0. \tag{2.5}$$

From (2.4) and (2.5) we infer that

$$XX'\left(\frac{S'_{n,c}}{S_{n,c}}\right)^2 + (4(n + c)X + 1)\,\frac{S'_{n,c}}{S_{n,c}} + 2nX' \leq 0. \tag{2.6}$$

This implies

$$\frac{S'_{n,c}(x)}{S_{n,c}(x)} \leq \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2XX'}, \tag{2.7}$$

and, since $S_{n,c}(0) = 1$,

$$\log S_{n,c}(t) \leq \int_0^t \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2XX'}\text{dx}. \tag{2.8}$$

Note that $X'^2 = 1 + 4cX$. Now (2.8) becomes

$$\log S_{n,c}(t) \leq \int_0^T \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2X(1 + 4cX)}\text{dX}. \tag{2.9}$$

where $T := t + ct^2$, $t \geq 0$.

Moreover, denoting $\rho := \sqrt{n^2 + c^2}$ and $R := \sqrt{16\rho^2T^2 + 8cT + 1}$, we have

Theorem 2.2. *The following inequalities hold in the case $c > 0$:*

$$S^2_{n,c}(t) \leq \frac{2}{1 + 4cT + R}\left(\frac{1}{R + 4nT}\right)^{n/c}\left(\frac{\rho R + 4\rho^2 T + c}{\rho + c}\right)^{\rho/c} \leq$$

$$\leq \frac{1}{1 + 4cT}\,(1 + 4(n + c)T)^{-n/c}\left(1 + 8\sqrt{n^2 + c^2}T\right)^{\sqrt{n^2+c^2}/c}. \tag{2.10}$$

Consequently,

$$S_{n,c}(t) = \mathcal{O}\left(t^{\frac{\sqrt{n^2+c^2}-n-c}{c}}\right), \quad t \to \infty. \tag{2.11}$$

Proof. The first inequality in (2.10) follows from (2.9) by a straightforward calculation. In order to get the second one it suffices to use the inequalities $1 + 4cT \leq R \leq 1 + 4\rho T$. □

Remark 2.1. The inequality (2.4) is stronger than (1.2); therefore, the bound for $S_{n,c}$ given in (2.10) is sharper than the bound given in (1.3). In particular, (1.3) yields

$$S_{n,c}(t) = \mathcal{O}\left(t^{-\frac{n}{n+c}}\right), \quad t \to \infty,$$

and comparing with (2.11) we see that

$$\frac{\sqrt{n^2 + c^2} - n - c}{c} < -\frac{n}{n + c}.$$

3. The case $c = 0$.

The relations (2.4) - (2.9) are still valid with obvious simplifications induced by $c = 0$. In particular, (2.9) reduces to

$$\log S_{n,0}(t) \le \int_0^t \frac{\sqrt{1+16n^2x^2} - 1 - 4nx}{2x} \mathrm{dx},$$

and this yields

$$S_{n,0}^2(t) \le \frac{2\exp\left(\sqrt{1+16n^2t^2} - 1 - 4nt\right)}{1+\sqrt{1+16n^2t^2}}, \quad t \ge 0. \tag{3.12}$$

This bound for $S_{n,0}$ is sharper than the bound furnished by (1.3) with $c = 0$.

By using (3.12) we get also

Theorem 3.3. *Let $I_0(t)$, $t \ge 0$, be the modified Bessel function of first kind of order 0. Then*

$$I_0^2(t) \le \frac{2\exp\left(\sqrt{1+4t^2} - 1\right)}{\sqrt{1+4t^2}+1}, \quad t \ge 0. \tag{3.13}$$

Proof. According to [15, (12)],

$$I_0(t) = e^t S_{n,0}\left(\frac{t}{2n}\right). \tag{3.14}$$

Now (3.13) is a consequence of (3.12) and (3.14). □

4. The case $c < 0$.

As mentioned in the Introduction, in this case Conjecture 1.1 was proved in [17]. Consequently, with the same notation and the same proof as in Theorem 2.2, we get

Theorem 4.4. *The following inequality holds for all $c < 0$ and $t \in \left[0, -\frac{1}{c}\right]$:*

$$S_{n,c}^2(t) \le \frac{2}{1+4cT+R}\left(\frac{1}{R+4nT}\right)^{n/c}\left(\frac{\rho R + 4\rho^2 T + c}{\rho + c}\right)^{\rho/c}.$$

Since the log-convexity of $S_{n,c}$ implies the convexity, the above inequality is sharper than (1.3). Remember that if $c < 0$, then $n = -cl$ for some $l \in \mathbb{N}$. It follows that

$$S_{n,c}(t) = S_{l,-1}(-ct), \quad t \in \left[0, -\frac{1}{c}\right].$$

Consequently, in what follows we shall investigate only the function $S_{n,-1}(x)$ with $n \in \mathbb{N}$ and $x \in [0,1]$.

