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Painlevé-Kuratowski convergences of the approximate solution sets for vector quasiequilibrium problems

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ABSTRACT. In this paper, we study vector quasiequilibrium problems. After that, the Painlevé-Kuratowski upper convergence, lower convergence and convergence of the approximate solution sets for these problems are investigated by using a sequence of mappings Γ_C-converging. As applications, we also consider the Painlevé-Kuratowski upper convergence of the approximate solution sets in the special cases of variational inequality problems of the Minty type and Stampacchia type. The results presented in this paper extend and improve some main results in the literature.

1. INTRODUCTION

The stability of the exact solution sets for various kinds of the optimization problems, variational inequality problems and equilibrium problems have been discussed in many different aspects, as the lower semicontinuity, upper semicontinuity and the convergence of efficient solution sets, etc, (see e.g. [1, 2, 3, 4, 5, 8, 9, 11, 13, 14, 16, 17, 19, 21]). However, in practically, there are many problems which theirs exact solutions may not exist. The reason is that the data of these problems are not sufficiently. So the approximate solutions are investigated by using the approximate methods to solve the mathematical models and produce approximations to the exact solutions. Similar to the stability of exact solution sets, there are many authors have been studied the stability of approximate solution sets for various kinds of the optimization-related problems (see e.g. [6, 7, 20], and the references therein).

In 1994, the Painlevé-Kuratowski convergence and Attouch-Wets convergence of the efficient and weak efficient solution sets for optimization problem was established by Luc et al. [10]. Since then, many authors considered the convergence of the solution sets for various kinds of the optimization problems, variational inequality problems and equilibrium problems. The notion of gamma convergence for sequences of vector-valued functions was introduced by Oppezzi and Rossi [13], which extend the continuous convergence for sequences of vector-valued functions. To develop the results in [13], Oppezzi and Rossi [14] generalized Mosco convergence for convex vector optimization problems in infinite dimensional space by using the concept of gamma convergence for a sequence of vector-valued functions. Recently, Lalitha and Chatterjee [5] developed the Painlevé-Kuratowski convergence of the solution sets for the a nonconvex vector optimization problem by using a sequence of mappings converging continuously and nonlinear scalarization function defined in terms of an improvement set. Very recently, Li et al. [7] established Painlevé-Kuratowski convergence of the approximate solution sets for generalized Ky Fan inequality problems by continuous convergence of the bifunction sequence and Painlevé-Kuratowski convergence of the set sequence.

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Motivated by the research works mentioned above, in this paper, we establish the Painlevé-Kuratowski convergence of the approximate solution sets for vector quasiequilibrium problems by using a sequence of mappings Γ_C-converging. Let X, Y, Z be three Banach spaces. We assume that C is a pointed closed convex cone in Y with nonempty interior, i.e., $\text{int}C \neq \emptyset$. Let $A \subseteq X$ be a nonempty, compact and convex subset and $B \subseteq Z$ be a nonempty subset. Let $K : A \to 2^A, T : A \to 2^B$ be set-valued mappings and $f : A \times B \times A \to Y$ be a vector-valued function. We consider the two following vector quasiequilibrium problems:
(QVEP^1) finding $x \in K(x)$ such that

$$f(x, z, y) \in Y \setminus -\text{int}C, \forall y \in K(x), \forall z \in T(y),$$

(QVEP^2) finding $x \in K(x)$ such that $\exists z \in T(x)$ satisfying

$$f(x, z, y) \in Y \setminus -\text{int}C, \forall y \in K(x).$$

For sequences of set-valued mappings $K_n : A \to 2^A, T_n : A \to 2^B$ and vector-valued mappings $f_n : A \times B \times A \to Y$, we consider the two following sequences of vector quasiequilibrium problems:
(QVEP^1_n) finding $x_n \in K_n(x_n)$ such that

$$f_n(x_n, z, y) \in Y \setminus -\text{int}C, \forall y \in K_n(x_n), \forall z \in T_n(y),$$

