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The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination

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ABSTRACT. In the present paper, we introduce and investigate the Fekete-Szegö functional associated with a new subclass of analytic functions, which we have defined here by using the principle of quasi-subordination between analytic functions. Some sufficient conditions for functions belonging to this class are also derived. The results presented here improve and generalize several known results.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{P}$ the class of functions ϕ, analytic in $\mathbb{U}$, such that

$$\phi(0) = 1 \quad \text{and} \quad \Re\big(\phi(z)\big) > 0 \quad (z \in \mathbb{U}).$$

In order to introduce the principles of subordination and quasi-subordination, we let f and g be two analytic functions in $\mathbb{U}$. We say that the function f is subordinate to the function g, written as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation (see also a recent investigation by Tang *et al.* [15] on applications of the principles of differential subordination and differential superordination between analytic functions):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further, for two functions f and g analytic in $\mathbb{U}$, the function f is said to be quasi-subordinate to the function g in $\mathbb{U}$, written as follows:

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}),$$

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if there exists an analytic function $\varphi(z)$, with

$$|\varphi(z)| \leqq 1 \qquad (z \in \mathbb{U}),$$

such that the function:

$$\frac{f(z)}{\varphi(z)}$$

is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{\varphi(z)} \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists the above-mentioned Schwarz function ω such that

$$f(z) = \varphi(z)\, g\big(\omega(z)\big).$$

The concept of quasi-subordination was given by Robertson [11]. It is clear that, in the special case when

$$\varphi(z) = 1 \qquad (z \in \mathbb{U}),$$

the quasi-subordination $\prec_q$ coincides with the usual subordination $\prec$. Furthermore, for the Schwarz function ω given by

$$\omega(z) = 1 \qquad (z \in \mathbb{U}),$$

the quasi-subordination $\prec_q$ becomes the majorization $\ll$, and we are led to the following relation:

$$f(z) \prec_q g(z) \implies f(z) = \varphi(z)\, g(z) \implies f(z) \ll g(z) \quad (z \in \mathbb{U}).$$

We begin by recalling each of the following subclasses of the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{U}$.

Definition 1.1. (see Owa *et al.* [10]) We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*(\alpha, b)$ of Sakaguchi type functions if it satisfies the following inequality:

$$\Re\left(\frac{(1-b)\, zf'(z)}{f(z) - f(bz)}\right) > \alpha \qquad (z \in \mathbb{U};\ 1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ 0 \leqq \alpha \leqq 1). \tag{1.2}$$

The Sakaguchi type function class $\mathcal{S}^*(\alpha, b)$ was studied by Owa *et al.* [10]. Obradović [9] introduced the class of non-Bazilević functions $f \in A$, which satisfy the following inequality:

$$\Re\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\rho}\right) > 0 \qquad (z \in \mathbb{U};\ 0 < \rho < 1). \tag{1.3}$$

More recently, by using of the above-defined concept of quasi-subordination, the following subclasses of the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{U}$ was defined by Sharma and Raina [13].

Definition 1.2. Let the function $\phi \in \mathcal{P}$ be univalent in $\mathbb{U}$ and let $\phi(\mathbb{U})$ be symmetrical about the real axis with

$$\phi'(0) > 0.$$

We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{G}_q^{\rho}(\phi, b)$ if the following quasi-subordination holds true:

$$f'(z)\left(\frac{(1-b)\, z}{f(z) - f(bz)}\right)^{\rho} - 1 \prec_q [\phi(z) - 1] \qquad (z \in \mathbb{U};\ 1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ \rho \geqq 0). \tag{1.4}$$

Motivated by each of the above definitions, we now make use of the concept of quasi-subordination to introduce the following subclass of the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{U}$.

Definition 1.3. Let the function $\phi \in \mathcal{P}$ be univalent in $\mathbb{U}$ and let $\phi(\mathbb{U})$ be symmetrical about the real axis with

$$\phi'(0) > 0 \qquad (z \in \mathbb{U}).$$

Then a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{J}_q^{\rho}(\phi, b, \alpha)$ if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho} - 1 \prec_q [\phi(z)-1] \tag{1.5}$$

$$(z \in \mathbb{U};\ 1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ \rho \geqq 0;\ 0 \leqq \alpha \leqq 1).$$

