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An iterative process for a hybrid pair of generalized I-asymptotically nonexpansive single-valued mappings and generalized nonexpansive multi-valued mappings in Banach spaces

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ABSTRACT. In this paper, an iterative process for a hybrid pair of a finite family of generalized I-asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings is established. Moreover, the weak convergence theorems and strong convergence theorems of the proposed iterative process in Banach spaces are proven. The examples are established for supporting our main results. The obtained results can be viewed as an improvement and extension of the several results in the literature.

1. INTRODUCTION

The fixed point theory for multi-valued nonexpansive mappings using the Pompeiu-Hausdorff metric was initiated by and Nadler [8] and Markin [7]. Since then there exist extensive literatures on multi-valued fixed point theory which have applications in diverse areas, such as control theory, convex optimization, differential inclusion, and economics (see [5] and references cited therein). Different iterative processes have been used to approximate fixed points of nonexpansive and multi-valued nonexpansive mappings. Sastry and Babu [12] proved that Mann and Ishikawa iterations for a multi-valued mapping T with a fixed point p converges to a fixed point of T.

Let X be a Banach space and let D be a nonempty subset of X. Let $CB(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets and nonempty compact convex subsets of D, respectively. The *Pompeiu-Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B)=\max\{\sup_{x\in A}\text{dist}(x,B),\sup_{y\in B}\text{dist}(y,A)\}\ \text{for } A,B\in CB(D),$$

where $\text{dist}(x,B)=\inf\{\|x-y\|:y\in B\}$ is the distance from a point x to a subset B, for more details see [2]. Let t be a single-valued mapping of D into D and T be a multi-valued mapping of D into $CB(D)$. The set of fixed points of t and T will be denoted by $F(t)=\{x\in D: x=tx\}$ and $F(T)=\{x\in D: x\in Tx\}$, respectively. A point x is called a *common fixed point* of t and T if $x=tx\in Tx$.

In 2011, Sokhuma and Kaewkhao [14] introduced the following iterative process of a pair of a nonexpansive single-valued mapping t and a nonexpansive multi-valued mapping T:

$$\begin{cases} y_n=(1-\alpha_n)x_n+\alpha_n z_n,\\ x_{n+1}=(1-\beta_n)x_n+\beta_n t y_n,\ n\in\mathbb{N}, \end{cases} \tag{1.1}$$

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where $x_1 \in D, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$. They assured the existence of a strong convergence theorem for the iterative process (1.1) in uniformly convex Banach spaces.

In 2011, Eslamian and Abkar [4] introduced the following iterative process for a pair of a finite family of asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^N$:

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ n \in \mathbb{N}, \end{array} \right. \tag{1.2}$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$.

In 2015, Suantai and Phuengrattana [16] extended the results of [3, 4, 14] in uniformly convex Banach spaces. They introduced the following iterative process for a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^N$:

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ n \in \mathbb{N}, \end{array} \right. \tag{1.3}$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$.

They [16] proved the weak convergence theorems and strong convergence theorems of the iterative process defined in (1.3) in Banach spaces.

Definition 1.1. ([17]) Let $t : D \to D$ and $I : D \to D$ be single-valued mappings. We say that t is generalized I-asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$ such that

$$\|t^n x - t^n y\| \leq k_n \|Ix - Iy\| + s_n,$$

for all $x, y \in D$ and $n \in \mathbb{N}$.

If I is an identity mapping, then a single-valued mapping t reduces to a generalized asymptotically nonexpansive mapping. If $s_n = 0$, for all $n \in \mathbb{N}$, and I is an identity mapping, then a single-valued mapping t is called an asymptotically nonexpansive mapping. In particular, if $k_n = 1$, $s_n = 0$, for all $n \in \mathbb{N}$, and I is an identity mapping, a single-valued mapping t reduces to a nonexpansive mapping. The fixed point theorems for generalized I-asymptotically nonexpansive single-valued mappings in uniformly convex Banach spaces can be found in [17].

Since the class of generalized I-asymptotically nonexpansive single-valued mappings is larger than the class of generalized asymptotically nonexpansive single-valued mappings, we are interested in extending and improving the above work for a pair of a finite family ofgeneralized I-asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^N$.

In this paper, an iterative process for a hybrid pair of a finite family of generalized I-asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings is established. Moreover, the weak convergence theorems and strong convergence theorems of the proposed iterative process in Banach spaces are proven. The obtained results can be viewed as an improvement and extension of the several results in [3, 4, 6, 14, 16, 20].