(QVEP^2_n) finding $x_n \in K_n(x_n)$ such that $\exists z_n \in T_n(x_n)$ satisfying

$$f_n(x_n, z_n, y) \in Y \setminus -\text{int}C, \forall y \in K_n(x_n).$$

The approximate solution set of (QVEP^1) is defined by

$$S^1(f, T, K, \varepsilon) = \{x \in K(x) : f(x, z, y) + \varepsilon e \in Y \setminus -\text{int}C, \forall y \in K(x), \forall z \in T(y)\}.$$

Similarly, the approximate solution sets of (QVEP^2), (QVEP^1_n) and (QVEP^2_n) are defined by $S^2(f, T, K, \varepsilon)$, $S^1(f_n, T_n, K_n, \varepsilon_n)$ and $S^2(f_n, T_n, K_n, \varepsilon_n)$ respectively, where $e \in \text{int}C$, $\varepsilon \geq 0$ and $\{\varepsilon_n\}$ is a nonnegative scalar-valued sequence.

Throughout this paper, the following notations are used. We denote by $\varepsilon_n \searrow \varepsilon$ when $\varepsilon_n > \varepsilon$ for all n and $\varepsilon_n \to \varepsilon$. We always assume that all solution sets considered in this paper are not equal empty sets.

2. Preliminaries

First of all, we recall some concepts of the convergence of set sequences and mapping sequences.

Definition 2.1. (See [18]) If X is a first countable topological space and $D, D_n \subseteq X$, $n \in \mathbb{N}$, let

$$\liminf_{n\to\infty} D_n := \{x \in X : \exists \bar{n} \in \mathbb{N} \quad \text{and} \quad x_n \in D_n, \forall n \geq \bar{n}, \text{ s.t. } \lim_{n\to\infty} x_n = x\},$$

$$\limsup_{n\to\infty} D_n := \{x \in X : \exists \{n_k\}_{k\in\mathbb{N}} \quad \text{and} \quad x_k \in D_{n_k}, \text{ s.t. } \lim_{k\to\infty} x_k = x\}.$$

We say that a sequence of sets $\{D_n\}$ is said to *upper converge* (resp. *lower converge*) in the sense of Painlevé-Kuratowski to D if $\limsup_{n\to\infty} D_n \subseteq D$ (resp. $D \subseteq \liminf_{n\to\infty} D_n$). $\{D_n\}$ is said to *converge* in the sense of Painlevé-Kuratowski to D, denoted as $D_n \xrightarrow{\text{P.K.}} D$, if $\limsup_{n\to\infty} D_n \subseteq D \subseteq \liminf_{n\to\infty} D_n$.

Definition 2.2. (See [18]) Let $G_n : X \to 2^Y$ be a sequence of set-valued mappings and $G : X \to 2^Y$ be a set-valued mapping. $\{G_n\}$ is said to *outer converge continuously* (resp. *inner converge continuously*) to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subseteq G(x_0)$ (resp. $G(x_0) \subseteq \liminf_{n\to\infty} G_n(x_n)$) $\forall x_n \to x_0$. $\{G_n\}$ is said to *converge continuously* to G at x_0 if $\limsup_{n\to\infty} G_n(x_n) \subseteq G(x_0) \subseteq \liminf_{n\to\infty} G_n(x_n) \quad \forall x_n \to x_0$. If $\{G_n\}$ converges continuously to G at every $x_0 \in X$, then it is said that $\{G_n\}$ converges continuously to G on X.

G is said to be *closed at* x_0, if for each sequence $\{(x_n, y_n)\} \subseteq \text{graph}G := \{(x, y)|y \in G(x)\}, (x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \text{graph}G$. G is said to be closed in X if it is closed at each $x_0 \in X$.

