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On some parameters in the space of regulated functions and their applications

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ABSTRACT. In this paper, we study a class of discontinuous functions being a space of solutions for some differential and integral equations. We investigate functions having finite one-sided limits, i.e. regulated functions. In the space of such functions, we introduce some new concepts like a modulus of equi-regularity or a measure of noncompactness, allowing us to unify the proofs for the results about existence for both continuous and discontinuous solutions. An example of applications for quadratic integral equations, essentially improving earlier ones, completes the paper.

1. INTRODUCTION

It is known, that for differential and integral equations we are looking for solutions in different function spaces. Functions being at least continuous are usually considered as solutions for differential problems, but for impulsive equations or generalized differential equations, one cannot expect continuous solutions. If necessary, some authors consider, as a solution space, the space of functions with bounded variation $BV([0,1],X)$ (cf. [27], for instance) or with generalized bounded variation (see [4] for recent results). This additional regularity requirement leads to some extra assumptions for considered functions, which are required for ensuring the boundedness of the variation of solutions. If we don't need such a property of solutions and we can drop it, we should look for solutions in a larger space (cf. [29]). On the other hand, for integral equations, the continuity of solutions seems to be too restrictive.

The main goal of the paper is to study the space of regulated functions, i.e. having finite one-side limits at every point, similarly as the classical space of continuous functions. It allows us to investigate in a unified manner continuous and regulated solutions. To do it we study the space of regulated functions and we introduce some indices corresponding to those known for the space of continuous functions. We define a modulus of equi-regularity and a measure of noncompactness in this space. Our approach allows to studying on a unified manner continuous and discontinuous solutions for some differential and integral problems.

A usefulness of our approach will be clarified by presenting an application. We will study some quadratic integral equations. When studying such problems we are able to prove the existence for both continuous and discontinuous solutions ([9, 17], for instance). If we need to find continuous solutions, then we can use some known properties of the space of continuous functions (like a compactness criterion, for instance). For discontinuous solutions, the problem is more subtle. If we try to keep the advantage of the first approach and the applicability of the second one, we should investigate the regularity of

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solutions for considered problems. We propose to investigate one of the spaces of discontinuous functions as solutions, namely regulated functions.

We will investigate the existence of regulated solutions of the considered quadratic Kurzweil-Stieltjes integral equation:

$$x(t) = g(t) + \lambda \cdot T(x)(t) \cdot \int_0^1 f(s, x(s))\, d_s K(t, s), \tag{1.1}$$

where the integral is considered to be the Kurzweil-Stietjes one (cf. [29], for instance). When we are looking for continuous solutions and $f(\cdot, x(\cdot))$ is continuous it is sufficient to consider the Riemann-Stieltjes or the Lebesgue-Stieltjes integral (cf. [11, 22]), but in our case the situation is more delicate (some common points of discontinuity for both functions are allowed). Note that such problems were also studied in the space of functions with bounded variation ([26, 27], for instance). Note that our results can be also applied for nonlinear generalized fractional quadratic integral equations studied recently in [1] and then their discontinuous solutions can be investigated. It is well-known, that even in the simplest case for Stieltjes-type integrals of the form $F(t) = \int_{[0,t]} f dg$ the function F for some discontinuous f and g being discontinuous is still sufficiently regular, i.e. regulated (cf. Section 3).

Recall that the space $G([0, 1])$ of all regulated functions on an interval $[0, 1]$ consists of functions having finite one-side limits at every point and consequently it contains the space of continuous functions as well as the space of functions with bounded variation (see [21, 22] for more details).

2. Regulated functions.

Let X be a Banach space. A function $u : [0, 1] \to X$ is said to be regulated if there exist the limits $u(t^+)$ and $u(s^-)$ for every points $t \in [0, 1)$ and $s \in (0, 1]$.

Lemma 2.1. [21] *The set of discontinuities of a regulated function is at most countable. Regulated functions are bounded and the space $G([0, 1], X)$ of regulated functions on $[0, 1]$ into the Banach space X is a Banach space too, endowed with the topology of uniform convergence, i.e. with the norm* $\|u\|_\infty = \sup_{t\in[0,1]} \|u(t)\|$.

Not all functions with countable set of discontinuity points are regulated. As an example we can specify the characteristic function $\chi_{\{1,1/2,1/3,...\}} \notin G([0, 1], \mathbb{R})$. In fact, a function is regulated if and only if it is a uniform limit of step functions. Clearly, $C([0, 1], X) \subset G([0, 1], X)$ and $BV([0, 1], X) \subset G([0, 1], X)$. When $(X, \|\cdot\|)$ is a Banach algebra with the multiplication $*$ the space $G([0, 1], X)$ becomes a Banach too endowed with the pointwise product, i.e. $(f \cdot g)(x) = f(x) * g(x)$ (cf. [13, 18]). In contrast to the case of continuous functions, it is worthwhile to note that the composition of regulated functions need not to be regulated. The simplest example is a composition $(g \circ f)$ of functions $f, g : [0, 1] \to \mathbb{R}$: $f(x) = x \cdot \sin \frac{1}{x}$ and $g(x) = \operatorname{sgn} x$ (both are regulated), which have no one-side limits at 0. Thus, even a composition of a regulated and continuous functions need not be regulated and one cannot expect, that we can simply replace continuous functions by regulated ones.

When we need to study some properties of regulated functions the notion of equi-regularity plays an important role.

Definition 2.1. A set $\mathcal{A} \subset G([0, 1], X)$ is said to be equi-regulated at $t_0 \in [0, 1]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in \mathcal{A}$

i): if $t_0 - \delta < s < t_0$ then $\|x(s) - x(t_0^-)\| < \varepsilon$;

ii): if $t_0 < \tau < t_0 + \delta$ then $\|x(\tau) - x(t_0^+)\| < \varepsilon$.

If such a number δ does not depend on the choice of t_0, then we will call the set *uniformly equi-regulated*.