2. PRELIMINARIES

In this section, we recall some definitions, propositions and lemmas that will be used in the sequel.

Recall that a multi-valued mapping $T : D \to CB(D)$ is called to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|,$$

for all $x \in D$ and $p \in F(T)$.

In 2011, Abkar and Eslamian [1] introduced a new condition on multi-valued mappings called condition (E_μ) as follows.

Definition 2.2. ([1]) A multi-valued mapping $T : D \to CB(D)$ is said to satisfy condition (E_μ) where $\mu \geq 0$ if for each $x, y \in D$,

$$\text{dist}(x, Ty) \leq \mu \text{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 2.1. We observe that if T is nonexpansive, then T satisfies the condition (E_1).

Recall that a Banach space X is said to be uniformly convex if for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$ imply $\|x + y\| \leq 2(1 - \delta)$.

A Banach space X is said to satisfy the Opial property (see [9]) whenever $\{x_n\}$ converges weakly to $x \in X$, then the following inequality holds:

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for each $y \in X$ with $y \neq x$. A Hilbert space is one of the examples of Banach spaces which satisfies the Opial property and $L^p[0, 2\pi]$ fails to satisfy the Opial property for all p with $1 < p \neq 2$ (see [16]).

Proposition 2.1. ([18]) *Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing continuous convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),$$

for all $x, y \in B_r = \{z \in X : \|z\| \leq r\}$ and $\lambda \in [0, 1]$.

Definition 2.3. ([10]) Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is of monotone type (I) with respect to F if there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} \varepsilon_n < \infty$, and $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + \varepsilon_n$ for all $n \in \mathbb{N}$ and $p \in F$.

Proposition 2.2. ([10]) *Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n \to \infty}$ dist$(x_n, F) = 0$, then $\lim_{n \to \infty} x_n = p$ for some $p \in X$ satisfying dist$(p, F) = 0$. In particular, if F is closed, then $p \in F$.*

Lemma 2.1. ([19]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then:

(i) $\lim_{n \to \infty} a_n$ *exists;*

(ii) *if* $\liminf_{n \to \infty} a_n = 0$, *then* $\lim_{n \to \infty} a_n = 0$.

Lemma 2.2. ([13]) *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$, for all $n \in \mathbb{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X satisfying, for some $r \geq 0$,*

(i) $\limsup_{n\to\infty} \|x_n\| \leq r$;
(ii) $\limsup_{n\to\infty} \|y_n\| \leq r$;
(iii) $\lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n)y_n\| = r$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. ([15]) *Let X be a Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3. Main results

Let D be a nonempty closed convex subset of a Banach space X. Suppose that $\{I_i\}_{i=1}^N$ is a finite family of asymptotically nonexpansive self-mappings of D into itself with sequence of real numbers $\{\nu_n^{(i)}\} \subset [1, \infty)$ with $\lim_{n\to\infty} \nu_n^{(i)} = 1$. Therefore

$$\|I_i^n x - I_i^n y\| \leq \nu_n^{(i)} \|x - y\|,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$. Assume that $\{t_i\}_{i=1}^N$ is a finite family of generalized I_i-asymptotically nonexpansive self-mappings of D into itself with the sequences of real numbers $\{k_n^{(i)}\} \subset [1, \infty)$ and $\{s_n^{(i)}\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n^{(i)} = 1$ and $\lim_{n\to\infty} s_n^{(i)} = 0$. Therefore

$$\|t_i^n x - t_i^n y\| \leq k_n^{(i)} \|I_i^n x - I_i^n y\| + s_n^{(i)},$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.
Letting $k_n = \max_{1\leq i\leq N}\{k_n^{(i)}\}$ and $s_n = \max_{1\leq i\leq N}\{s_n^{(i)}\}$. It follows that $\lim_{n\to\infty} k_n = 1$, and $\lim_{n\to\infty} s_n = 0$ and

$$\|t_i^n x - t_i^n y\| \leq k_n \|I_i^n x - I_i^n y\| + s_n,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.
Let $\nu_n = \max_{1\leq i\leq N}\{\nu_n^{(i)}\}$. It is clear that $\lim_{n\to\infty} \nu_n = 1$ and

$$\|I_i^n x - I_i^n y\| \leq \nu_n \|x - y\|,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.
Put $r_n = \max\{k_n, \nu_n\}$. Thus we have $\lim_{n\to\infty} r_n = 1$, $\|I_i^n x - I_i^n y\| \leq r_n \|x - y\|$ and

$$\|t_i^n x - t_i^n y\| \leq k_n \|I_i^n x - I_i^n y\| + s_n \leq r_n^2 \|x - y\| + s_n,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

The following lemma plays a crucial role in the sequel.

