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A new modification of Durrmeyer type mixed hybrid operators

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ABSTRACT. In 2008 V. Miheşan constructed a general class of linear positive operators generalizing the Szász operators. In this article, a Durrmeyer variant of these operators is introduced which is a method to approximate the Lebesgue integrable functions. First, we derive some indispensable auxiliary results in the second section. We present a quantitative Voronovskaja type theorem, local approximation theorem by means of second order modulus of continuity and weighted approximation for these operators. The rate of convergence for differential functions whose derivatives are of bounded variation is also obtained.

1. INTRODUCTION

Miheşan [23] constructed an important generalization of the well known Szász operators depending on $\alpha \in \mathbb{R}$ as

$$\mathcal{G}_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right),\ x \in [0,\infty), \tag{1.1}$$

where $\alpha + nx > 0$, $m_{n,k}^{(\alpha)}(x) = \dfrac{(\alpha)_k}{k!} \dfrac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}}$ and $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, $(\alpha)_0 = 1$, is the rising factorial. The operators $\mathcal{G}_n^{(\alpha)}$ preserve the linear functions and they reduce to the following well-known operators in special cases:

(1) If $\alpha = -n, x \in [0,1]$, we get the Bernstein operators [9].
(2) If $\alpha = n$, we obtain Baskakov operators [8].
(3) If $\alpha = nx, x > 0$, we get the Lupaş operators [22].
(4) If $\alpha \to \infty$, we obtain Szász-Mirakjan operators [26].

A different form of the operators (1.1) were also discussed in [1].
The Szász-Mirakyan operators and their modifications have been intensively studied in recent years. We can mention some of them as: in [27] Varma and Taşdelen proposed a modification of Szász operators based on Charlier polynomials and studied some convergence properties of the operators with the help of Korovkin theorem, in [6] Agrawal et al. considered the Baskakov-Szász type operators depending on a nonnegative parameters and estimated order of approximation and simultaneous approximation, in [16] Gupta and Rassias presented a Durrmeyer type modification of Szász type operators and derived some direct results e.g. weighted approximation, asymptotic formula and error estimation in terms of modulus of smoothness, in [12] Goyal et al. constructed one parameter family of linear positive operators and obtained their approximation theorems, in [14] Gupta introduced a sequence of mixed operators with weights of the Păltănea basis function and gave some approximation properties for the operators, in [5] Acu and

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Gupta defined hybrid operators involving two parameters and they proved the Korovkin type approximation theorem and the rate of approximation for unbounded functions with derivatives of bounded variation. Very recently, Kajla et al. [21] constructed the hybrid operators based on inverse Pólya-Eggenberger distribution and investigated their approximations properties. In the literature, many researchers have derived the approximation properties of different mixed hybrid operators, among the others, we can refer the readers to (cf. [3], [7], [10], [11], [15], [19], [20], [24], [25], [28] etc.)

Inspired by the above work, we present a new sequence of mixed hybrid operators as follows:

For $\gamma > 0$ and $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma), \text{ as } t \to \infty\}, c > 0$, we define

$$\mathcal{H}_{n,c}^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) \int_0^\infty b_{n,k}^c(t) f(t)dt, \tag{1.2}$$

where $b_{n,k}^c(t) = \dfrac{c}{B\left(k+1,\frac{n}{c}\right)} \dfrac{(ct)^k}{(1+ct)^{\frac{n}{c}+k+1}}$, $m_{n,k}^{(\alpha)}$ is given in (1.1) and $B(k+1,n)$ is the beta function defined by $B(x,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x,y > 0$.
We observe that for $f \in C_\gamma[0,\infty)$, the integral in the right hand side of (1.2) exists for all $n > \gamma$, and hence $\mathcal{H}_{n,c}^{\alpha}$ is well-defined. The operators include many well-known Durrmeyer operators in special cases, hence they allow us to describe the approximation properties of those operators at the same time. As a special cases, for $\alpha \to \infty$ and $c = 1$ these operators reduce the well-known Szász-Beta operators [18]. For $\alpha = n$ and $c = 1$ these operators include well known Baskakov-Beta operators [13]. For $\alpha = nx$ and $c = 1$ these operators reduce the Lupaş-Beta operators [17].