Let us begin by considering a kind of modulus of equi-regularity in $G([0,1], X)$ in such a way to obtain similar properties as the modulus of continuity for the space $C([0,1], X)$. Recall its definition:

$$\omega_\delta^C(A) = \sup_{x\in A}\{\|x(t) - x(s)\| : \ t, s \in [0,1], |t-s| < \delta\} = \sup_{x\in A} \sup_{t,s\in[0,1],|t-s|<\delta} \|x(t) - x(s)\|$$

and

$$\omega^C(A) = \lim_{\delta\to 0} \omega_\delta^C(A). \tag{2.2}$$

It is sometimes called uniform modulus of equicontinuity, in contrast to the pointwise modulus of equicontinuity:

$$\omega_\delta^C(A,t) = \sup_{x\in A}\{\|x(t) - x(s)\| : \ s \in [0,1], |t-s| < \delta\} = \sup_{x\in A} \sup_{s\in[0,1],|t-s|<\delta} \|x(t) - x(s)\|.$$

Let us mention an important fact, that for $X = \mathbb{R}$ the above formula for $\omega^C(A)$ define a measure of noncompactness in the space $C([0,1], \mathbb{R})$. Moreover, the Hausdorff measure of noncompactness of a bounded subset A, i.e. $\beta_C(A)$ in this space is equal to $\frac{1}{2}\omega^C(A)$ ([2, Section 1.1.10], for instance).

For the space $C([0,1], X)$ of vector-valued functions the following formula $\mu_c(A) = \omega^C(A) + \sup_{t\in[0,1]} \mu(A(t))$ defines a measure of noncompactness provided that μ is a measure of noncompactness in X. As measures of noncompactness form a useful tool for the studies of several integral or differential problems (see [8, 11, 17], for instance) and the particular form of such a measure in some function spaces are very useful in such studies, we will extend these notions to the space $G([0,1], X)$. We will stress on illustrative form of a paper.

Definition 2.2. For a bounded subset $A \subset G([0,1], X)$, $t \in [0,1]$ and $\delta > 0$ we define

$$\begin{aligned} \omega_\delta^G(A,t) &= \sup_{x\in A} \sup_{s\in(0,1],t-\delta<s<t} \|x(s) - x(t^-)\| \\ &+ \sup_{x\in A} \sup_{s\in[0,1),t<s<t+\delta} \|x(s) - x(t^+)\|, \end{aligned}$$

(with the convention $x(1+) = x(1)$ and $x(0-) = x(0)$) and

$$\begin{aligned} \omega_\delta^G(A) &= \sup_{x\in A} \sup_{t\in[0,1]} \sup_{s\in(0,1],t-\delta<s<t} \|x(s) - x(t^-)\| \\ &+ \sup_{x\in A} \sup_{t\in[0,1]} \sup_{s\in[0,1),t<s<t+\delta} \|x(s) - x(t^+)\|. \end{aligned}$$

Then a function

$$\omega^G(A) = \lim_{\delta\to 0} \omega_\delta^G(A).$$

will be called a (uniform) modulus of equi-regularity of the set A. Similarly we define the pointwise modulus of equi-regularity at the point $t_0 \in (0,1)$ by

$$\begin{aligned} \omega^G(A,t_0) &= \lim_{\delta\to 0} \Bigg(\sup_{x\in A} \sup_{s\in[0,1],t_0-\delta<s<t_0} \|x(s) - x(t_0^-)\| \\ &+ \sup_{x\in A} \sup_{s\in[0,1],t_0<s<t_0+\delta} \|x(s) - x(t_0^+)\| \Bigg). \end{aligned}$$

A version of these definitions for an arbitrary compact interval $[a,b]$ is immediate. As a consequence of the above definition and [19, Corollary 2.4] we get:

Proposition 2.1. *For a subset A of $G([0,1],X)$ we have $\omega^G(A)=0$ if and only if A is uniformly equi-regulated. Consequently, for any relatively compact subsets B of $G([0,1],X)$ we have $\omega^G(B)=0$.*

In the case when both indices are well-defined, i.e. when $A \subset C([0,1],X)$, we have:

Lemma 2.2. *Let A be a subset of $C([0,1],X)$. Then:*

$$\omega^G(A) \leq 2\omega^C(A).$$

Proof. Clearly for any $x \in A$ and $\delta > 0$

$$\sup_{t,s\in[0,1],t-\delta<s<t} \|x(t)-x(s)\| \leq \sup_{t,s\in[0,1],|t-s|<\delta} \|x(t)-x(s)\|$$

and similar estimation for the right neighbourhood of t. Thus

$$\sup_{x\in A} \sup_{t,s\in[0,1],t-\delta<s<t} \|x(t)-x(s)\| \leq \omega^C_\delta(A),$$

$$\sup_{x\in A} \sup_{t,s\in[0,1],t<s<t+\delta} \|x(t)-x(s)\| \leq \omega^C_\delta(A)$$

and then

$$\omega^G_\delta(A) \leq 2\cdot\omega^C_\delta(A).$$

Taking a limit when $\delta \to 0$ we get the thesis. □

To the best of our knowledge, there was the only a few attempts to study such a type of moduli. In [24] such a definition for the space of cádlàg functions $D([0,1]) \subset G([0,1])$ was presented (in a different manner, based on partitions of the interval). It is also discussed for $D([0,1])$ in [14, Chapter 3, Section 14].

Corollary 2.1. *If the set $A \subset C([0,1],X)$ is uniformly equicontinuous, then it is uniformly equi-regulated (as a subset of $G([0,1],X)$).*

Our modulus is, in some sense, uniform. Following the idea from [30, Proposition 12.2] for ω^C we immediately get the following lemma:

Lemma 2.3. *Let A be a subset of $G([0,1],X)$. Then*

$$\sup_{t_0\in(0,1)} \omega^G(A,t_0) \leq \omega^G(A) \leq 2\cdot \sup_{t_0\in(0,1)} \omega^G(A,t_0).$$

The above estimation is best possible. Namely, we have

Example 2.1. Consider the set of simple functions defined on $[-1,1]$ by the following manner: $x_n(t)=-1$ for $t \leq -\frac{1}{n}$, $x_n(t)=1$ for $t > \frac{1}{n}$ and $x_n(t)=0$ for $t \in (\frac{1}{n},\frac{1}{n})$.

Let $A=\{x_n : n \geq 2\} \subset G([0,1],\mathbb{R})$. Then $\omega^G(A,t)=0$ for $t \neq 0$, $\omega^G(A,0)=2$, but the uniform modulus $\omega^G(A)=4$ (cf. the Nussbaum example in [30, Example 12.3]).