Lemma 3.4. *Let D be a nonempty closed convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^\infty (r_n^3 - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings of D into itself with sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(I_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_n^{(i)}\}$ *and* $\{\beta_n^{(i)}\}$ *are sequences in* $[0,1]$ *for all* $i = 1, 2, ..., N$ *such that* $\sum_{i=0}^{N} \alpha_n^{(i)} = 1$ *and* $\sum_{i=0}^{N} \beta_n^{(i)} = 1$. *Then* $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in \mathcal{F}$.

Proof. Assume that $p \in \mathcal{F}$. Therefore

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - p\| \\
&= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - \sum_{i=0}^{N} \alpha_n^{(i)} p\| \\
&= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - \alpha_n^0 p - \sum_{i=1}^{N} \alpha_n^{(i)} p\| \\
&\le \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&\le \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 \|y_n - p\| + s_n \Big),
\end{aligned}$$

for all $i = 1, 2, ..., N$.
Since

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)} - p\| \\
&= \|\beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)} - \sum_{i=0}^{N} \beta_n^{(i)} p\| \\
&\le \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} \|I_i^n z_n^{(i)} - p\| \\
&\le \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|z_n^{(i)} - p\| \\
&= \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \text{dist}(z_n^{(i)}, T_i p) \\
&\le \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n H(T_i x_n, T_i p) \\
&\le \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|x_n - p\|,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 \|y_n - p\| + s_n \Big) \\
&\le \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 (\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|x_n - p\|) + s_n \Big)
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha_n^{(0)} + r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} s_n \\
&\leq (\alpha_n^{(0)} + r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&\leq \alpha_n^{(0)}\|x_n - p\| + r_n(r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&= \alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=0}^{N} \beta_n^{(i)})\|x_n - p\| + s_n \\
&= \alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&\leq r_n^3 \alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&= r_n^3 \sum_{i=0}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&= r_n^3 \|x_n - p\| + s_n \\
&= (1 + (r_n^3 - 1))\|x_n - p\| + s_n.
\end{aligned}
$$

It now follows from Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. This completes the proof. □

Theorem 3.1. *Let D be a nonempty closed convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings of D into itself with sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(I_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$
(3.5) \qquad \begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \;\; z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \;\; n \in \mathbb{N}, \end{cases}
$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} dist(x_n, \mathcal{F}) = 0$.

Proof. The necessity is obvious. For proving the converse, suppose that $\liminf_{n\to\infty}$ dist $(x_n, \mathcal{F}) = 0$. It follows from the proof of Lemma 3.4 that the sequence can conclude that the sequence $\{x_n\}$ is of monotone type (I) with respect to $\mathcal{F}$. By Proposition 2.2, we obtain that the sequence $\{x_n\}$ converges to a point in $\mathcal{F}$. □

Recall that a single-valued mapping $t : D \to D$ is said to be a uniformly L-Lipschitzian mapping if there exists a constant $L > 0$ such that $\|t^n x - t^n y\| \leq L\|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$.

The following lemma is a main tool for proving our results.

Lemma 3.5. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(I_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1,2,...,N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\text{(3.6)} \qquad \begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1,2,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then we have the followings:

(i) $\lim_{n\to\infty} \|x_n - I_i^n z_n^{(i)}\| = 0$ *for all* $i = 1,2,...,N$;
(ii) $\lim_{n\to\infty} \|x_n - t_i x_n\| = 0$ *for all* $i = 1,2,...,N$;
(iii) if $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$, *then* $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$ *for all* $i = 1,2,...,N$.

Proof. (i) Let $p \in \mathcal{F}$. We conclude from Lemma 3.4 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - p\| = c$. This implies that

$$\begin{aligned} \|t_i^n y_n - p\| &\le r_n^2 \|y_n - p\| + s_n \\ &= r_n^2 \|\beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)} - p\| + s_n \\ &\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^2 \sum_{i=1}^N \beta_n^{(i)} \|I_i^n z_n^{(i)} - p\| + s_n \\ &\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^2 \sum_{i=1}^N \beta_n^{(i)} \nu_n \|z_n^{(i)} - p\| + s_n \\ &\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - p\| + s_n \\ &= r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^N \beta_n^{(i)} \text{dist}(z_n^{(i)}, T_i p) + s_n \\ &\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^N \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\ &\le r_n^3 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^N \beta_n^{(i)} \|x_n - p\| + s_n \\ &= r_n^3 \|x_n - p\| + s_n. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \le \limsup_{n\to\infty} (r_n^2 \|y_n - p\| + s_n) \le \limsup_{n\to\infty} (r_n^3 \|x_n - p\| + s_n).$$