Definition 2.3. (See [13]) Let $f_n, f : X \to Y$ and let $\mathcal{U}(x)$ be the family of neighborhoods of x. We say that f_n Γ_C*-converges* to f, denoted as $f_n \xrightarrow{\Gamma_C} f$, if for every $x \in X$:

(i) $\forall U \in \mathcal{U}(x), \forall \eta \in \text{int}C, \exists n_{\eta,U} \in \mathbb{N}$ such that $\forall n \geq n_{\eta,U}, \exists x_n \in U$, $f_n(x_n) \in f(x) + \eta - C$;
(ii) $\forall \eta \in \text{int}C, \exists U_\eta \in \mathcal{U}(x), k_\eta \in \mathbb{N}$ such that $\forall x' \in U_\eta, \forall n \geq k_\eta$, $f_n(x') \in f(x) - \eta + C$.

If only the condition (i) (resp. (ii)) is satisfied, f_n outer (resp. inner) Γ_C*-converges* to f, denoted as $f_n \xrightarrow{\Gamma_C^u} f$ (resp. $f_n \xrightarrow{\Gamma_C^i} f$)

Definition 2.4. (See [12], Definition 4.7) A sequence of mappings $\{f_n\}$ where $f_n : X \to Y$ is said to *converge continuously* to a mapping $f : X \to Y$ if $\forall x \in X$ and $\forall V$ neighborhood of $f(x)$ in Y, there exists $k \in \mathbb{N}$ and $U \in \mathcal{U}(x)$ such that $f_n(y) \in V$ for every $n \geq k$ and for every $y \in U$.

Lemma 2.1. (See [15], Lemma 4.1) *Let $C \subseteq Y$ be a pointed cone having int$C \neq \emptyset$. If $\eta \in \text{int}C$ and $y \in Y$, then the sets $\{z \in Y : z \in y - \eta + C\backslash\{0\}\}$ and $\{z \in Y : z \in y + \eta - C\backslash\{0\}\}$ are neighborhoods of y.*

Remark 2.1. From Definitions 2.3, 2.4 and Lemma 2.1, clearly such a continuous convergence implies Γ_C-convergence, but the reverse implication does not hold.

Indeed, we assume that $\{f_n\}$ converges continuously to f, then for every $x \in X$, if $U \in \mathcal{U}(x)$ and $\eta \in \text{int}C$, it follows from Lemma 2.1 that $V = \{z \in Y : z \in f(x) + \eta - C\backslash\{0\}\}$ is a neighborhood of $f(x)$. By $\{f_n\}$ converges continuously to f, there exist $n_{\eta,U}$ and $x_n \in U$ such that $f_n(x_n) \in V$, for every $n \geq n_{\eta,U}$, i.e. $f_n(x_n) \in f(x) + \eta - C\backslash\{0\} \subseteq f(x) + \eta - C\}$. Hence, condition (i) of Definition 2.3 holds.

Let us consider a countable base $\mathcal{F}(x) = \{U_n, n \in \mathbb{N}\}$ of neighborhoods of x such that $U_{n+1} \subseteq U_n$. Now, we prove that condition (ii) of Definition 2.3 holds. By contradiction we suppose that there exists $\eta \in \text{int}C$ such that for every $U_n \in \mathcal{F}(x)$ there exists $x_{k_n} \in U_n$ with $k_n > n$ such that $f_{k_n}(x_{k_n}) \notin f(x) - \eta + C$. Then we can define

$$x'_k = \begin{cases} x & \text{if } k \neq k_n, \forall n \\ x_{k_n} & \text{if } k = k_n. \end{cases}$$

So it follows that $x'_k \to x$ and $f_k(x'_k) \notin f(x) - \eta + C$. This implies $f_k(x'_k) \notin f(x) - \eta + C\backslash\{0\}$. Then $f_k(x'_k) \notin V' = \{z \in Y : z \in f(x) - \eta + C\backslash\{0\}\}$ (by Lemma 2.1, V' is a neighborhood of $f(x)$) which is a contradiction (as the condition $\{f_n\}$ converges continuously to f).

The following example shows that the converse is not true.