As a consequence of the above theorem we get a result of Ambrosetti-type (cf. [16, Theorem 3.1], for the Kuratowski measure of noncompactness in $G([0,1])$):

Corollary 2.2. *If $A \subset G([0,1],X)$ is bounded and equi-regulated, then $t \to \omega^G(A,t)$ is a regulated function.*

Example 2.2. Some examples of sets $A \subset G([0,1],X)$.

A) *A being uniformly equicontinuous subset of $C([0,1],X)$ is also equi-regulated,*

B) *Let $A=\{\chi_E : E-$ finite subset of $X\}$. Then $\omega^G(A)=0$.*

C) *Let* $A = \{\chi_{[a,1]} : a \in [0,1]\}$. *Then* $\omega^G(A) = 0$. *Note that* $\omega^C(A)$ *is also defined, but* $\omega^C(A) = 1$.

D) *Let* $A = \{y_n : n \in \mathbb{N}\}$ *with* $y_n(t) = \int_0^t f_n(s)\, dg(s)$, *with* $Var_0^1\, f_n \leq M$ *for all* n *and continuous* g *with* $\|g\|_\infty \leq L$. *Then* $\omega^G(A) \leq M \cdot L$.

A mapping μ defined on a family of all nonempty and bounded subsets of E with non-negative values is said to be a regular measure of noncompactness in a Banach space E (cf. [8]) if it satisfies the following conditions:

(i) $\mu(X) = 0 \Leftrightarrow X$ is relatively compact in E,
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
(iii) $\mu(\overline{X}) = \mu(conv\ X) = \mu(X)$,
(iv) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$,
(vi) $\mu(X \cup \{y\}) = \mu(X)$.

In fact, in the book [8] the definition of a regular measure of noncompactness is less restrictive and only axioms (i), (ii) and (vi) are required. Note that all measures of noncompactness considered in this paper have always all the above properties, so we will use more restrictive definition given above.

In an axiomatic theory of measures of noncompactness one more axiom (called the generalized Cantor intersection property) is usually considered. It is the following property of measures of noncompactness:

(vii) Let X_n be a sequence of nonempty, bounded and closed subsets of X such that $X_n \supset X_{n+1}$ $(n = 1, 2, ...)$ and $\lim_{n\to\infty} \mu(X_n) = 0$. Then the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty

This property, in particular, is necessary in the proof of the Darbo fixed point theorem (also for for axiomatic measures of noncompactness (cf. [6]). Since we are interested in particular functions being measures of noncompactness in the sense of the above definition, the last axiom will follow from other properties. Namely, we have

Lemma 2.4. ([28, p.19]) *If a measure of noncompactness* μ *has the properties (i), (ii) and (vi), then it has also the property (vii).*

In applications, it is important to find an analytical formula for measures of noncompactness (cf. [2, 30]). To the best of our knowledge, such a formula does not exist for the space of regulated functions. We fulfil this gap and we will show how to apply this new notion. By μ_X denote an arbitrary regular measure of noncompactness satisfying (i)-(vi) in the space X (cf. also [6]).

Theorem 2.1. *The following function* μ_G *is a regular measure of noncompactness satisfying (i)-(vii) in the space* $G([0,1], X)$:

$$\mu_G(A) = \omega^G(A) + \sup_{t\in[0,1]} \mu_X(A(t)),$$

provided μ_X *is a regular measure of noncompactness on* X *satisfying (i)-(vi).*

Proof. (i) As the functions are non-negative, we have: $\mu_G(A) = 0$ if and only if $\omega^G(A) = 0$ and $\sup_{t\in[0,1]} \mu_X(A(t)) = 0$. By Proposition 2.1 the set A is equi-regulated. Since μ_X is a regular measure of noncompactness, all the sets $A(t)$ are relatively compact. By using the compactness characterization in $G([0,1], X)$ (see [19]) we get the thesis.

Conditions (i) - (vi) follow from the known properties of the supremum and the norm in X. Recall that μ_X has the required properties (cf. [8] for more details). □

In the sequel we will always assume, that μ_X has properties (i)-(vi), and consequently μ_G will have the same properties on $G([0,1],X)$ (including (vii) due to Lemma 2.4).

As an immediate consequence we have:

Corollary 2.3. *If a bounded subset A of $G([0,1],X)$ is equi-regulated, then so is $\overline{conv\ A}$.*

Denote by β_E the classical Hausdorff measure of noncompactness in the space E, i.e. $\beta_E(A) = \inf\{r > 0 : \exists_{x_1,x_2,...,x_k \in E}$ such that $A \subset \bigcup_i B_r(x_i)\}$. Then we are able to estimate the Hausdorff measure of noncompactness β_G in $G([0,1],X)$ by the above defined measures.

Theorem 2.2. *For any subset A of $G([0,1],X)$ we have*

$$\beta_G(A) \leq \omega^G(A) + \sup_{t\in[0,1]} \beta_X(A(t))$$

and

$$\beta_G(A) \leq \sup_{t\in[0,1]} \left[\omega^G(A,t) + \beta_X(A(t))\right].$$

Proof. We follow the idea of Nussbaum [25] and Väth [30, Theorem 12.5]. Let $\varepsilon > 0$ be arbitrary. By the definition of $\omega^G(A)$ there exists a $\delta > 0$ such that for each $s \in (t-\delta, t)$ and $\tau \in (t, t+\delta)$ we have $\|x(s) - x(t-)\| + \|x(t+) - x(\tau)\| \leq \omega^G(A) + \varepsilon/2$ for every $x \in A$.

Since $[0,1]$ is compact we can take a finite set $\{t_1, t_2, ..., t_k\} \subset [0,1]$ such that $[0,1] \subset \bigcup_{i=1}^k (t_i - \delta, t_i + \delta)$. Take a partition of unity subordinated to this cover: $\lambda_1, \lambda_2, ..., \lambda_k$ with $\sum_{i=1}^k \lambda_i(t) = 1$ for any $t \in [0,1]$.