Since $\lim_{n\to\infty} r_n = 1$ and $\lim_{n\to\infty} s_n = 0$, we have

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c. \tag{3.7}$$

Because of $\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|\alpha_n^{(0)}(x_n - p) + \sum_{i=1}^N \alpha_n^{(i)}(t_i^n y_n - p)\|$ and by Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|x_n - t_i^n y_n\| = 0 \text{ for all } i = 1, 2, ..., N. \tag{3.8}$$

Since

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&= \Big(1 - \sum_{i=1}^N \alpha_n^{(i)}\Big) \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&\leq \Big(1 - \sum_{i=1}^N \alpha_n^{(i)}\Big) \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} (r_n^2 \|y_n - p\|) + s_n),
\end{aligned}$$

we have

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \sum_{i=1}^N \alpha_n^{(i)} (r_n^2 \|y_n - p\| - \|x_n - p\| + s_n).$$

This implies that

$$\begin{aligned}
\frac{\|x_{n+1} - p\| - \|x_n - p\|}{bN} + \|x_n - p\| &\leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\sum_{i=1}^N \alpha_n^{(i)}} + \|x_n - p\| \\
&\leq r_n^2 \|y_n - p\| - \|x_n - p\| + s_n + \|x_n - p\| \\
&= r_n^2 \|y_n - p\| + s_n.
\end{aligned}$$

By (3.7), this yields

$$\begin{aligned}
c &= \liminf_{n\to\infty} \Big(\frac{\|x_{n+1} - p\| - \|x_n - p\|}{bN} + \|x_n - p\| \Big) \\
&\leq \liminf_{n\to\infty} (r_n^2 \|y_n - p\| + s_n) \\
&= \liminf_{n\to\infty} \|y_n - p\| \\
&\leq \limsup_{n\to\infty} \|y_n - p\| \leq c.
\end{aligned}$$

Since

$$\|I_i^n z_n^{(i)} - p\| \leq \nu_n \|z_n^{(i)} - p\| = \nu_n \text{dist}(z_n^{(i)}, T_i p) \leq \nu_n H(T_i x_n, T_i p) \leq \nu_n \|x_n - p\|,$$

it follows that

$$\limsup_{n\to\infty} \|I_i^n z_n^{(i)} - p\| \leq \limsup_{n\to\infty} \nu_n \|x_n - p\| = c.$$

Therefore

$$c = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|\beta_n^{(0)}(x_n - p) + \sum_{i=1}^N \beta_n^{(i)} (I_i^n z_n^{(i)} - p)\|.$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - I_i^n z_n^{(i)}\| = 0 \ \text{ for all } \ i = 1, 2, ..., N.$$

(ii) Since t_i is generalized I_i-asymptotically nonexpansive, for all $i = 1, 2, ..., N$, we obtain that

$$\begin{aligned}\|t_i^n x_n - x_n\| &= \|t_i^n x_n - t_i^n y_n + t_i^n y_n - x_n\| \\ &\leq \|t_i^n x_n - t_i^n y_n\| + \|t_i^n y_n - x_n\| \\ &\leq r_n^2 \|x_n - y_n\| + s_n + \|t_i^n y_n - x_n\|.\end{aligned}$$

Using the definition of $\{x_n\}$, we have $y_n - x_n = \sum_{i=1}^N \beta_n^{(i)} (I_i^n z_n^{(i)} - x_n)$. This implies that

$$\begin{aligned}\|t_i^n x_n - x_n\| &\leq r_n^2 \sum_{i=1}^N \beta_n^{(i)} \|I_i^n z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n \\ &\leq r_n^2 \|I_i^n z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n.\end{aligned}$$

By (i) and (3.8), we obtain that

$$\lim_{n\to\infty} \|x_n - t_i^n x_n\| = 0 \ \text{ for all } \ i = 1, 2, ..., N. \tag{3.9}$$

For each $i = 1, 2, ..., N$, we have

$$\begin{aligned}\|x_n - t_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + \|t_i^{n+1} x_{n+1} - t_i^{n+1} x_n\| + \|t_i^{n+1} x_n - t_i x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L\|x_{n+1} - x_n\| + \|t_i^{n+1} x_n - t_i x_n\| \\ &\leq (1 + L)\|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L\|t_i^n x_n - x_n\| \\ &\leq (1 + L) \sum_{i=1}^N \alpha_n^{(i)} \|x_n - t_i^n y_n\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L\|t_i^n x_n - x_n\|.\end{aligned}$$