Example 2.1. (See [13], Remark 2.8) Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$, $f_n(x) = (x, nxe^{-2n^2x^2})$ and

$$f(x) = \begin{cases} (x, 0) & x \neq 0; \\ \left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right) & x = 0. \end{cases}$$

We have f_n Γ_C-converges to f, but $\{f_n\}$ does not converge continuously to f. Indeed, we may consider the two sequences: $x_n = \frac{1}{2n}, x'_n = -\frac{1}{2n}$, clearly $x_n \to 0$ and $x'_n \to 0$, but $f_n(x_n) \to (0, \frac{1}{2}e^{-\frac{1}{2}})$ and $f_n(x'_n) \to (0, -\frac{1}{2}e^{-\frac{1}{2}})$.

3. Main results

In this section, we establish the sufficient conditions for Painlevé-Kuratowski upper convergence, Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence of the approximate solution sets for (QVEP^1), (QVEP^1_n) and (QVEP^2), (QVEP^2_n).

Theorem 3.1. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n inner converges continuously to T in A;*
(iii) *$-f_n$ Γ^i_C-converges to $-f$.*

Then, for any $\varepsilon_n \to \varepsilon_0$, $\limsup_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n) \subseteq S^1(f, T, K, \varepsilon_0)$.

Proof. Suppose to the contrary that there exists $\varepsilon_n \to \varepsilon_0$,

$$\limsup_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n) \nsubseteq S^1(f, T, K, \varepsilon_0),$$

i.e., there exists $x_0 \in \limsup_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n)$, but x_0 not belong to $S^1(f, T, K, \varepsilon_0)$. As $x_0 \in \limsup_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n)$, there exists a sequence $\{x_{n_k}\}$, $x_{n_k} \in S^1(f_{n_k}, T_{n_k}, K_{n_k}, \varepsilon_{n_k})$, $x_{n_k} \to x_0$, as $k \to \infty$. Then, for each $y \in K_{n_k}(x_{n_k})$, we have

$$f_{n_k}(x_{n_k}, z, y) + \varepsilon_{n_k} e \in Y \setminus -\mathrm{int}C, \forall z \in T_{n_k}(y). \tag{3.1}$$

By K_n outer converges continuously to K with compact values, we have $x_0 \in K(x_0)$. Now we prove that $x_0 \in S^1(f, T, K, \varepsilon_0)$. If $x_0 \notin S^1(f, T, K, \varepsilon_0)$, there exist $y_0 \in K(x_0)$ and $z_0 \in T(y_0)$ such that

$$f(x_0, z_0, y_0) + \varepsilon_0 e \notin Y \setminus -\mathrm{int}C. \tag{3.2}$$

Since K_n inner converges continuously to K, for all $y_0 \in K(x_0)$, there exists a sequence $\{y_{n_k}\}$, $y_{n_k} \in K_{n_k}(x_{n_k})$ such that $y_{n_k} \to y_0$, as $k \to \infty$. As T_n inner converges continuously to T, for all $z_0 \in T(y_0)$, there exists a sequence $\{z_{n_k}\}$, $z_{n_k} \in T_{n_k}(y_{n_k})$ such that $z_{n_k} \to z_0$, as $k \to \infty$. As $x_{n_k} \in S^1(f_n, T_n, K_n, \varepsilon_n)$, we have

$$f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) + \varepsilon_{n_k} e \in Y \setminus -\mathrm{int}C. \tag{3.3}$$

By the condition (iii), $-f_n \xrightarrow{\Gamma^1_C} -f$ and $(x_{n_k}, z_{n_k}, y_{n_k}) \to (x_0, z_0, y_0)$, it follows from Definition 2.3 (ii) that for any $\eta \in \mathrm{int}C$,

$$-f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) \in -f(x_0, z_0, y_0) - \eta + C. \tag{3.4}$$

We choose $\eta = |\varepsilon_{n_k} - \varepsilon_0|e$, then from (3.3) and (3.4), we have

$$\begin{aligned} f(x_0, z_0, y_0) + \varepsilon_{n_k} e + |\varepsilon_{n_k} - \varepsilon_0| e &\in f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) + \varepsilon_{n_k} e + C \\ &\subseteq Y \setminus -\mathrm{int}C + C \subseteq Y \setminus -\mathrm{int}C \end{aligned} \tag{3.5}$$