Denote by c the number $\sup_{t\in[0,1]} \beta_X(A(t))$. By using the definition of the Hausdorff measure of noncompactness β_X we are able to find a finite covering of each set $A(t_n)$ ($n = 1, 2, ..., k$) by balls with radius $r > 0$ less than $c + \varepsilon/2$, say $(B_r(w_m^n))_{m=1,...,p}$ with centers denoted by (w_m^n).

For any point t_n and each choice of the points w_m^n define a function

$$y_p(t) = \sum_{m=1}^{p} \lambda_i(t) w_m^n.$$

Clearly, $y_p \in G([0,1],X)$. Now define a (finite) set N as the set of all such functions y_p ($n = 1, ..., k$, $m = 1, ..., p$).

To prove the first inequality it suffices to show, that the set of balls $(B_{r_0}(y_p)) \subset G([0,1],X)$ with $y_p \in N$ is a finite covering for A, where $r_0 = \omega^G(A) + c + \varepsilon$.

Indeed, for any $x \in A$ and any t_n, by the definition of $\beta_X(A(t_n))$, there exist $v_n \in N$ such that $\|x(t_n) - v_n\| \leq \beta_X(A(t_n)) + \varepsilon$ for $n = 1, 2, ..., p$. By using that elements, define a function from N

$$z_p(t) = \sum_{i=1}^{p} \lambda_i(t) v_i.$$

Take arbitrary $s \in (t_n - \delta, t_n)$ and $\tau \in (t_n, t_n + \delta)$. Let $c_n = \sup_{t\in[0,1]} \beta_X(A(t_n))$. Then

$$\|v_n - x(s)\| \leq \|v_n - x(t_n-)\| + \|x(t_n-) - x(s)\| \leq (c_n + \varepsilon/2) + \omega^G(A) + \varepsilon/2$$

and

$$\|v_n - x(\tau)\| \leq \|v_n - x(t_n+)\| + \|x(t_n+) - x(\tau)\| \leq (c_n + \varepsilon/2) + \omega^G(A) + \varepsilon/2.$$

Thus for any $t \in (t_n - \delta, t_n + \delta)$ we have $\|v_n - x(t)\| \leq (c_n + \varepsilon/2) + \omega^G(A) + \varepsilon/2$.

As z_p is constructed in a basis of a partition of unity $z_p(t) = 0$ outside of this interval, i.e. for the remaining n we have $\lambda_n(t) = 0$. We need to estimate the distance between x and z_p in $G([0,1],X)$.. For any $t \in [0,1]$ we have

$$\begin{aligned}\|x(t) - z_p(t)\| &= \left|\sum_{i=1}^{p} \lambda_i(t) \cdot (v_n - x(t))\right| \\ &\leq \sum_{i=1}^{p} \lambda_i(t) \cdot \left|c_n + \varepsilon + \omega^G(A)\right| \\ &\leq c + \varepsilon + \omega^G(A).\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the function $x \in B_r(v_n)$ with $r = c+\varepsilon+\omega^G(A) = \sup_{t\in[0,1]} \beta_X(A(t))+\omega^G(A)$, so we have a finite covering of A by balls of radius less than $\sup_{t\in[0,1]} \beta_X(A(t)) + \omega^G(A)$ and by the definition of the Hausdorff measure of noncompactness

$$\beta_G(A) \leq \sup_{t\in[0,1]} \beta_X(A(t)) + \omega^G(A).$$

The proof of the second inequality runs as above. The only exception lies in the fact, that we need to use an estimation $\|x(s) - x(t_n-)\| + \|x(t_n+) - x(\tau)\| \leq \omega^G(A, t_n) + \varepsilon/2$ instead of that one with $\omega^G(A)$. □

For the Kuratowski measure of noncompactness $\alpha_X(A)$ (i.e. the infimum over all $\varepsilon > 0$ for which there exists a finite cover of A by sets with the diameter less than ε, cf. [8]) we are able to present now a result being an extension for [16, Theorem 3.2]:

Corollary 2.4. *For any subset A of $G([0,1],X)$ we have*

$$\alpha_G(A) \leq \omega^G(A) + 2 \cdot \sup_{t\in[0,1]} \alpha_X(A(t))$$

and

$$\alpha_G(A) \leq 2 \cdot \sup_{t\in[0,1]} \left[\omega^G(A,t) + \alpha_X(A(t))\right].$$

The above estimates are direct consequences of Theorem 2.2 and some relationships between measures of noncompactness (cf. [2, 30]).

As an immediate consequence of Theorem 2.1 we get the Ascoli-type theorem (cf. also [19, Corollary 2.4]):

Theorem 2.3. *A bounded subset A of $G([0,1],X)$ is relatively compact if and only if $\mu_G(A) = 0$. Consequently, it is relatively compact iff is equi-regulated and $A(t)$ are relatively compact in X for $t \in [0,1]$.*

In particular, for the case $X = \mathbb{R}$, we will denote $G([0,1],\mathbb{R})$ by $G([0,1])$. We have:

Corollary 2.5. *A bounded subset A of $G([0,1])$ is relatively compact if and only if $\omega_G(A) = 0$. Consequently, it is relatively compact iff is equi-regulated and $A(t)$ are bounded for $t \in [0,1]$.*

Let us summarize this section by presenting some comments about the Nemytskii superposition operator acting on $G([0,1],X)$. It is one of the most important nonlinear operators, which is investigated in different function spaces (cf. [3]) (acting conditions, boundedness, continuity, for instance).

The following theorem is proved by Michalak [23].

Theorem 2.4. [23, Theorem 3.1] *A superposition operator $F(x) = f(\cdot, x(\cdot))$ maps $G([0,1])$ into itself if and only if the function f has the following properties:*

(1) *the limit* $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $(s,x) \in (0,1] \times \mathbb{R}$,
(2) *the limit* $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $(t,x) \in [0,1) \times \mathbb{R}$.

In particular, it means that for the composition operator (autonomous superposition operator) $F(x)(t) = f(x(t))$ maps $G([0,1])$ into itself iff f is continuous. It is worthwhile to recall that an earlier condition (sufficient, but not necessary) was presented in [5].