By using (3.8) and (3.9), we can conclude that $\lim_{n\to\infty} \|x_n - t_i x_n\| = 0$ for all $i = 1, 2, ..., N$.
(iii) Since

$$\begin{aligned}\|I_i^n x_n - x_n\| &\leq \|I_i^n x_n - I_i^n z_n^{(i)}\| + \|I_i^n z_n^{(i)} - x_n\| \\ &\leq \nu_n \|x_n - z_n^{(i)}\| + \|I_i^n z_n^{(i)} - x_n\| \\ &\leq \nu_n (\|x_n - I_i^n z_n^{(i)}\| + \|I_i^n z_n^{(i)} - z_n^{(i)}\|) + \|I_i^n z_n^{(i)} - x_n\|\end{aligned}$$

and by (i), we obtain that

$$\lim_{n\to\infty} \|I_i^n x_n - x_n\| = 0. \tag{3.10}$$

Since

$$\begin{aligned}\|x_n - I_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1} x_{n+1}\| + \|I_i^{n+1} x_{n+1} - I_i^{n+1} x_n\| + \|I_i^{n+1} x_n - I_i x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1} x_{n+1}\| + \Gamma\|x_{n+1} - x_n\| + \|I_i^{n+1} x_n - I_i x_n\| \\ &\leq (1 - \Gamma)\|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1} x_{n+1}\| + \Gamma\|I_i^n x_n - x_n\| \\ &\leq (1 - \Gamma) \sum_{i=1}^N \alpha_n^{(i)} \|x_n - t_i^n y_n\| + \|x_{n+1} - I_i^{n+1} x_{n+1}\| + \Gamma\|I_i^n x_n - x_n\|,\end{aligned}$$

and by (3.8) and (3.10), we can conclude that $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$ for all $i = 1, 2, ..., N$. □

Next, we prove a strong convergence theorem of the proposed iterative process in uniformly convex Banach spaces.

Theorem 3.2. *Let D be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^\infty (r_n^3 - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(I_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1,2,...,N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{array} \right. \tag{3.11}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1,2,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Suppose that $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1,2,...,N$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof. Using Lemma 3.4, we obtain that $\{x_n\}$ is bounded. By the compactness of D, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to $p \in D$. By condition (E), there exists $\mu \geq 1$ such that

$$\begin{aligned} \mathrm{dist}(p, T_i p) &\leq \|p - x_{n_j}\| + \mathrm{dist}(x_{n_j}, T_i p) \\ &\leq \|p - x_{n_j}\| + \mu \mathrm{dist}(x_{n_j}, T_i x_{n_j}) + \|x_{n_j} - p\| \\ &= 2\|x_{n_j} - p\| + \mu \mathrm{dist}(x_{n_j}, T_i x_{n_j}) \\ &\leq 2\|x_{n_j} - p\| + \mu \|x_{n_j} - z_{n_j}^{(i)}\| \\ &\leq 2\|x_{n_j} - p\| + \mu \|x_{n_j} - I_i^{n_j} z_{n_j}^{(i)}\| + \mu \|I_i^{n_j} z_{n_j}^{(i)} - z_{n_j}^{(i)}\|, \end{aligned}$$

for all $i = 1,2,...,N$. By using Lemma 3.5 (i), we obtain that $p \in T_i p$ for all $i = 1,2,...,N$. This implies that $p \in \bigcap_{i=1}^N F(T_i)$. Since t_i is uniformly L-Lipschitzian, we have

$$\begin{aligned} \|t_i p - p\| &\leq \|t_i p - t_i x_{n_j}\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &\leq L\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &= (L+1)\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\|, \end{aligned}$$

for all $i = 1,2,...,N$. By Lemma 3.5 (ii), we obtain that $t_i p = p$ for all $i = 1,2,...,N$. This implies that $p \in \bigcap_{i=1}^N F(t_i)$. Since I_i is uniformly Γ-Lipschitzian, we have

$$\begin{aligned} \|I_i p - p\| &\leq \|I_i p - I_i x_{n_j}\| + \|I_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &\leq \Gamma\|x_{n_j} - p\| + \|I_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &= (\Gamma+1)\|x_{n_j} - p\| + \|I_i x_{n_j} - x_{n_j}\|, \end{aligned}$$

for all $i = 1,2,...,N$. By Lemma 3.5 (iii), we obtain that $I_i p = p$ for all $i = 1,2,...,N$. It follows that $p \in \bigcap_{i=1}^N F(I_i)$. Thus $p \in \mathcal{F}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have $\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j} - p\| = 0$. Hence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. □

The following example illustrates Theorem 3.2.