In the present paper, we investigate approximation properties of the operators $\mathcal{H}_{n,c}^{\alpha}$ such as, rate of convergence via modulus of continuity, weighted approximation, pointwise convergence of the operators in terms of quantitative Voronovskaya type theorem and the rate of approximation for functions having derivatives of bounded variation.

2. Direct results

In this section we prove certain results, which are necessary to derive the main results. Let $e_i(t) = t^i, i = \overline{0,6}$.

Lemma 2.1. *For the operators $\mathcal{G}_n^{(\alpha)}(f;x)$, we have*

(i) $\mathcal{G}_n^{(\alpha)}(e_0;x) = 1$,

(ii) $\mathcal{G}_n^{(\alpha)}(e_1;x) = x$,

(iii) $\mathcal{G}_n^{(\alpha)}(e_2;x) = x^2 + \frac{x(nx+\alpha)}{n\alpha}$,

(iv) $\mathcal{G}_n^{(\alpha)}(e_3;x) = \frac{x^3(1+\alpha)(2+\alpha)}{\alpha^2} + \frac{3x^2(1+\alpha)}{n\alpha} + \frac{x}{n^2}$,

(v) $\mathcal{G}_n^{(\alpha)}(e_4;x) = \frac{x^4(1+\alpha)(2+\alpha)(3+\alpha)}{\alpha^3} + \frac{6x^3(1+\alpha)(2+\alpha)}{n\alpha^2} + \frac{7x^2(1+\alpha)}{n^2\alpha} + \frac{x}{n^3}$,

(vi) $\mathcal{G}_n^{(\alpha)}(e_5;x) = \frac{x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{\alpha^4} + \frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n\alpha^3} + \frac{25x^3(1+\alpha)(2+\alpha)}{n^2\alpha^2} + \frac{15x^2(1+\alpha)}{n^3\alpha} + \frac{x}{n^4}$,

(vii) $\mathcal{G}_n^{(\alpha)}(e_6;x) = \frac{x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{\alpha^5} + \frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{n\alpha^4}$
$+ \frac{65x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n^2\alpha^3} + \frac{90x^3(1+\alpha)(2+\alpha)}{n^3\alpha^2} + \frac{31x^2(1+\alpha)}{n^4\alpha} + \frac{x}{n^5}$.

Proof. The proofs of the parts $(i)-(iii)$ are given in ([23], Lemma 4.1). The proof of $(iv)-(vii)$ can be computed following the same idea of proof of ([23], Lemma 4.1). □

Lemma 2.2. *For the operators $\mathcal{H}^{\alpha}_{n,c}(f;x)$, we have*

(i) $\mathcal{H}^{(\alpha)}_{n,c}(e_0;x)=1;$

(ii) $\mathcal{H}^{(\alpha)}_{n,c}(e_1;x)=\frac{nx+1}{(n-c)};$

(iii) $\mathcal{H}^{(\alpha)}_{n,c}(e_2;x)=\frac{n^2x^2(1+\alpha)}{(n-2c)(n-c)\alpha}+\frac{4nx}{(n-2c)(n-c)}+\frac{2}{(n-2c)(n-c)};$

(iv) $\mathcal{H}^{(\alpha)}_{n,c}(e_3;x)=\frac{n^3x^3(1+\alpha)(2+\alpha)}{(n-3c)(n-2c)(n-c)\alpha^2}+\frac{9n^2x^2(1+\alpha)}{(n-3c)(n-2c)(n-c)\alpha}+\frac{18nx}{(n-3c)(n-2c)(n-c)}$
$+\frac{6}{(n-3c)(n-2c)(n-c)};$

(v) $\mathcal{H}^{(\alpha)}_{n,c}(e_4;x)=\frac{n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha^3}+\frac{16n^3x^3(1+\alpha)(2+\alpha)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha^2}$
$+\frac{72n^2x^2(1+\alpha)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha}+\frac{96nx}{(n-4c)(n-3c)(n-2c)(n-c)}+\frac{24}{(n-4c)(n-3c)(n-2c)(n-c)},$