G. Nikolov proved in [10, Theorem 3] that the Legendre polynomials $P_n(t)$ satisfy the inequalities

$$\frac{n(n+1)}{2t + (n-1)\sqrt{t^2-1}} \le \frac{P_n'(t)}{P_n(t)} \le \frac{n^2(2n+1)}{(n+1)t + (2n^2-1)\sqrt{t^2-1}}, \quad t \ge 1. \tag{4.15}$$

Let

$$X := x(1-x), \quad t = \frac{2x^2 - 2x + 1}{1-2x} = \frac{1-2X}{X'}, \quad x \in \left[0, \frac{1}{2}\right).$$

Then $t \ge 1$ and (see [13, (2.9)], [15, Section 4])

$$\frac{P_n'(t)}{P_n(t)} = \frac{nX'}{2X} + \frac{1-4X}{4X}\frac{S_{n,-1}'(x)}{S_{n,-1}(x)}. \tag{4.16}$$

From (4.15) and (4.16) we obtain

$$-\frac{2nX'}{1+(n-3)X} \le \frac{S'_{n,-1}(x)}{S_{n,-1}(x)} \le -\frac{2n(n+1)X'}{n+1+(4n^2-2n-4)X}, \quad x \in \left[0, \frac{1}{2}\right]. \tag{4.17}$$

Let $t \in \left[0, \frac{1}{2}\right]$. By integrating in (4.17) with respect to $x \in [0,t]$ it follows that

$$(1+(n-3)T)^{-\frac{2n}{n-3}} \le S_{n,-1}(t) \le \left(1 + \frac{4n^2-2n-4}{n+1}T\right)^{-\frac{n(n+1)}{2n^2-n-2}}, \tag{4.18}$$

where $T = t(1-t)$ and for $n=3$ the left-hand side is e^{-6T}. Since $S_{n,-1}(1-t) = S_{n,-1}(t)$, (4.18) is valid for $t \in [0,1]$.

Remark 4.2. For $c=-1$, (1.3) is a consequence of the inequality

$$\frac{S'_{n,-1}(x)}{S_{n,-1}(x)} \le -\frac{2nX'}{1+4(n-1)X}, \quad x \in \left[0, \frac{1}{2}\right]. \tag{4.19}$$

Comparing (4.19) with (4.17), we get

$$-\frac{2n(n+1)X'}{n+1+(4n^2-2n-4)X} \le -\frac{2nX'}{1+4(n-1)X}, \quad x \in \left[0, \frac{1}{2}\right],$$

and so the second inequality (4.18) is sharper than (1.3) with $c=-1$.

Remark 4.3. According to [15, (29)],

$$S_{n,-1}(t) = \frac{1}{\pi}\int_0^1 \left(x+(1-x)(1-2t)^2\right)^n \frac{dx}{\sqrt{x(1-x)}}, \quad t \in [0,1].$$

It follows that

$$S_{n,-1}(t) \ge \frac{2}{\pi}\int_0^1 \left(x+(1-x)(1-2t)^2\right)^n dx,$$

which leads to

$$\frac{1-(1-4T)^{n+1}}{2\pi(n+1)T} \le S_{n,-1}(t), \quad t \in [0,1].$$

This inequality is comparable with the first inequality (4.18).

The following results ca be found also in [13].

Consider the inequality

$$\frac{P'_n(t)}{P_n(t)} \le \frac{2n^2}{t+(2n-1)\sqrt{t^2-1}}, \quad t \ge 1, \tag{4.20}$$

which was established in [10, Theorem 2]. As remarked in [10], (4.15) is stronger than (4.20).

From (4.20) we get by integration

$$P_n(t) \le (t+\sqrt{t^2-1})^{\frac{n(2n-1)}{2(n-1)}} \left(t+(2n-1)\sqrt{t^2-1}\right)^{-\frac{n}{2(n-1)}}, \quad t \ge 1, n \ge 2.$$

The stronger inequality

$$P_n(t) \le (t+\sqrt{t^2-1})^{\frac{n(2n^2-1)}{2n^2-n-2}} \left(t+\frac{2n^2-1}{n+1}\sqrt{t^2-1}\right)^{-\frac{n(n+1)}{2n^2-n-2}}, \quad t \ge 1$$

is a consequence of (4.15).

5. Concluding remarks and further work

The index of coincidence $S_{n,c}$ is intimately related with the Renyi entropy $R_{n,c}$, Tsallis entropy $T_{n,c}$ and Legendre polynomial P_n. We established new bounds for $S_{n,c}$ and, consequently, for $R_{n,c}$, $T_{n,c}$ and P_n. Certain convexity properties of $S_{n,c}$ were instrumental in our proofs. In fact, $S_{n,c}$ has also other useful convexity properties. For example, for each integer j in $[1,n]$, $S_{n,-1}$ is $(2j-1)$-strongly convex with modulus

$$4^{j-n}\binom{2j}{j}\binom{2n-2j}{n-j}$$

(see the pertinent definition in [5]), and for each $j \geq 1$, $S_{n,0}$ is approximately $(2j-1)$-concave with modulus

$$n^{2j}\binom{4j}{2j}\frac{1}{(2j)!}$$

(see the definition in [9]).

On the other hand, according to (1.1), $S_{n,c}$ is a Heun function. By comparing two different expressions of this Heun function it is possible to derive combinatorial identities generalizing some classical ones from [6]. Sample results are

$$\sum_{j=k}^{n}\binom{j}{k}\binom{2j}{j}\binom{2n-2j}{n-j} = 4^{n-k}\binom{n}{k}\binom{2k}{k}, \quad 0 \leq k \leq n,$$

$$\sum_{i=0}^{n-j}\left(-\frac{1}{4}\right)^{i}\binom{n-j}{i}\binom{2i+2j}{i+j} = 4^{j-n}\binom{2j}{j}\binom{2n-2j}{n-j}\binom{n}{j}^{-1}, \quad 0 \leq j \leq n.$$

All these investigations will be presented in forthcoming papers.

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