Letting $k \to \infty$, $\varepsilon_{n_k} \to \varepsilon_0$ and by the closedness of $Y \setminus -\mathrm{int}C$ that from (3.5),

$$f(x_0, z_0, y_0) + \varepsilon_0 e \in Y \setminus -\mathrm{int}C, \tag{3.6}$$

which contradicts (3.2) and so completed the proof. □

Passing to the (QVEP^2), (QVEP^2_n), we obtain a similar conclusion as that Theorem 3.1.

Theorem 3.2. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n outer converges continuously to T with compact values in A;*
(iii) *$-f_n$ Γ_C^i-converges to $-f$.*

Then, for any $\varepsilon_n \to \varepsilon_0$, $\limsup_{n\to\infty} S^2(f_n, T_n, K_n, \varepsilon_n) \subseteq S^2(f, T, K, \varepsilon_0)$.

Remark 3.2. (i) Let X and Y be two normed linear spaces and $K(x) = K, K_n(x) = K_n, \forall x \in X, \varepsilon_n = \varepsilon_0 = 0$. Let $f(x, z, y) = g(y) - g(x), f_n(x, z, y) = g_n(y) - g_n(x)$ (with g, g_n be functions from X into Y) for any $x, y \in X$. The problems (QVEP)^2 and (QVEP)$_n^2$ reduce to vector optimization problems (P) and (P)$_n$, respectively which were studied in [5]. Then, Theorem 3.3 and Theorem 4.6 in [5] are special cases of Theorem 3.2. Moreover, the assumption (iii) in our Theorem 3.2 is weaker than the assumption (iii) in Theorem 3.3 and Theorem 4.6 in [5] (see, Example 2.1).

(ii) Let $X = \mathbb{R}^m, Y = \mathbb{R}^l, K(x) \equiv K, K_n(x) \equiv K_n, \forall x \in X, f(x, z, y) = f(x, y), f_n(x, z, y) = f_n(x, y)$ for any $x, y \in X$. The problems (QVEP)^2 and (QVEP)$_n^2$ reduce to the generalized Ky Fan inequality GKF$(C, f)_2$ and GKF$(C_n, f_n)_2$, respectively which were studied in [7]. Then, Theorem 3.2 improves and extends Lemma 3.1 and Theorem 3.1 in [7]. Our Theorem 3.1 is new.

Next, we establish the sufficient conditions for Painlevé-Kuratowski lower convergence for (QVEP)^1 and (QVEP)$_n^1$.

Theorem 3.3. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n outer converges continuously to T with compact values in A;*
(iii) *f_n Γ_C^i-converges to f.*

Then, for any $\varepsilon_n \searrow \varepsilon_0$, $S^1(f, T, K, \varepsilon_0) \subseteq \liminf_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n)$.

Proof. Letting any $x_0 \in S^1(f, T, K, \varepsilon_0)$. Then for all $y \in K(x_0)$,

$$f(x_0, z, y) + \varepsilon_0 e \in Y \setminus -\text{int}C, \forall z \in T(y). \tag{3.7}$$

By the condition (i), there exists $x_n \in K_n(x_n)$ such that $x_n \to x_0$. To prove that $x_0 \in \liminf_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n)$. We first prove the following property for the sequence $\{x_n\}$:

$$\forall \varepsilon > \varepsilon_0, \exists n_\varepsilon, \forall n \geq n_\varepsilon : x_n \in S^1(f_n, T_n, K_n, \varepsilon) \tag{3.8}$$

On the contrary, suppose that

$$\exists \varepsilon^* > \varepsilon_0, \forall k, \exists n_k \geq k : x_{n_k} \notin S^1(f_{n_k}, T_{n_k}, K_{n_k}, \varepsilon^*). \tag{3.9}$$