(vi) $\mathcal{H}^{(\alpha)}_{n,c}(e_5;x)=\frac{n^5x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^4}+\frac{25n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^3}$
$+\frac{200n^3x^3(1+\alpha)(2+\alpha)}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^2}+\frac{600n^2x^2(1+\alpha)}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha}$
$+\frac{600nx}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)}+\frac{120}{(n-5c)(n-4c)(n-3c)(n-2c)(n-c)},$

(vii) $\mathcal{H}^{(\alpha)}_{n,c}(e_6;x)=\frac{n^6x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^5}$
$+\frac{36n^5x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^4}+\frac{450n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^3}$
$+\frac{2400n^3x^3(1+\alpha)(2+\alpha)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^2}+\frac{5400n^2x^2(1+\alpha)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha}$
$+\frac{4320nx}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)}+\frac{720}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)}.$

Proof. The lemma follows easily using the relation (1.2) and Lemma 2.1. Hence the details are omitted. □

Lemma 2.3. *For $m=1,2,4,6$, the m^{th} order central moments of $\mathcal{H}^{(\alpha)}_{n,c}$ defined as $\beta^c_{n,\alpha,m}(x)=\mathcal{H}^{(\alpha)}_{n,c}((t-x)^m;x)$, we have*

(i) $\beta^c_{n,\alpha,1}(x)=\frac{xc+1}{(n-c)};$

(ii) $\beta^c_{n,\alpha,2}(x)=\frac{x^2(n^2+c(2c+n)\alpha)}{(n-2c)(n-c)\alpha}+\frac{2x(n+2c)}{(n-2c)(n-c)}+\frac{2}{(n-2c)(n-c)};$

(iii) $\beta^c_{n,\alpha,4}(x)=\frac{x^4\left(24c^4\alpha^3+46c^3n\alpha^3+3n^4(2+\alpha)+3c^2n^2\alpha^2(24+\alpha)+2cn^3\alpha(16+3\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha^3}$
$+\frac{4x^3\left(24c^3\alpha^2+46c^2n\alpha^2+3cn^2\alpha(12+\alpha)+n^3(8+3\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha^2}$
$+\frac{12x^2\left(6n^2+12c^2\alpha+17cn\alpha+n^2\alpha\right)}{(n-4c)(n-3c)(n-2c)(n-c)\alpha}$
$+\frac{x(96c+72n)}{(n-4c)(n-3c)(n-2c)(n-c)}+\frac{24}{(n-4c)(n-3c)(n-2c)(n-c)},$

(iv) $\beta^c_{n,\alpha,6}(x)=\frac{1}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^5}\Bigg[x^6\bigg(720c^6\alpha^5+2556c^5n\alpha^5$
$+40c^4n^2\alpha^4(135+19\alpha)+9cn^5\alpha(96+5\alpha(18+\alpha))+15c^3n^3\alpha^3(320+\alpha(138+\alpha))+$
$5n^6(24+\alpha(26+3\alpha))+5c^2n^4\alpha^2(540+\alpha(398+9\alpha))\bigg)\Bigg]$
$+\frac{1}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^4}\Bigg[6x^5$
$\bigg(720c^5\alpha^4+2556c^4n\alpha^4+40c^3n^2\alpha^3(90+19\alpha)+3n^5(48+5\alpha(10+\alpha))+15c^2n^3\alpha^2(160+$
$\alpha(92+\alpha))+10cn^4\alpha(90+\alpha(77+3\alpha))\bigg)\Bigg]$

$$+\frac{1}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^3}\Bigg[30x^4\Bigg(360c^4\alpha^3 + 1098c^3n\alpha^3+c^2n^2\alpha^2(1080+311\alpha)+6cn^3\alpha(80+\alpha(54+\alpha))+n^4(90+\alpha(85+6\alpha))\Bigg)\Bigg]$$
$$\frac{1}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha^2}\Bigg[120x^3\Bigg(120c^3\alpha^2+286c^2n\alpha^2+9cn^2\alpha(20+7\alpha)+ n^3(40+\alpha(30+\alpha))\Bigg)\Bigg]+\frac{360x^2\left(30c^2\alpha+49cn\alpha+3n^2(5+2\alpha)\right)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)\alpha}$$
$$+\frac{720x(6c+5n)}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)}+\frac{720}{(n-6c)(n-5c)(n-4c)(n-3c)(n-2c)(n-c)}.$$