It is clear from Definition 1.3 that $f \in \mathcal{J}_q^{\rho}(\phi, b, \alpha)$ if and only if there exists a function φ with

$$|\varphi(z)| \leqq 1 \qquad (z \in \mathbb{U})$$

such that

$$\frac{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho} - 1}{\varphi(z)} \prec [\phi(z)-1] \qquad (z \in \mathbb{U}). \tag{1.6}$$

For different choices of the parameters involved in Definition 1.3 and (1.5), we have the following specialized subclasses of the analytic function class $\mathcal{A}$:

(i) For

$$\varphi(z) = 1 \qquad (z \in \mathbb{U})$$

in the subordination (1.6), we have the class $\mathcal{J}^{\rho}(\phi, b, \alpha)$ defined by the following condition:

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho} \prec \phi(z) \qquad (z \in \mathbb{U}). \tag{1.7}$$

(ii) For $\alpha = 1$ in the subordination (1.6), we have the class $\mathcal{G}_q^{\rho}(\phi, b)$ studied by Sharma and Raina [13].

(iii) For

$$\phi(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}),$$

the class $\mathcal{J}^2(\phi, 0, 1)$ reduces to the class studied by Nunokawa *et al.* [8].

(iv) The classes

$$\mathcal{J}_q^1(\phi, 0, 1) \qquad \text{and} \qquad \mathcal{J}_q^0(\phi, 0, 1)$$

coincide with the classes

$$\mathcal{S}_q^*(\phi) \qquad \text{and} \qquad \mathcal{R}_q(\phi),$$

respectively, defined in [6].

(v) The class $\mathcal{J}^1(\phi, b, 1)$ reduces to the class $\mathcal{S}^*(\phi, b)$ studied by Goyal and Gowswami [4].

(vi) For

$$\phi(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}),$$

we get the class $\mathcal{S}_s^*$ given by

$$\mathcal{J}^1(\phi, -1, 1) =: \mathcal{S}_s^*,$$

which happens to be the class of starlike functions with respect to symmetric points introduced by Sakaguchi [12].

The Fekete-Szegö functional given, for the Taylor-Maclaurin coefficients in (1.1), by

$$\left|a_3 - \mu a_2^2\right|$$

is due to Fekete and Szegö [3]. It has indeed been studied extensively (see, for example, [14] and the references cited therein). Our main aim in this paper is to study the corresponding Fekete-Szegö inequality for the above-defined analytic function class $\mathcal{J}_q^{\rho}(\phi, b, \alpha)$.

In proving our results, we use an inequality due to Keogh and Merkes [5] which is given in following lemma.

Lemma. *Let the Schwarz function* $\omega(z)$ *be given by*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots \qquad (z \in \mathbb{U}). \tag{1.8}$$

Then

$$|\omega_1| \leqq 1 \qquad \text{and} \qquad \left|\omega_2 - \kappa\omega_1^2\right| \leqq 1 + (|\kappa| - 1)|\omega_1|^2 \leqq \max\{1, |\kappa|\} \quad (\kappa \in \mathbb{C}). \tag{1.9}$$

The result is sharp for the function given by

$$\omega(z) = z \qquad \text{or} \qquad \omega(z) = z^2.$$

2. Main results and their consequences

Let the function $f \in \mathcal{A}$ be of the form (1.1). Then it is clearly seen that

$$\frac{f(z) - f(bz)}{1-b} = z + \sum_{n=2}^{\infty} \delta_n a_n z^n \qquad (z \in \mathbb{U}),$$

where

$$\delta_n = \frac{1-b^n}{1-b} = 1 + b + b^2 + \cdots + b^{n-1} \qquad (n \in \mathbb{N}). \tag{2.10}$$

Therefore, for $\rho \geqq 0$, we find that

$$\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\rho} = 1 - \rho\delta_2 a_2 z + \rho\left(\frac{1+\rho}{2}\,\delta_2^2 a_2^2 - \delta_3 a_3\right) z^2 + \cdots. \tag{2.11}$$

Throughout this paper, we assume that ρ and b are such that

$$\rho\delta_n \neq n \quad (z \in \mathbb{U}) \qquad \text{and} \qquad \rho\delta_n < n \quad (n \in \mathbb{N}\setminus\{1\};\ b \in \mathbb{R}).$$

We also suppose that the function $\phi \in \mathcal{P}$ is of the form:

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (c_1 > 0;\ z \in \mathbb{U}) \tag{2.12}$$

and that the $\psi(z)$, analytic in $\mathbb{U}$, is of the form:

$$\psi(z) = b_0 + b_1 z + b_2 z^2 + \cdots (z \in \mathbb{U}). \tag{2.13}$$

Theorem 2.1. *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{J}_q^{\rho}(\phi, b, \alpha)$. *Then*

$$|a_2| \leqq \frac{c_1}{|1 + \alpha - \alpha\rho\delta_2|}$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{|1 + 2\alpha - \alpha\rho\delta_3|}\ \max\left\{1, \left|c_1 L - \frac{c_2}{c_1}\right|\right\}, \tag{2.14}$$

where

$$L=\frac{\mu\left(1+2\alpha-\alpha\rho\delta_3\right)}{\left(1+\alpha-\alpha\rho\delta_2\right)^2}-\frac{\rho\left(1+\dfrac{3\alpha-1-\alpha\delta_2}{1+\alpha-\alpha\rho\delta_2}\right)\delta_2}{2\left(1+\alpha-\alpha\rho\delta_2\right)} \tag{2.15}$$

and $\delta_n \quad (n\in\mathbb{N})$ *is given by* (2.10).

Proof. Suppose that the function $f\in\mathcal{J}_q^{\rho}(\phi,b,\alpha)$ is given by (1.1). Then, by using the concept of subordination with Schwarz function $\omega(z)$ given by (1.8) and for the analytic function $\psi(z)$ defined in (2.13), it follows that

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho}-1=\psi(z)\left[\phi(z)-1\right] \qquad (z\in\mathbb{U}). \tag{2.16}$$

Moreover, in view of (2.12), it is easy to see that

$$\begin{aligned}\psi(z)\left(\phi(z)-1\right)&=\left(b_0+b_1z+b_2z^2+\cdots\right)\left[c_1\omega_1z+\left(c_1\omega_2+c_2\omega_1^2\right)z^2+\cdots\right]\\&=b_0c_1\omega_1z+\left[b_0\left(c_1\omega_2+c_2\omega_1^2\right)+b_1c_1\omega_1\right]z^2+\cdots.\end{aligned} \tag{2.17}$$

Since $f\in\mathcal{A}$, by using (1.1) together with (2.11) and after simple computations, we obtain

$$\begin{aligned}&(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho}-1\\&\quad=\left(1+\alpha-\alpha\rho\delta_2\right)a_2z+\left[\left(1+2\alpha-\alpha\rho\delta_3\right)a_3-\alpha\rho\left(2-\frac{1}{2}\left(1+\rho\right)\ \delta_2\right)\delta_2a_2^2\right]z^2+\cdots.\end{aligned} \tag{2.18}$$

Now, upon substituting the values from (2.17) and (2.18) into (2.16) and equating the coefficients of z and z^2, we have

$$\left(1+\alpha-\alpha\rho\delta_2\right)a_2=b_0c_1\omega_1 \tag{2.19}$$

and

$$\left(1+2\alpha-\alpha\rho\delta_3\right)a_3-\alpha\rho\left[2-\frac{1}{2}\left(1+\rho\right)\ \delta_2\right]\delta_2a_2^2=b_0\left(c_1\omega_2+c_2\omega_1^2\right)+b_1c_1\omega_1. \tag{2.20}$$

Since $|b_0|\leqq 1$ and $|c_1|\leqq 1$, we find from (2.19) that

$$|a_2|\leqq\frac{c_1}{\left|1+\alpha-\alpha\rho\delta_2\right|}. \tag{2.21}$$

Now, from (2.19), (2.20) yields

$$\left(1+2\alpha-\alpha\rho\delta_3\right)a_3=\alpha\rho\left(\frac{4-\left(1+\rho\right)\delta_2}{2\left(1+\alpha-\alpha\rho\delta_2\right)^2}\right)\delta_2b_0^2c_1^2\omega_1^2+b_0\left(c_1\omega_2+c_2\omega_1^2\right)+b_1c_1\omega_1,$$

which implies that

$$a_3=\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[b_1\omega_1+b_0\left\{\omega_2+\left(\frac{b_0\rho\left(1+\dfrac{3\alpha-1-\alpha\delta_2}{1+\alpha-\alpha\rho\delta_2}\right)\delta_2c_1}{2\left(1+\alpha-\alpha\rho\delta_2\right)}+\frac{c_2}{c_1}\right)\omega_1^2\right\}\right]. \tag{2.22}$$