Corollary 2.6. [5, Theorem 2.3] *Suppose that the function* $h(\cdot,u)$ *is regulated on* $[0,1]$ *for all* $u \in \mathbb{R}$, *and the function* $h(t,\cdot)$ *is continuous on* $\mathbb{R}$, *uniformly with respect to* $t \in I$. *Then the superposition operator* $F(u)(t) = h(t,u(t))$ *maps* $G([0,1])$ *into itself and is (norm) bounded.*

We will need also a continuity property for the superposition operator:

Theorem 2.5. [23, Corollary 3.6] *A superposition operator* $F(x) = f(\cdot, x(\cdot))$ *maps* $G([0,1])$ *into itself is continuous if and only if a function* $\tilde{f} : \mathbb{R} \to G([0,1])$ *given by the formula* $\tilde{f}(x)(t) = f(t,x)$ *is continuous.*

Finally, let us present an acting condition for linear operators on $G([0,1])$. Put $I = [0,1]$.

Theorem 2.6. [26, Theorem 1] *Assume, that* $K : I \times I \to \mathbb{R}$ *satisfies*

(1) $K(t,\cdot)$ *is a function of bounded variation for every* $t \in I$ *say by* M, *i.e.* $\|K(t,\cdot)\|_{BV(I)} \leq M$, *for some* $M > 0$,
(2) $K(\cdot,s) \in G(I)$.

Then the linear operator $H(x)(t) = \int_I x(s)\ d_sK(t,s)$ *maps* $G(I)$ *into itself and is bounded with* $\|H\| \leq 2\sup_{t\in I} \|K(t,\cdot)\|_{BV(I)}$.

3. Quadratic integral equations.

We will present an example of applications for the presented theory. For simplicity, we put in this section $X = \mathbb{R}$, but it is easy to notice that similar results hold true in the case when X is a commutative Banach algebra of vector-valued functions.

For the considered quadratic problem the use of a fixed point theorem of Darbo type [8, 11] seems to be most appropriate, but its use in practice require an analytical formula for a measure of noncompactness in $G([0,1])$, i.e. ω^G, which is defined in this paper. By considering quadratic problems one cannot expect, in general, the compactness of considered operators and further the use of the Banach fixed point theorem seems to be too restrictive. Thus, some assumptions guaranteeing, that at least one of the operators is a contraction with respect to a measure of noncompactness are optimal in our method of the proof. The study of measures of noncompactness seems to be important in such a case. Let us recall a general form of such fixed point theorems of Darbo type:

Theorem 3.7. *Let* W *be a nonempty, bounded, closed and convex subset of* E *and let* $V : W \to W$ *be a continuous transformation which is a contraction with respect to the regular measure of noncompactness* μ *satisfying (i)-(vi), i.e. there exists* $k \in [0,1)$ *such that*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X *of* E. *Then* V *has at least one fixed point in the set* W *and the set of all fixed points for* V *is compact in* E.

We will study the problem (1.1) under the assumptions allowing us to prove the existence of discontinuous solutions, but not necessarily being of bounded variation. As the existence of finite values for the function $t \to \int_0^t x(s)\ dg(s)$ implies that it is a regulated function ([29]), the space $G(I)$ seems to be a natural space for solutions of the considered problem (1.1). Recall that such a kind of problems, till now, was investigated under the conditions allowing to find either continuous solutions ([11], for instance) or being of bounded variation ([4], for instance).

Let us describe some operators from the equation (1.1):

$$x(t) = g(t) + \lambda \cdot T(x)(t) \cdot \int_0^1 f(s, x(s))\, d_s K(t, s).$$

Note that in a special case $K(t,s) = \int_0^s \int_0^t p(\tau,\rho)\, d\rho\, d\tau$ for some p we have usual quadratic Hammerstein integral equations, but considered under less restrictive assumptions. More special cases can be found in the paper [10]. Note that we will study the case of quadratic integral equations having solutions not necessarily neither continuous nor of bounded variation (so the operators do not preserve these properties).

Denote by H the operator associated with the right-hand side of equation (1.1) i.e. $H(x) = g + H_1(x)$, where $H_1(x) = \lambda \cdot T(x) \cdot B(x) = \lambda \cdot T(x) \cdot (D \circ F)(x)$, where $D(x)(t) = \int_I x(s)\, d_s K(t,s)$ is the linear integral operator and $F(x)(t) = f(t, x(t))$ is the superposition operator generated by f.

The operator A can be treated as a pointwise product (multiplication) of the operator T and the Hammerstein operator B. As $G(I)$ is a Banach algebra with the pointwise multiplication (cf. [19]), we will consider the case when all mentioned operators are acting from $G(I)$ into itself.

We shall treat equation (1.1) under the following assumptions listed below:

(i) $g \in G(I)$,

(ii) Assume, that $K : I \times I \to \mathbb{R}$ satisfies $K(t,\cdot)$ is a function of bounded variation for every $t \in I$ with $\|K(t,\cdot)\|_{BV(I)} \le M$, for some $M > 0$, $K(\cdot, s) \in G(I)$ and

$$\lim_{\varepsilon\to 0^+} \left(\sup \left\{ Var_0^1[K(t^-,\cdot) - K(\tau,\cdot)] :\ t \in (0,1], \tau \in (t-\varepsilon, t) \right\}\right) = 0 \tag{3.3}$$

$$\lim_{\varepsilon\to 0^+} \left(\sup \left\{ Var_0^1[K(t^+,\cdot) - K(\tau,\cdot)] :\ t \in [0,1), \tau \in (t, t+\varepsilon) \right\}\right) = 0, \tag{3.4}$$

(iii) $f : I \times \mathbb{R} \to \mathbb{R}$ is regulated in the first variable and satisfies the Lipschitz condition in the second one with Lipschitz constant L,

(iv) $T : G(I) \to G(I)$ is continuous and $\|T(x)\|_\infty \le m_1 + m_2 \|x\|_\infty$ for any $x \in G(I)$ and for some non-negative constants m_1, m_2 and assume, that T satisfies the condition:

$$\sup \omega_\varepsilon^G(\{T(x) : x \in A\}) \le Q \cdot \omega_\varepsilon^G(A) \tag{3.5}$$

for some $Q \in \mathbb{R}_+$,

(v) for any $r > 0$ satisfying the quadratic inequality $s \le \|g\|_\infty + \lambda(m_1 + m_2 \ \cdot\ s) \cdot M \cdot\ (\|f(t,0)\|_\infty + L \cdot s)\ ,\ s \in [0, r]$, assume, that

$$2 \cdot \lambda \cdot (\|f(t,0)\|_\infty + L \cdot r) \cdot M \cdot Q < 1.$$

Remark 3.1. Let us remark, that the condition (3.5) is satisfied when T is compact (by Theorem 2.1) as acting between $G(I)$ and $G(I)$ or $C(I)$ (by Lemma 2.2) or when T is Lipschitz with constant Q. If T maps bounded sets into equi-regulated sets in $G(I)$ or equicontinuous sets in $C(I)$, then this condition is also satisfied (see also Section 2).