Remark 2.1. *If $\alpha=\alpha(n)\to\infty$, as $n\to\infty$ and $\lim\limits_{n\to\infty}\dfrac{n}{\alpha(n)}=l\in\mathbb{R}$, then*

$$\begin{aligned}\lim_{n\to\infty} n\beta^c_{n,\alpha,1}(x) &= cx+1;\\ \lim_{n\to\infty} n\beta^c_{n,\alpha,2}(x) &= x(2+cx+lx);\\ \lim_{n\to\infty} n^2\beta^c_{n,\alpha,4}(x) &= 3x^2(2+cx+lx)^2.\end{aligned}$$

3. Main results

Theorem 3.1. *Let $f\in C_\gamma[0,\infty)$ and $\alpha=\alpha(n)\to\infty$ as $n\to\infty$. Then $\lim\limits_{n\to\infty}\mathcal{H}^{(\alpha)}_{n,c}(f;x)=f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. From Lemma 2.2, $\mathcal{H}^{(\alpha)}_{n,c}(e_0;x)=1$, $\mathcal{H}^{(\alpha)}_{n,c}(e_1;x)\to x$, $\mathcal{H}^{(\alpha)}_{n,c}(e_2;x)\to x^2$, as $n\to\infty$ uniformly in each compact subset of $[0,\infty)$. By Bohman-Korovkin Theorem, it follows that $\mathcal{H}^{(\alpha)}_{n,c}(f;x)\to f(x)$, as $n\to\infty$ uniformly on every compact subset of $[0,\infty)$. □

Let $\tilde{C_B}[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm

$$||f||_{\tilde{C_B}[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|.$$

For $f\in\tilde{C_B}[0,\infty)$, the Steklov mean is defined as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+u+v)-f(x+2(u+v))\right]du\,dv. \tag{3.3}$$

By simple computation, it is observed that

a) $\|f_h-f\|_{\tilde{C_B}[0,\infty)}\le\omega_2(f,h)$.

b) $f'_h,f''_h\in\tilde{C_B}[0,\infty)$ and $\|f'_h\|_{\tilde{C_B}[0,\infty)}\le\frac{5}{h}\omega(f,h)$, $\quad\|f''_h\|_{\tilde{C_B}[0,\infty)}\le\frac{9}{h^2}\omega_2(f,h)$,

where the second order modulus of continuity is defined as

$$\omega_2(f,\delta)=\sup_{x,u,v\ge0}\sup_{|u-v|\le\delta}|f(x+2u)-2f(x+u+v)+f(x+2v)|,\ \delta\ge0.$$

The usual modulus of continuity of $f\in\tilde{C_B}[0,\infty)$ is given by

$$\omega(f,\delta)=\sup_{x,u,v\ge0}\sup_{|u-v|\le\delta}|f(x+u)-f(x+v)|.$$

Theorem 3.2. *Let $f\in\tilde{C_B}[0,\infty)$. Then for every $x\ge0$, the following inequality holds*

$$\left|\mathcal{H}^{(\alpha)}_{n,c}(f;x)-f(x)\right|\le5\omega\left(f,\sqrt{\beta^c_{n,\alpha,2}(x)}\right)+\frac{13}{2}\omega_2\left(f,\sqrt{\beta^c_{n,\alpha,2}(x)}\right).$$

Proof. For $x \geq 0$, applying the Steklov mean f_h that is given by (3.3), we can write

$$\left|\mathcal{H}_{n,c}^{(\alpha)}(f;x)-f(x)\right| \leq \mathcal{H}_{n,c}^{(\alpha)}\left(|f-f_h|;x\right)+\left|\mathcal{H}_{n,c}^{(\alpha)}\left(f_h-f_h(x);x\right)\right|+|f_h(x)-f(x)|. \tag{3.4}$$