Example 3.1. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D = [0,1]$. Define single-valued mappings t_1, t_2, I_1, and I_2 on D as follows:

$$t_1 x = \arctan x,\ t_2 x = x^2,\ I_1 x = x \text{ and } I_2 x = \frac{x}{2}.$$

Define multi-valued mappings T_1 and T_2 on D by

$$T_1 x = [0, \frac{x}{3}] \text{ and } T_2 x = [\frac{x}{4}, \frac{x}{2}].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{2} \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{2} \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{array} \right. \tag{3.12}$$

where $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$, for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i)$.

Solution We first show that t_1 is a generalized I_1-asymptotically nonexpansive and uniformly L-Lipschitzian single-valued mapping. Let $k_n = 1$ and $s_n = (\frac{2}{3})^n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$. Since

$$\frac{|t_1^n x - t_1^n y|}{|x-y|} \le 1 + (\frac{2}{3|x-y|})^n \ \text{ for all } \ x, y \in D,$$

we have

$$|t_1^n x - t_1^n y| \le |x-y| + s_n \ \text{ for all } n \in \mathbb{N},$$

and we can show that t_1 is a uniformly L-Lipschitzian mapping with $L \ge 2$. Since t_2 is a single-valued nonexpansive mapping of D, we have t_2 is a uniformly L-Lipschitzian and generalized I_2-asymptotically nonexpansive single-valued mapping of D. Moreover $\bigcap_{i=1}^{2} F(t_i) = \{0\} = \bigcap_{i=1}^{2} F(I_i)$. Both T_1 and T_2 are quasi-nonexpansive multi-valued mappings satisfying condition (E) and $\bigcap_{i=1}^{2} F(T_i) = \{0\}$ (see [16]). Thus $\bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i) = \{0\}$. For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$. Therefore the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ satisfy all assumptions in Theorem 3.2. By putting $z_n^{(1)} = \frac{x_n}{3}$, and $z_n^{(2)} = \frac{x_n}{2}$ for all $n \in \mathbb{N}$ and by using the algorithm 3.12 with the initial point $x_1 = 0.5$. The sequence $\{x_n\}$ converges strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i)$.

Finally, we prove a weak convergence theorem of the proposed iterative process in uniformly convex Banach spaces.

Theorem 3.3. *Let D be a nonempty closed convex subset of a uniformly convex Banach spaces X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(t_i) \cap \bigcap_{i=1}^{N} F(I_i) \cap \bigcap_{i=1}^{N} F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{array} \right. \tag{3.13}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^{N} \alpha_n^{(i)} = 1$ and $\sum_{i=0}^{N} \beta_n^{(i)} = 1$. Suppose that $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1, 2, ..., N$. Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$.

Proof. By using Lemma 3.4, we obtain that $\{x_n\}$ is bounded. Since X is a uniformly convex Banach space, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $p \in D$. By Lemma 3.5, we have $\lim_{j\to\infty} \|x_{n_j} - I_i^{n_j} z_{n_j}^{(i)}\| = 0$, $\lim_{j\to\infty} \|x_{n_j} - t_i x_{n_j}\| = 0$ and $\lim_{n\to\infty} \|I_i x_{n_j} - x_{n_j}\| = 0$ for all $i = 1, 2, ..., N$. We will show that $p \in \mathcal{F}$. Since $T_1 p$ is compact, for all $j \in \mathbb{N}$, we can choose $w_{n_j} \in T_1 p$ such that $\|x_{n_j} - w_{n_j}\| = \text{dist}(x_{n_j}, T_1 p)$ and the sequence $\{w_{n_j}\}$ has a convergent subsequence $\{w_{n_k}\}$ with $\lim_{k\to\infty} w_{n_k} = w \in T_1 p$. By using condition (E), we obtain that

$$\text{dist}(x_{n_k}, T_1 p) \le \mu \text{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\|.$$