Also, for some $\mu\in\mathbb{C}$, we find from (2.21) and (2.22) that

$$a_3-\mu a_2^2=\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[b_1\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)b_0-c_1L\omega_1^2b_0^2\right], \tag{2.23}$$

where L is given in (2.15). Since the function ψ defined in (2.13) is analytic and bounded in $\mathbb{U}$, by using a result recorded by Nehari [7], we have

$$|b_0| \leqq 1 \qquad \text{and} \qquad b_1 = \left(1 - b_0^2\right) x \tag{2.24}$$

for some $x \quad (|x| \leqq 1)$. Thus, upon substituting the value of b_1 from (2.24) into (2.23), we get

$$a_3 - \mu a_2^2 = \frac{c_1}{1 + 2\alpha - \alpha\rho\delta_3}\left[x\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right) b_0 - \left(c_1 L\omega_1^2 + x\omega_1\right) b_0^2\right]. \tag{2.25}$$

For $b_0 = 0$ in (2.25), it follows that

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{|1 + 2\alpha - \alpha\rho\delta_3|}.$$

For the case when $b_0 \neq 0$, we consider

$$g(b_0) = x\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right) b_0 - \left(c_1 L\omega_1^2 + x\omega_1\right) b_0^2,$$

which is a polynomial in b_0 and is, therefore, analytic in $|b_0| \leqq 1$, and the maximum of $|g(b_0)|$ is attained at $b_0 = e^{i\theta} \quad (0 \leqq \theta < 2\pi)$. We find that

$$\max_{0 \leqq \theta < 2\pi} |g(b_0)| = |g(1)| \qquad \left(b_0 = e^{i\theta}\right)$$

and

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{|1 + 2\alpha - \alpha\rho\delta_3|}\left|\omega_2 - \left(c_1 L - \frac{c_2}{c_1}\right)\omega_1^2\right| \tag{2.26}$$

Finally, by using the Lemma in Section 1, we have the assertion (2.14) of Theorem 2.1. It is easily seen that the result is sharp for the function $f(z)$ given by

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\rho} = \phi(z) \tag{2.27}$$

$$(1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ \rho \geqq 0;\ z \in \mathbb{U})$$

or

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\rho} = \phi\left(z^2\right) \tag{2.28}$$

$$(1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ \rho \geqq 0;\ z \in \mathbb{U})$$

or

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\rho} - 1 = z\left[\phi(z) - 1\right] \tag{2.29}$$

$$(1 \neq b \in \mathbb{C};\ |b| \leqq 1;\ \rho \geqq 0;\ z \in \mathbb{U})$$

This complete the proof of Theorem 2.1 □

By suitably specializing Theorem 2.1, we obtain the following sharp results for the classes

$$\mathcal{S}_q^*(\phi) \qquad \text{and} \qquad \mathcal{R}_q(\phi),$$

which were introduced in Section 1.

Corollary 2.1. *Let the function $f \in \mathcal{A}$ of the form* (1.1) *be in the class $\mathcal{S}_q^*(\phi)$. Then*

$$|a_2| \leqq c_1$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{2} \max\left\{1, \left|\frac{c_2}{c_1} + (1-2\mu)\, c_1\right|\right\}.$$

The result is sharp.

Corollary 2.2. *Let the function $f \in \mathcal{A}$ of the form* (1.1) *be in the class $\mathcal{R}_q(\phi)$. Then*

$$|a_2| \leqq \frac{c_1}{2},$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{3} \max\left\{1, \left|\frac{c_2}{c_1} - \frac{3\mu}{4}\, c_1\right|\right\}.$$

The result is sharp.

Remark 2.1. In the special case of Theorem 2.1 when $b = 0$, $\alpha \neq 1$ and $\rho = 1$, we obtain the following result:

$$|a_2| \leqq c_1$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{1+\alpha} \max\left\{1, \left|\frac{c_2}{c_1} + (\alpha - \mu(1+\alpha))\, c_1\right|\right\}.$$

Remark 2.2. In the special case of Theorem 2.1 when $b = 0$, $\alpha \neq 1$ and $\rho = 0$, we obtain the following result:

$$|a_2| \leqq \frac{c_1}{1+\alpha}$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{1+2\alpha} \max\left\{1, \left|\frac{c_2}{c_1} - \frac{(2\alpha+1)\,\mu}{(1+\alpha)^2} c_1\right|\right\}.$$