From (1.2), for every $f \in \tilde{C}_B[0,\infty)$ we have

$$\left|\mathcal{H}_{n,c}^{(\alpha)}(f;x)\right| \leq ||f||_{\tilde{C_B}[0,\infty)}. \tag{3.5}$$

Using property (a) of Steklov mean and (3.5), we get

$$\mathcal{H}_{n,c}^{(\alpha)}\left(|f-f_h|;x\right) \leq \|\mathcal{H}_{n,c}^{(\alpha)}\left(f-f_h\right)\|_{\tilde{C_B}[0,\infty)} \leq \|f-f_h\|_{\tilde{C_B}[0,\infty)} \leq \omega_2(f,h).$$

By Taylor's expansion and Cauchy-Schwarz inequality, we have

$$\left|\mathcal{H}_{n,c}^{(\alpha)}\left(f_h-f_h(x);x\right)\right| \leq \|f_h'\|_{\tilde{C_B}[0,\infty)}\sqrt{\mathcal{H}_{n,c}^{(\alpha)}\left((t-x)^2;x\right)}+\frac{1}{2}\|f_h''\|_{\tilde{C_B}[0,\infty)}\mathcal{H}_{n,c}^{(\alpha)}\left((t-x)^2;x\right).$$

By Lemma 2.3 and property (b) of Steklov mean, we obtain

$$\left|\mathcal{H}_{n,c}^{(\alpha)}\left(f_h-f_h(x);x\right)\right| \leq \frac{5}{h}\omega(f,h)\sqrt{\beta_{n,\alpha,2}^{c}(x)}+\frac{9}{2h^2}\omega_2(f,h)\beta_{n,\alpha,2}^{c}(x).$$

Choosing $h=\sqrt{\beta_{n,\alpha,2}^{c}(x)}$, and substituting the values of the above estimates in (3.4), we get the desired relation. □

Remark 3.2. To show the advantage of the new construction of operators with respect to the order of approximation, we consider the following well-known Korovkin type inequalities.

$$|\,\mathcal{G}_n^{(\alpha)}(f;x)-f(x)\,| \leq \left(1+\frac{\mathcal{G}_n^{(\alpha)}((t-x)^2;x)}{\delta}\right)\omega(f,\delta)$$

$$|\,\mathcal{H}_{n,c}^{(\alpha)}(f;x)-f(x)\,| \leq \left(1+\frac{\mathcal{H}_{n,c}^{(\alpha)}((t-x)^2;x)}{\delta}\right)\omega(f,\delta)$$

For uniformly continuous functions on $[0,\infty)$, the sequences of linear positive operators $\mathcal{H}_{n,c}^{(\alpha)}$ will present a rate of convergence at least as good as that of operators $\mathcal{G}_n^{(\alpha)}$ whenever

$$\mathcal{H}_{n,c}^{(\alpha)}((t-x)^2;x) \leq \mathcal{G}_n^{(\alpha)}((t-x)^2;x).$$

With the help of Maple, if we choose $\alpha=-2, c=3, n=100$ the above inequality holds for $x \in [\frac{6041}{12300}+\frac{1}{12300}\sqrt{38953681},\infty)$. Hence we conclude that our new construction present better rate of convergence on this interval. Of course, this interval can be extended with different selection of α and c.

4. Weighted approximation

Let $\varrho(x)=1+x^2$ be a weight function and $B_\varrho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)| \leq N_f\varrho(x)$, where N_f is a positive constant which may depend only on f. Let $C_\varrho[0,\infty)$ represents the space of all continuous functions in $B_\varrho[0,\infty)$ endowed with the norm

$$\|f\|_\varrho := \sup_{x\in[0,\infty)}\frac{|f(x)|}{\varrho(x)} \text{ and } C_\varrho^0[0,\infty) := \left\{f \in C_\varrho[0,\infty) : \lim_{x\to\infty}\frac{|f(x)|}{\varrho(x)} \text{ exists and is finite}\right\}.$$

Let $f \in C_\varrho^0[0,\infty)$. The weighted modulus of continuity is given [see [28]],

$$\Omega(f;\delta)=\sup_{x\in[0,\infty),0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Lemma 4.4. [28] *Let $f \in C^0_\varrho[0,\infty)$, then:*

i) $\Omega(f;\delta)$ *is a monotone increasing function of* δ;
ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;
iii) *for each* $m \in \mathbb{N}, \Omega(f, m\delta) \leq m\Omega(f;\delta)$;
iv) *for each* $\lambda \in [0,\infty), \Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.