This yields

$$\begin{aligned}
\|x_{n_k} - w\| &\le \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - w\| \\
&= \text{dist}(x_{n_k}, T_1 p) + \|w_{n_k} - w\| \\
&\le \mu \text{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\
&\le \mu \|x_{n_k} - z_{n_k}^{(1)}\| + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\
&\le \mu \|x_{n_k} - I_1^{n_k} z_{n_k}^{(1)}\| + \mu \|I_1^{n_k} z_{n_k}^{(1)} - z_{n_k}^{(1)}\| + \|x_{n_k} - p\| + \|w_{n_k} - w\|.
\end{aligned}$$

It follows that

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \le \limsup_{k\to\infty} \|x_{n_k} - p\|.$$

From the Opial property, we can conclude that $p = w \in T_1 p$. Similarly, it can be shown that $p \in T_i p$ for all $i = 1, 2, ..., N$. Therefore $p \in \bigcap_{i=1}^N F(T_i)$. By mathematical induction, we obtain that

$$\lim_{j\to\infty} \|x_{n_j} - t_i^m x_{n_j}\| = 0 \text{ for each } m \in \mathbb{N}, \tag{3.14}$$

for all $i = 1, 2, ..., N$. Indeed, the conclusion it true for $m = 1$. Suppose the conclusion holds for $m \ge 1$. Since t_i is a uniformly L-Lipschitzian single-valued mapping, we obtain that

$$\begin{aligned}
\|x_{n_j} - t_i^{m+1} x_{n_j}\| &\le \|x_{n_j} - t_i^m x_{n_j}\| + \|t_i^m x_{n_j} - t_i^{m+1} x_{n_j}\| \\
&= \|x_{n_j} - t_i^m x_{n_j}\| + L\|x_{n_j} - t_i x_{n_j}\|.
\end{aligned}$$

This implies that $\lim_{j\to\infty} \|x_{n_j} - t_i^{m+1} x_{n_j}\| = 0$ for all $i = 1, 2, ..., N$. Therefore (3.14) holds. By (3.14), we have for each $x \in D$, $m \in \mathbb{N}$ and $i = 1, 2, ..., N$,

$$\limsup_{j\to\infty} \|x_{n_j} - x\| = \limsup_{j\to\infty} \|t_i^m x_{n_j} - x\|. \tag{3.15}$$

Since t_i is generalized I_i-asymptotically nonexpansive, we obtain that

$$\begin{aligned}
\limsup_{j\to\infty} \|t_i^m x_{n_j} - t_i^m p\| &\le \limsup_{j\to\infty} (k_m \|I_i^m x_{n_j} - p\| + s_m) \\
&\le \limsup_{j\to\infty} (k_m (\nu_m \|x_{n_j} - p\|) + s_m) \\
&\le \limsup_{j\to\infty} (r_m^2 \|x_{n_j} - p\| + s_m).
\end{aligned}$$

It follows that

$$\limsup_{m\to\infty} \left(\limsup_{j\to\infty} \|t_i^m x_{n_j} - t_i^m p\| \right) \le \limsup_{j\to\infty} \|x_{n_j} - p\|. \tag{3.16}$$

By Proposition 2.1, there exists a strictly increasing continuous convex function $g : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned}\Big\|x_{n_j} - \frac{p + t_i^m p}{2}\Big\|^2 &= \Big\|\frac{1}{2}(x_{n_j} - p) + \frac{1}{2}(x_{n_j} - t_i^m p)\Big\|^2 \\ &\le \frac{1}{2}\|x_{n_j} - p\|^2 + \frac{1}{2}\|x_{n_j} - t_i^m p\|^2 - \frac{1}{4}g(\|p - t_i^m p\|).\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{j\to\infty} \Big\|x_{n_j} - \frac{p + t_i^m p}{2}\Big\|^2 \le \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2 \\ - \frac{1}{4}g(\|p - t_i^m p\|).\end{aligned} \tag{3.17}$$

Since X satisfies the Opial property and $\{x_{n_j}\}$ converges weakly to p, we obtain that

$$\limsup_{j\to\infty}\|x_{n_j} - p\|^2 \le \limsup_{j\to\infty}\|x_{n_j} - \frac{p + t_i^m p}{2}\|^2.$$

By using (3.17), we have

$$\begin{aligned}\frac{1}{4}g(\|p - t_i^m p\|) &\le \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2 \\ &- \limsup_{j\to\infty}\Big\|x_{n_j} - \frac{p + t_i^m p}{2}\Big\|^2 \\ &\le \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2 \\ &- \limsup_{j\to\infty}\|x_{n_j} - p\|^2.\end{aligned}$$

Therefore

$$g(\|p - t_i^m p\|) \le 2\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2 - 2\limsup_{j\to\infty}\|x_{n_j} - p\|^2. \tag{3.18}$$