Theorem 3.8. *Let assumptions (i)–(v) be satisfied. Then the set of solutions for the equation (1.1) is nonempty and compact as a subset of $G(I)$.*

Proof. First of all observe that the considered operators are well-defined on $G(I)$. Our assumption (iii) form a sufficient acting condition for F (see Theorem 2.4 and Corollary 2.6) and then $F(G(I)) \subset G(I)$. Taking into account Theorem 2.6, the assumption (ii) implies, that B maps $G(I)$ into itself. Since $g \in G(I)$, we get $H : G(I) \to G(I)$.

We need to prove, that the operator satisfies the assumptions of Theorem 3.7. To do it, we will construct an invariant bounded, closed and convex set $W \subset G(I)$. Then we will prove, that H is continuous on W and is a contraction with respect to the measure of noncompactness ω^G.

Fix an arbitrary $x \in G(I)$ and $t \in I$. Since by (iii) $f(t,x)$ satisfies the Lipschitz condition with constant L, we get $\|f(t,x)\| \leq \|f(t,x) - f(t,0)\| + \|f(t,0)\| \leq L \cdot \|x\| + \|f(t,0)\|$. Thus

$$\|f(t,x)\| \leq a(t) + L \cdot \|x\|$$

for the regulated real-valued and non-negative function $a(t) = \|f(t,0)\|$.

In view of the assumptions (i)-(iv) and by using the properties of the Henstock-Stieltjes integral, we have

$$\begin{aligned} |(H(x))(t)| &\leq |g(t)| + \lambda \cdot |T(x)(t)| \cdot \int_0^1 |f(s,x(s))| \, d_s K(t,s) \\ &\leq |g(t)| + \lambda \|T(x)(t)\|_\infty \|K(t,\cdot)\|_{BV(I)} \|F(x)\|_\infty \\ &\leq |g(t)| + \lambda(m_1 + m_2\|x\|_\infty) \cdot M \cdot \|F(x)\|_\infty \\ &\leq |g(t)| + \lambda(m_1 + m_2\|x\|_\infty) \cdot M \cdot (\|a\|_\infty + L\|x\|_\infty). \end{aligned}$$

Then $\|(H(x))\|_\infty \leq \|g\|_\infty + \lambda(m_1 + m_2\|x\|_\infty) \cdot M \cdot (\|a\|_\infty + L\|x\|_\infty)$. Note that for the quadratic inequality $s \leq \|g\|_\infty + \lambda(m_1 + m_2 \cdot s) \cdot M \cdot (\|a\|_\infty + L \cdot s)$, there always exists a number $r > 0$ such that for any $s \in [0,r]$ this inequality holds true, so the above estimations give us that H transforms the ball B_r into itself. Obviously, the set B_r is nonempty bounded closed and convex and we can put $W = B_r$ in Theorem 3.7.

Now, we show that H is continuous on the set $B_r \subset G(I)$. As T is continuous (assumption (iv)), for arbitrary number $\varepsilon > 0$ there exists an $\delta > 0$ be such that $\|T(x) - T(y)\|_\infty < \varepsilon$ whenever $\|x - y\|_\infty < \delta$, $x, y \in B_r \subset G(I)$. Without loss of generality, we assume $\delta < \varepsilon$.

Then, for $t \in I$ and $\|x - y\|_\infty < \delta$, we have the following estimates:

$$\begin{aligned} & |(H(x))(t) - (H(y))(t)| \\ \leq\ & \lambda |T(x)(t) \cdot \int_0^1 f(s,x(s)) \, d_s K(t,s) - T(y)(t) \cdot \int_0^1 f(s,y(s)) \, d_s K(t,s)| \\ \leq\ & \lambda |T(x)(t) \cdot \int_0^1 f(s,x(s)) \, d_s K(t,s) - T(y)(t) \cdot \int_0^1 f(s,x(s)) \, d_s K(t,s)| \\ +\ & \lambda |T(y)(t) \cdot \int_0^1 f(s,x(s)) \, d_s K(t,s) - T(y)(t) \cdot \int_0^1 f(s,y(s)) \, d_s K(t,s)| \\ \leq\ & \lambda |T(x)(t) - T(y)(t)| \cdot |\int_0^1 f(s,x(s)) \, d_s K(t,s)| \\ +\ & \lambda |T(y)(t)| \cdot |\int_0^1 (f(s,x(s)) - f(s,y(s))) \, d_s K(t,s) \\ \leq\ & \lambda \|T(x) - T(y)\|_\infty \cdot \|F(x)\|_\infty \|K(t,\cdot)\|_{BV(I)} \\ +\ & \lambda \|T(y)\|_\infty |\int_0^1 (f(s,x(s)) - f(s,y(s))) \, d_s K(t,s)| \\ \leq\ & \lambda \cdot \varepsilon \cdot (\|a\|_\infty + L \cdot r) + \lambda(m_1 + m_2 r) \cdot |\int_0^1 L \cdot |x(s) - y(s)| \, d_s K(t,s)| \\ \leq\ & \lambda \cdot \varepsilon \cdot (\|a\|_\infty + L \cdot r) \cdot M + \lambda(m_1 + m_2 r) \cdot L \cdot \delta \cdot \|K(t,\cdot)\|_{BV(I)} \\ \leq\ & \lambda \cdot \varepsilon \cdot (\|a\|_\infty + L \cdot r) \cdot M + \lambda(m_1 + m_2 r) \cdot L \cdot \varepsilon \cdot M. \end{aligned}$$

Thus we obtain the following inequality for $x, y \in B_r$:

$$\|H(x) - H(y)\|_\infty \le \lambda \cdot \varepsilon \cdot (\|a\|_\infty + L \cdot r) \cdot M + \lambda(m_1 + m_2 r) \cdot L \cdot \varepsilon \cdot M,$$

which implies the continuity of the operator H on the set B_r.