Theorem 2.2. *Let $f \in \mathcal{J}^\rho(\phi, b, \alpha)$. Then*

$$|a_2| \leqq \frac{c_1}{|1+\alpha-\alpha\rho\delta_2|}$$

and, for some $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{c_1}{|1+2\alpha-\alpha\rho\delta_3|} \max\left\{1, \left|\frac{c_2}{c_1} - c_1 L\right|\right\}, \tag{2.30}$$

where L is given by (2.15) *and $\delta_n \quad (n \in \mathbb{N})$ is given by* (2.10). *The result is sharp.*

Proof. Suppose that the function $f \in \mathcal{J}^\rho(\phi, b, \alpha)$ is given by (1.1). Then $\psi(z) \equiv 1$, which implies that

$$b_0 = 1 \qquad \text{and} \qquad b_n = 0 \quad (n \in \mathbb{N}).$$

Now, by using (2.21) and (2.23) in conjunction with the Lemma in Section 1, we obtain the required result (2.30). Sharpness of this result can be verified for the function $f(z)$ given by

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^\rho = \phi(z) \tag{2.31}$$

or

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho}=\phi\left(z^2\right). \tag{2.32}$$

□

Our next result (Theorem 2.3 below) is associated with the familiar concept of majorization.

Theorem 2.3. *Let* $1\neq b\in\mathbb{C}$, $|b|\leqq 1$ *and* $0\leqq\alpha\leqq 1$. *If the function* $f\in\mathcal{A}$ *of form given by* (1.1) *satisfies the following majorization condition*:

$$(1-\alpha)\frac{f(z)}{z}+\left[\alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho}-1\right]\ll[\phi(z)-1] \qquad (z\in\mathbb{U}), \tag{2.33}$$

then

$$|a_2|\leqq\frac{c_1}{|1+\alpha-\alpha\rho\delta_2|}$$

and, for some $\mu\in\mathbb{C}$,

$$\left|a_3-\mu a_2^2\right|\leqq\frac{c_1}{|1+2\alpha-\alpha\rho\delta_3|}\ \max\left\{1,\left|\frac{c_2}{c_1}-c_1L\right|\right\},$$

where $\delta_n\ \ (n\in\mathbb{N})$ *is given by* (2.10) *and* L *is given by* (2.15). *The result is sharp.*

Proof. By using the concept of majorization and Theorem 2.1, we have $\omega(z)\equiv z$ in (1.8). This implies that

$$\omega_1=1 \qquad \text{and} \qquad \omega_n=0 \quad (n\in\mathbb{N}\setminus\{1\}).$$

Thus, if we make use of (2.21) and (2.23), we get

$$|a_2|\leqq\frac{c_1}{|1+\alpha-\alpha\rho\delta_2|}$$

and

$$a_3-\mu a_2^2=\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left(b_1+\frac{c_2}{c_1}b_0-c_1Lb_0^2\right). \tag{2.34}$$

Upon substituting the value of b_1 from (2.24) into (2.34), we find that

$$a_3-\mu a_2^2=\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left(x+\frac{c_2}{c_1}b_0-(c_1L+x)\,b_0^2\right). \tag{2.35}$$

If $b_0=0$, then

$$\left|a_3-\mu a_2^2\right|\leqq\frac{c_1}{|1+2\alpha-\alpha\rho\delta_3|}. \tag{2.36}$$

Also, for $b_0\neq 0$, we consider

$$H(b_0)=x+\frac{b_0c_2}{c_1}-(c_1L+x)\,b_0^2,$$

which is a polynomial in b_0 and is, therefore, analytic in $|b_0|\leqq 1$, and the maximum of $|H(b_0)|$ is attained at $b_0=e^{i\theta}\ \ (0\leqq\theta<2\pi)$. We thus find that

$$\max_{0\leqq\theta<2\pi}\left|H\left(e^{i\theta}\right)\right|=|H(1)|.$$

Therefore, we have

$$\left|a_3-\mu a_2^2\right|\leqq\frac{c_1}{|1+2\alpha-\alpha\rho\delta_3|}\left|c_1L-\frac{c_2}{c_1}\right|. \tag{2.37}$$

Now, from the inequality (2.37) together with (2.36), we have the result asserted by Theorem 2.3. The result is sharp for the function given by

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\rho}=\phi(z).$$

□

We now find the bounds of the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ when μ and b are real numbers. We first obtain the following result for the class $\mathcal{J}_q^{\rho}(\phi,b,\alpha)$.