Using (3.15), (3.16), and (3.18, these yield

$$\begin{aligned}\limsup_{m\to\infty} g(\|p - t_i^m p\|) &\le 2\limsup_{m\to\infty}(\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2) - 2\limsup_{j\to\infty}\|x_{n_j} - p\|^2 \\ &\le 0.\end{aligned}$$

Therefore $\lim_{m\to\infty} g(\|p - t_i^m p\|) = 0$ for all $i = 1, 2, ..., N$. By using the properties of g, we have $\lim_{m\to\infty}\|p - t_i^m p\| = 0$ for all $i = 1, 2, ..., N$. This implies that

$$\begin{aligned}\|t_i p - p\| &\le \|t_i p - t_i^{m+1} p\| + \|t_i^{m+1} p - p\| \\ &\le L\|p - t_i^m p\| + \|t_i^{m+1} p - p\|.\end{aligned}$$

This implies that $t_i p = p$ for all $i = 1, 2, ..., N$. Therefore $p \in \bigcap_{i=1}^N F(t_i)$. Similarly, we can prove that $p \in \bigcap_{i=1}^N F(I_i)$. Thus we obtain $p \in \mathcal{F}$. We now show that $\{x_n\}$ converges weakly to p. Suppose on the contrary. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\{x_{n_l}\}$ converges weakly to $q \in D$ and $q \neq p$. By the same argument as above, we can show that $q \in \mathcal{F}$. By Lemma 3.4, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty}\|x_n - q\|$ exist. Using Lemma 2.3, we obtain that $q = p$. Hence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. This completes the proof. □

We now present the following example for supporting Theorem 3.3.

Example 3.2. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D = [0, +\infty)$. Define single-valued mappings t_1, t_2, I_1, and I_2 on D as follows:

$$t_1 x = \frac{x}{1+x},\ t_2 x = \frac{x}{1+2x},\ I_1 x = \frac{x}{1+2x} \text{ and } I_2 x = \frac{x}{1+3x}.$$

Define multi-valued mappings T_1 and T_2 on D by

$$T_1 x = [0, \frac{x}{5}] \text{ and } T_2 x = [\frac{x}{2}, \frac{x}{4}].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\left\{ \begin{array}{l} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{2} \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{2} \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{array} \right. \tag{3.19}$$

where $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$, for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges weakly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i)$.

Solution We first show that t_1 is a generalized I_1-asymptotically nonexpansive and uniformly L-Lipschitzian single-valued mapping. Let $k_n = 1$ and $s_n = (\frac{1}{2})^n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$. Since

$$\begin{aligned} \frac{|t_1^n x - t_1^n y|}{|x-y|} &\leq |\frac{x}{1+2x} - \frac{y}{1+2y}| \leq |\frac{x}{1+2x} - \frac{y}{1+2y}| + \frac{1}{2^n} \\ &\leq k_n |I_1 x - I_1 y| + \frac{1}{2^n} \end{aligned}$$

for all $x, y \in D$ and $n \in \mathbb{N}$. Next, we show that t_2 is a generalized I_2-asymptotically nonexpansive and uniformly L-Lipschitzian single-valued mapping. Let $k_n = 1$ and $s_n = (\frac{1}{2})^n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$. Since

$$\begin{aligned} \frac{|t_1^n x - t_1^n y|}{|x-y|} &\leq |\frac{x}{1+3x} - \frac{y}{1+3y}| \leq |\frac{x}{1+3x} - \frac{y}{1+3y}| + \frac{1}{2^n} \\ &\leq k_n |I_2 x - I_2 y| + \frac{1}{2^n} \end{aligned}$$

for all $x, y \in D$ and and $n \in \mathbb{N}$. Moreover, we obtain that t_1, t_2, I_1 and I_2 are uniformly L-Lipschitzian single-valued mappings (see [11]). It can verify that T_1 and T_1 are quasi-nonexpansive multi-valued mappings satisfying condition (E). Furthermore, we have $\bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i) = \{0\}$ For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$. Therefore the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\alpha_n^{(2)}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ satisfy all assumptions in Theorem 3.3. By putting $z_n^{(1)} = \frac{x_n}{5}$ and $z_n^{(2)} = \frac{x_n}{2}$ for all $n \in \mathbb{N}$ and by using the algorithm 3.19 with the initial point $x_1 = 5$. Then the sequence $\{x_n\}$ converges weakly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(I_i) \cap \bigcap_{i=1}^{2} F(T_i)$.

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