Since $x_{n_k} \notin S^1(f_{n_k}, T_{n_k}, K_{n_k}, \varepsilon^*)$, there exists $y_{n_k} \in K_{n_k}(x_{n_k})$ and for some $z_{n_k} \in T_{n_k}(y_{n_k})$ such that

$$f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) + \varepsilon^* e \in -\text{int}C. \tag{3.10}$$

Since K_n outer converges continuously to K with compact values, we can assume, without loss of generality, that $y_{n_k} \to y_0 \in K(x_0)$. Since T_n outer converges continuously to T with compact values, we can assume, without loss of generality, that $z_{n_k} \to z_0 \in T(y_0)$. Since f_n Γ_C^i-converges to f and $(x_{n_k}, z_{n_k}, y_{n_k}) \to (x_0, z_0, y_0)$ as $k \to \infty$, it follows from Definition 2.3 (ii) that for any $\eta \in \text{int}C$,

$$f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) \in f(x_0, z_0, y_0) - \eta + C.$$

Since the arbitrariness of $\eta \in \text{int}C$, we choose $\eta = (\varepsilon^* - \varepsilon_0)e$, then by (3.10) we have

$$f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) + \varepsilon^* e \in f(x_0, z_0, y_0) + \varepsilon_0 e + C$$

or

(3.11) $$f(x_0, z_0, y_0) + \varepsilon_0 e \in f_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) + \varepsilon^* e - C \subseteq -\text{int}C - C \subseteq -\text{int}C.$$

which contradicts (3.7) and so we obtain (3.8).

We now consider a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \searrow \varepsilon_0$ as $m \to \infty$. From (3.8) we have

$$\forall m > 0, \exists n_m, \forall n \geq n_m : x_n \in S^1(f_n, T_n, K_n, \varepsilon_m)$$

where the mapping $m \mapsto n_m$ can be assumed to be strictly increasing. For every $n \geq n_1$, there exists a unique m such that $n_m \leq n \leq n_{m+1}$. Calling it $m(n)$ and setting $\varepsilon_n := \varepsilon_{m(n)}$, we have $\forall n \geq n_1, x_n \in S^1(f_n, T_n, K_n, \varepsilon_n)$ and, as $m \mapsto n_m$ is increasing, we have $\varepsilon_n \searrow \varepsilon_0$. Hence, $x_0 \in \liminf_{n\to\infty} S^1(f_n, T_n, K_n, \varepsilon_n)$. Thus, the proof is complete. □

Passing to the (QVEP^2), (QVEP^2_n), we obtain a similar conclusion as that Theorem 3.3.

Theorem 3.4. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n inner converges continuously to T in A;*
(iii) *f_n Γ^i_C-converges to f.*

Then, for any $\varepsilon_n \searrow \varepsilon_0$, $S^2(f, T, K, \varepsilon_0) \subseteq \liminf_{n\to\infty} S^2(f_n, T_n, K_n, \varepsilon_n)$.

Combining Theorems 3.1 and 3.3, we derive the sufficient conditions for the Painlevé-Kuratowski convergence of solution sets for (QVEP^1) and (QVEP^1_n).

Theorem 3.5. *Suppose that all conditions in Theorems 3.1 and 3.3 are satisfied. Then, for any $\varepsilon_n \searrow \varepsilon_0$, $S^1(f_n, T_n, K_n, \varepsilon_n) \xrightarrow{\text{P.K.}} S^1(f, T, K, \varepsilon_0)$.*

Combining Theorems 3.2 and 3.4, we derive the sufficient conditions for the Painlevé-Kuratowski convergence of solution sets for (QVEP^2) and (QVEP^2_n).

Theorem 3.6. *Suppose that all conditions in Theorems 3.2 and 3.4 are satisfied. Then, for any $\varepsilon_n \searrow \varepsilon_0$, $S^2(f_n, T_n, K_n, \varepsilon_n) \xrightarrow{\text{P.K.}} S^2(f, T, K, \varepsilon_0)$.*

Remark 3.3. (i) In special case as in Remark 3.2(i). Then, Theorem 3.4 improves and extends Theorem 3.5 in [5].
(ii) In special case as in Remark 3.2(ii). Then, Theorem 3.4 and Theorem 3.6 improve and extend Theorem 3.3(a) and Theorem 3.4, respectively in [7].
(iii) Our Theorems 3.3 and 3.5 are new.