In what follows, let us take a nonempty set $A \subset B_r$. Further, fix arbitrarily a number $\varepsilon > 0$ and choose an arbitrary $x \in A$, $t \in (0,1]$ and $\tau \in (t-\varepsilon, t)$. Since x is regulated and $H : G(I) \to G(I)$, one-side limits $H(x)(t^-)$ and $H(x)(t^+)$ exist at every point t. Let us estimate :

$$\begin{aligned}
& |H(x)(t^-) - H(x)(\tau)| \\
\le\ & |g(t^-) - g(\tau)| + \lambda \cdot | \left(T(x)(t^-) \cdot \int_0^1 f(s, x(s))\, d_s K(t^-, s) \right) \\
-\ & \left(T(x)(\tau) \cdot \int_0^1 f(s, x(s))\, d_s K(\tau, s) \right) | \\
\le\ & |g(t^-) - g(\tau)| + \lambda \cdot | \left(T(x)(t^-) - T(x)(\tau) \right) | \cdot | \int_0^1 f(s, x(s))\, d_s K(t^-, s)| \\
+\ & \lambda |T(x)(\tau)| \cdot | \left(\int_0^1 f(s, x(s))\, d_s K(t^-, s) - \int_0^1 f(s, x(s))\, d_s K(\tau, s) \right) | \\
\le\ & |g(t^-) - g(\tau)| + \lambda | \int_0^1 f(s, x(s))\, d_s K(t^-, s)| \cdot |T(x)(t^-) - T(x)(\tau)| \\
+\ & \lambda |T(x)(\tau)| \cdot | \int_0^1 f(s, x(s))\, d_s [K(t^-, s) - K(\tau, s)] \,| \\
\le\ & \omega_\varepsilon^G(\{g\}, t) + \lambda(\|a\|_\infty + L \cdot r) \cdot M \cdot |T(x)(t^-) - T(x)(\tau)| \\
+\ & \lambda(m_1 + m_2 r) \cdot (\|F(x)\|_\infty) \cdot Var_0^1[K(t^-, \cdot) - K(\tau, \cdot)] \\
\le\ & \omega_\varepsilon^G(\{g\}, t) + \lambda(\|a\|_\infty + L \cdot r) \cdot M \cdot \omega_\varepsilon^G(\{T(x)\}, t) \\
+\ & \lambda(m_1 + m_2 r) \cdot (\|a\|_\infty + L \cdot r) \cdot \sup_{\tau \in (t-\varepsilon, t)} \{Var_0^1[K(t^-, \cdot) - K(\tau, \cdot)]\} \\
\le\ & \omega_\varepsilon^G(\{g\}) + \lambda(\|a\|_\infty + L \cdot r) \cdot M \cdot Q \cdot \omega_\varepsilon^G(A) + \lambda(m_1 + m_2 r) \cdot (\|a\|_\infty + L \cdot r) \cdot \gamma_r^-(\varepsilon),
\end{aligned}$$

where

$$\gamma_r^-(\varepsilon) = \sup_{t \in (0,1], \rho \in (t-\varepsilon, t)} \left\{ Var_0^1[K(t^-, \cdot) - K(\rho, \cdot)] \right\}.$$

Recall, that here $\omega_\varepsilon^G(\{g\}, t) = \sup_{s \in (0,1], t-\varepsilon < s < t} \|g(s) - g(t^-)\| + \sup_{s \in [0,1), t < s < t+\varepsilon} \|g(s) - g(t^+)\|$ (cf. Definition 2.2) and $\omega_\varepsilon^G(\{g\})$ denotes the modulus of a set $\{g\}$ being compact (as a singelton), so in view of Proposition 2.1 and by Theorem 2.3, $\lim_{\varepsilon \to 0} \omega_\varepsilon^G(\{g\}) = 0$. Similarly, for $t \in [0,1)$ and $\rho \in (t, t+\varepsilon)$ we are able to obtain the same estimation

$$|H(x)(t^+) - H(x)(\rho)| \le \omega_\varepsilon^G(\{g\}) + \lambda(\|a\|_\infty + br) \cdot MQ\omega_\varepsilon^G(A) + \lambda(m_1 + m_2 r)(\|a\|_\infty + L \cdot r)\gamma_r^+(\varepsilon).$$

for

$$\gamma_r^+(\varepsilon) = \sup_{t \in [0,1), \tau \in (t, t+\varepsilon)} \left\{ Var_0^1[K(t^+, \cdot) - K(\tau, \cdot)] \right\}.$$

Thus

$$\begin{aligned}
\omega_\varepsilon^G(H(A)) \ \le\ & \sup_{x \in A} \sup_{t \in (0,1], \tau \in (t-\varepsilon, t)} |H(x)(t^-) - H(x)(\tau)| \\
+\ & \sup_{x \in A} \sup_{t \in [0,1), \rho \in (t, t+\varepsilon)} |H(x)(t^+) - H(x)(\rho)| \\
\le\ & 2 \cdot \omega_\varepsilon^G(\{g\}) + 2\lambda(\|a\|_\infty + L \cdot r) \left(MQ\omega_\varepsilon^G(A) + (m_1 + m_2 r)\gamma_r(\varepsilon) \right).
\end{aligned}$$

Here $\gamma_r(\varepsilon) = \max(\gamma_r^-(\varepsilon), \gamma_r^+(\varepsilon))$.