Corollary 2.3. *Let the function $f\in\mathcal{J}_q^{\rho}(\phi,b,\alpha)$ be of the form (1.1). Then, for real values of μ and b,*

(2.38)

$$\left|a_3-\mu a_2^2\right|\leqq\begin{cases}\dfrac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[c_1\left(\dfrac{\rho\{4\alpha-\alpha(1+\rho)\delta_2\}\delta_2-2\mu(1+2\alpha-\alpha\rho\delta_3)}{2(1+\alpha-\alpha\rho\delta_2)^2}+\dfrac{c_2}{c_1}\right)\right] & (\mu\leqq\nu)\\[2ex] \dfrac{B_1c_1}{1+2\alpha-\alpha\rho\delta_3} & (\nu\leqq\mu\leqq\nu+2\gamma)\\[2ex] \dfrac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[c_1\left(\dfrac{2s(1+2\alpha-\alpha\rho\delta_3)-\rho\{4\alpha-\alpha(1+\rho)\delta_2\}\delta_2}{2(1+\alpha-\alpha\rho\delta_2)^2}\right)-\dfrac{c_2}{c_1}\right] & (\mu\geqq\nu+2\gamma),\end{cases}$$

where

$$\nu=\frac{\rho[4\alpha-\alpha(1+\rho)\delta_2]\delta_2}{2(1+2\alpha-\alpha\rho\delta_3)}-\frac{(1+\alpha-\alpha\rho\delta_2)^2}{1+2\alpha-\alpha\rho\delta_3}\left(\frac{1}{c_1}-\frac{c_2}{c_1^2}\right), \tag{2.39}$$

$$\gamma=\frac{(1+\alpha-\alpha\rho\delta_2)^2}{c_1(1+2\alpha-\alpha\rho\delta_3)} \tag{2.40}$$

and δ_n $(n\in\mathbb{N})$ is given by (2.10). The result is sharp.

Proof. For real values of μ and b, by applying (2.14), we get the result asserted by Corollary 2.3 under the following cases:

$$c_1L-\frac{c_2}{c_1}\leqq-1,\quad -1\leqq c_1L-\frac{c_2}{c_1}\leqq1\qquad\text{and}\qquad c_1L-\frac{c_2}{c_1}\geqq1,$$

where L is given by (2.15).

(1) For $\mu<\nu$ or $\mu>\nu+2\gamma$, the equality holds true if and only if $\omega(z)=z$ or one of its rotations.

(2) For $\nu<\mu<\nu+2\gamma$, the equality holds true if and only if $\omega(z)=z^2$ or one of its rotations. □

Theorem 2.4. *Let the function $f\in\mathcal{J}_q^{\rho}(\phi,b,\alpha)$ be of the form (1.1). Then, for real values of μ and b,*

$$\left|a_3-\mu a_2^2\right|+(\mu-\nu)\left|a_2\right|^2\leqq\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\qquad(\nu<\mu\leqq\nu+\gamma) \tag{2.41}$$

and

$$\left|a_3-\mu a_2^2\right|+(\nu+2\gamma-\mu)\left|a_2\right|^2\leq\frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\qquad(\gamma+\nu<\mu<\nu+2\gamma), \tag{2.42}$$

where ν and γ are given by (2.39) and (2.40), respectively, and δ_3 is given by (2.10).

Proof. Suppose that the function $f \in \mathcal{J}_q^{\rho}(\phi, b, \alpha)$ is given by (1.1). If $\nu < \mu \leqq \nu + \gamma$, then we find from (2.21) and (2.26) that

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| + (\mu - \nu)\left|a_2^2\right| \\ &\quad \leqq \frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[\left|\omega_2\right| - \frac{c_1\left(1+2\alpha-\alpha\rho\delta_3\right)}{\left(1+\alpha-\alpha\rho\delta_2\right)^2}(\mu-\nu-\gamma)\left|\omega_1\right|^2 \right.\\ &\qquad\qquad \left. + \frac{c_1\left(1+2\alpha-\alpha\rho\delta_3\right)}{1+\alpha-\alpha\rho\delta_2}(\mu-\nu)\left|\omega_1\right|^2\right].\end{aligned}$$