4. Applications

Since vector quasiequilibrium problems contain many problems related to optimization, namely, optimization problems, variational inequality problems, fixed-point problems, etc, the obtained results of the previous sections can be employed to derive the corresponding results for such special cases. In this section we discuss only some corollaries for vector quasivariational inequality problems in the types of Minty and Stampacchia as examples.

Let X and Y be two Banach spaces, C, A, K, T, K_n, T_n be as in Sect. 2 and $Z = L(X; Y)$ be the space of all linear continuous operators from X into Y. Denoted by $\langle z, x\rangle$ the value of a linear operator $z \in L(X; Y)$ at $x \in X$. We consider the two following vector quasivariational inequality problems in the types of Minty and Stampacchia.

(MQVIP) finding $x \in K(x)$ such that

$$\langle z, y - x\rangle \in Y \setminus -\text{int}C, \forall y \in K(x), \forall z \in T(y),$$

(SQVIP) finding $x \in K(x)$ such that $\exists z \in T(x)$ satisfying

$$\langle z, y - x\rangle \in Y \setminus -\mathrm{int}C, \forall y \in K(x).$$

We also consider the two following vector quasivariational inequality problems in the types of Minty and Stampacchia (in short, (MQVIP)$_n$ and (SQVIP)$_n$, respectively).

(MQVIP)$_n$ finding $x_n \in K_n(x_n)$ such that

$$\langle z, y - x_n\rangle \in Y \setminus -\mathrm{int}C, \forall y \in K_n(x_n), \forall z \in T_n(y),$$

(SQVIP)$_n$ finding $x_n \in K_n(x_n)$ such that $\exists z_n \in T_n(x_n)$ satisfying

$$\langle z_n, y - x_n\rangle \in Y \setminus -\mathrm{int}C, \forall y \in K_n(x_n).$$

The approximate solution set of (MQVIP) is defined by

$$\Psi^1(T, K, \varepsilon) = \{x \in K(x) : \langle z, y - x\rangle + \varepsilon e \in Y \setminus -\mathrm{int}C, \forall y \in K(x), \forall z \in T(y)\}.$$

Similarly, the approximate solution sets of (SQVIP), (MQVIP)$_n$ and (SQVIP)$_n$ are defined by $\Psi^2(T, K, \varepsilon)$, $\Psi^1(T_n, K_n, \varepsilon_n)$ and $\Psi^2(T_n, K_n, \varepsilon_n)$, respectively.

Corollary 4.1. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n inner converges continuously to T in A.*

Then, for any $\varepsilon_n \to \varepsilon_0$, $\limsup_{n\to\infty} \Psi^1(T_n, K_n, \varepsilon_n) \subseteq \Psi^1(T, K, \varepsilon_0)$.

Proof. Setting $f(x, z, y) = \langle z, y-x\rangle$ and $f_n(x_n, z_n, y) = \langle z_n, y-x_n\rangle$, the problems (MQVIP) and (MQVIP)$_n$ become the particular cases of (QVEP)^1 and (QVEP)^1_n, respectively. It is clear that all assumptions of Theorem 3.1 are fulfill, and hence by applying Theorem 3.1 we establish the conclusion of Corollary 4.1. □

Passing to the (SQVIP), (SQVIP)$_n$, we obtain a similar conclusion as that Corollary 4.1.

Corollary 4.2. *Suppose that*

(i) *K_n converges continuously to K with compact values in A;*
(ii) *T_n outer converges continuously to T with compact values in A.*

Then, for any $\varepsilon_n \to \varepsilon_0$, $\limsup_{n\to\infty} \Psi^2(T_n, K_n, \varepsilon_n) \subseteq \Psi^2(T, K, \varepsilon_0)$.

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