By passing to the limit with $\varepsilon \to 0$ we get

$$\omega^G(H(A)) \leq 2\lambda(\|a\|_\infty + L \cdot r) \cdot \left(M \cdot Q \cdot \omega^G(A) + (m_1 + m_2 r) \cdot \lim_{\varepsilon \to 0} \gamma_r(\varepsilon) \right).$$

Notice that, in view of our assumption (ii) we have that $\gamma_r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Finally, by the assumption (v) the number r is sufficiently small to get $2\lambda(\|a\|_\infty + L \cdot r) \cdot M \cdot Q < 1$ and

$$\mu^G(H(A)) \leq \omega^G(H(A)) \leq 2\lambda(\|a\|_\infty + L \cdot r) \cdot M \cdot Q \cdot \omega^G(A).$$

Now, taking into account the above inequality, H is a contraction with respect μ^G and as claimed above, the assumption (v) allows us to apply Theorem 3.7, which completes the proof. □

Remark 3.2. We should present some comments about the assumption (ii). Observe, that the conditions (3.3) and (3.4) do not result from other assumptions. Define a function $K : I \times I \to \mathbb{R}$ in the following way:

$$K(t,s) = \begin{cases} (t - 1/2) \sin \frac{s}{t-1/2} & \text{for} \quad t, s \in I, t \neq 1/2, \\ 1 & \text{for} \quad t = 1/2, s \in I \end{cases}$$

(cf. [10, Example 8] for a basic idea). Clearly $Var_0^1 K(t,\cdot) \leq (\sin 1) < \infty$, so $\|K(t,\cdot)\|_{BV(I)} \leq (\sin 1)$ and $K(\cdot, s) \in G(I)$ (but not continuous at $t = 1/2$). However, the remaining parts of this assumption, i.e. conditions (3.3) and (3.4) are not satisfied, so they are independent on the initial part of (ii).

Take two points $t = \frac{1}{2} + \frac{1}{2n}$ and $\tau = \frac{1}{2} + \frac{1}{n}$ for some big enough, but fixed $n \in \mathbb{N}$. Let $s_0 = 1/2 < s_1 < s_2 < \ldots < s_{k_0} < s_{k_0+1} = 1$ be the points from $[1/2, 1]$, where $k_0 = Ent\left(\frac{n}{2\pi}\right)$ (i.e. the integer part of $\frac{n}{2\pi}$) and such that $s_{2k} = \frac{2\pi+2k\pi}{n}$, $s_{2k-1} = \frac{\pi/2+2\pi+2k\pi}{n}$, for $k = 0, 1, 2, \ldots, k_0$. Thus

$$Var_0^1[K(t^-,\cdot) - K(\tau,\cdot)] \geq \sum_{k=0}^{k_0} | \left[\left(K(t^-, s_{k+1}) - K(\tau, s_{k+1}) \right) - \left(K(t^-, s_k) - K(\tau, s_k) \right) \right] |.$$

Let us observe that the choice of points s_k, we get $\sin \frac{s_k}{\rho} = 0$ for even k and $\rho = t, \tau$, $\sin \frac{s_k}{t} = 1$, $\sin \frac{s_k}{\tau} = 1$ for odd k, so $|K(t^-, s_{k+1}) - K(\tau, s_{k+1}) - K(t^-, s_k) + K(\tau, s_k)| \geq \frac{1}{2n}$ and finally $Var_0^1 [K(t^-,\cdot) - K(\tau,\cdot)] \geq \frac{k_0}{2n} = \frac{Ent\left(\frac{n}{2\pi}\right)}{n}$. Observe, that the last term is not convergent to zero as $n \to \infty$. Indeed, we have $nx \leq Ent(nx) + 1$ for any $x > 0$, so $\frac{Ent(nx)}{nx} \geq \frac{Ent(nx)}{Ent(nx)+1}$ and finally $\frac{Ent(nx)}{n} \geq x \cdot \frac{Ent(nx)}{Ent(nx)+1} = x \cdot \left(1 - \frac{1}{Ent(nx)+1}\right)$.

Thus $\lim_{n\to\infty} \frac{Ent(nx)}{n} \geq x$. For $x = \frac{1}{2\pi}$ we get our thesis, so (3.3) is not satisfied. Similar calculation holds true for (3.4).

Remark 3.3. Recall that for $K(t,s) = \int_0^s \int_0^t p(\tau,\rho)\, d\rho\, d\tau$ with sufficiently regular function p our equations is a usual quadratic Hammerstein integral equations, nonetheless it can be solved in $G(I)$. Moreover, some interesting special cases of our problem (1.1) can be obtained by putting some special functions K. All the result can be obtained in $G(I)$ and even discontinuity of g is sufficient to earlier results could not be applied.

a) (cf. [12, Section 4]) For $K(t,s) = \frac{1}{\alpha}[t^\alpha - (t-s)^\alpha]$ for $s \leq t$ and $K(t,s) = 0$ for $s > t$, $t, s \in I$ we have $d_s K(t,s) = \frac{ds}{(t-s)^{(1-\alpha)}}$ for $s \leq t$ and $d_s K(t,s) = 0$ elsewhere. Put $T(x)(t) = x(t)$. In such a case we have fractional quadratic integral equation $x(t) = g(t) + \lambda \cdot x(t) \cdot \int_0^1 \frac{1}{(t-s)^{(1-\alpha)}} f(s, x(s))\, ds$ which can be considered for $x \in C(I)$ or $x \in G(I)$.

b) For a given continuously differentiable function h let us consider the kernel $K(t,s) = \frac{1}{\alpha}\left[h(t)^\alpha - (h(t)-h(s))^\alpha\right]$ for $s \le t$ and $K(t,s) = 0$ for $s > t$, $t,s \in I$ we have $d_s K(t,s) = \frac{h'(s)ds}{(h(t)-h(s))^{1-\alpha}}$ for $s \le t$ and $d_s K(t,s) = 0$ elsewhere. Put $T(x)(t) = g(t, x(t))$. In such a case we have generalized fractional quadratic integral equation considered in [1] which can be now investigated both in $C(I)$ (as in [1]) or $G(I)$.

c) (cf. [10, p.47-48], [12, Section 5]) Put $K(t,s) = t \cdot \ln \frac{t+s}{t}$ for $t \in (0,1]$ and $s \in I$ and $K(t,s) = 0$ for $t = 0$, $s \in I$. Let $f(t,x) = x \cdot \varphi(t)$, $T(x) = x$ and $g(t) \equiv 1$. Thus $d_s K(t,s) = \frac{t}{t+s} ds$ and we get the classical Chandrasekhar quadratic integral equation $x(t) = 1 + x(t) \cdot \int_0^1 \frac{t}{t+s}\varphi(s)x(s)\, ds$ (cf. [9, 10, 17]).

For more examples of functions K we refer to [7, Section 3], [10, Section 5] or to [12].

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