Hence, by applying the Lemma in Section 1, we get

$$\left|a_3 - \mu a_2^2\right| + (\mu - \nu)\left|a_2^2\right| \leqq \frac{c_1}{1+2\alpha-\alpha\rho\delta_3},$$

If $\gamma + \nu < \mu < \nu + 2\gamma$, then we again make use of (2.21) and (2.26) in conjunction with the Lemma in Section 1. We thus find that

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| + (\nu + 2\gamma - \mu)\left|a_2\right|^2 \\ &\quad \leqq \frac{c_1}{1+2\alpha-\alpha\rho\delta_3}\left[\left|\omega_2\right| + \frac{c_1\left(1+2\alpha-\alpha\rho\delta_3\right)}{\left(1+\alpha-\alpha\rho\delta_2\right)^2}\ (\mu-\nu-\gamma)\left|\omega_1\right|^2 \right.\\ &\qquad\qquad \left. + \frac{c_1\left(1+2\alpha-\alpha\rho\delta_3\right)}{(1+\alpha-\alpha\rho\delta_2)^2}\ (\nu+2\gamma-\mu)\left|\omega_1\right|^2\right],\end{aligned}$$

so that, by using the Lemma in Section 1 once again, we obtain

$$\left|a_3 - \mu a_2^2\right| + (\nu + 2\gamma - \mu)\left|a_2\right|^2 \leqq \frac{c_1}{1+2\alpha-\alpha\rho\delta_3}.$$

Hence we have established the result asserted by Theorem 2.4. □

Remark 2.3. We observe that, on choosing $\alpha = 1$ and $\rho = 1$, the inequality (2.38) and its subsequently-improved version given by Theorem 2.4 would coincide with a known result of Goyal and Goswami [4].

3. Conclusion

In our present investigation, we have introduced and systematically studied the familiar Fekete-Szegö functional which is associated with a new subclass of the normalized analytic function class $\mathcal{A}$. We have defined this interesting subclass by using the principle of quasi-subordination between analytic functions. In particular, we have derived some sufficient conditions for functions belonging to this subclass. Our results in this paper are shown to improve and generalize several known results which were derived in a number of recent works. We choose also to cite some recent investigations on the subject of this paper by (for example) Altınkaya and Yalçın (see [1] and [2]).

References

[1] Altınkaya, Ş. and Yalçın, S., *Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points*, J. Funct. Spaces, **2015** (2015), Article ID 145242, 1–5

[2] Altınkaya, Ş. and Yalçın, S., *On a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions*, Malaya J. Math., **5** (2017), 305–309

[3] Fekete, M. and Szegö, G., *Eine Bemerkung Über Ungerade Schlichte Funktionen*, J. London Math. Soc., **8** (1933), No. 2, 85–89

[4] Goyal, S. P. and Goswami, P., *Certain coefficient inequalities for Sakaguchi type functions and applications to fractional derivative operator*, Acta Univ. Apulensis, **19** (2009), 159–166
[5] Keogh, F. R. and Merkes, E. P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8–12
[6] Mohammad, M. H. and Darus, M., *Fekete-Szegö problems for quasi-subordination classes*, Abstr. Appl. Anal., **2012** (2012), Article ID 192956, 1–14
[7] Nehari, Z., *Conformal Mapping*, McGraw-Hill Book Company, New York, 1952
[8] Nunokawa, M., Obradović, M. and Owa, S., *One criterion for univalency*, Proc. Amer. Math. Soc., **106** (1989), 1035–1037
[9] Obradović, M., *A class of univalent functions*, Hokkaido Math. J., **2** (1988), 329–335
[10] Owa, S., Sekine, T. and Yamakawa, R., *On Sakaguchi type functions*, Appl. Math. Comput., **187** (2007), 356–361
[11] Robertson, M. S., *Quasi-subordination and coefficient conjectures*, Bull. Amer. Math. Soc., **76** (1970), 1–9
[12] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan, **11** (1959), 72–75
[13] Sharma, P. and Raina, R. K., *On a Sakaguchi type class of analytic functions associated with quasi-subordination*, Comment. Math. Univ. St. Paul., **64** (2015), 59–70
[14] Srivastava, H. M., Mishra, A. K. and Das, M. K., *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl., **44** (2001), 145–163
[15] Tang, H., Srivastava, H. M. and Deng, G.-T., *Some families of analytic functions in the upper half-plane and their associated differential subordination and differential superordination properties and problems*, Appl. Math. Inform. Sci., **11** (2017), 1247–1257

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