Theorem 4.3. *Let $f \in C^0_\varrho[0,\infty)$. If $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} \frac{n}{\alpha(n)} = l \in \mathbb{R}$, then there exists $\tilde{n} \in \mathbb{N}$ and a positive constant $M(c,l)$ depending on c and l such that*

$$\sup_{x\in(0,\infty)} \frac{|\mathcal{H}^{(\alpha)}_{n,c}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq M(c,l)\Omega\left(f; n^{-1/2}\right), \text{ for } n > \tilde{n}. \tag{4.6}$$

Proof. For $t > 0, x \in (0,\infty)$ and $\delta > 0$, by the definition of $\Omega(f;\delta)$ and Lemma 4.4, we can write

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (x + |x-t|)^2)\Omega(f; |t-x|) \\ &\leq 2(1+x^2)(1+(t-x)^2)\left(1 + \frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

$\mathcal{H}^{(\alpha)}_{n,c}$ is linear and positive, we find that

$$\begin{aligned} |\mathcal{H}^{(\alpha)}_{n,c}(f;x) - f(x)| \leq 2(1+x^2)\Omega(f;\delta)\Big\{&1 + \mathcal{H}^{(\alpha)}_{n,c}((t-x)^2;x) \\ &+ \mathcal{H}^{(\alpha)}_{n,c}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\Big\}. \end{aligned} \tag{4.7}$$

From the Remark 2.1 it follows that there is $n_1 \in \mathbb{N}$ such that

$$\mathcal{H}^{(\alpha)}_{n,c}((t-x)^2;x) \leq M_1(c,l)\frac{(1+x^2)}{n}, \text{ for } n > n_1, \tag{4.8}$$

where $M_1(c,l)$ is a positive constant depending on c and l. Using Cauchy-Schwarz inequality, we can write

$$\begin{aligned} \mathcal{H}^{(\alpha)}_{n,c}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) &\leq \frac{1}{\delta}\sqrt{\mathcal{H}^{(\alpha)}_{n,c}((t-x)^2;x)} \\ &+ \frac{1}{\delta}\sqrt{\mathcal{H}^{(\alpha)}_{n,c}((t-x)^4;x)}\sqrt{\mathcal{H}^{(\alpha)}_{n,c}((t-x)^2;x)}. \end{aligned} \tag{4.9}$$

In view of the Remark 2.1 it follows that there is $n_2 \in \mathbb{N}$ such that

$$\sqrt{\mathcal{H}^{(\alpha)}_{n,c}\left((t-x)^4;x\right)} \leq M_2(c,l)\frac{(1+x^2)}{n}, \text{ for } n > n_2, \tag{4.10}$$

where $M_2(c,l)$ is a positive constant depending on c and l.

Let $\tilde{n} = \max\{n_1, n_2\}$. Combining (4.7)-(4.10) and choosing

$$M(c,l) = 2\left(1 + M_1(c,l) + \sqrt{M_1(c,l)} + M_2(c,l)\sqrt{M_1(c,l)}\right),\ \delta = \frac{1}{\sqrt{n}}, \text{ for } n > \tilde{n},$$

we get (4.6). □

5. Pointwise Convergence of $\mathcal{H}^{(\alpha)}_{n,c}$

Voronovskaya theorem in quantitative mean is given in this section. This kind of theorem decsribes the rate of convergence and an upper bound for the error of approximation simultaneously. Very recently in [4] Acar et al. (See also [2]) presented the quantitative Voronovskaya theorem for positive linear operators acting on unbounded intervals as:

Theorem 5.4. *Let E be a subspace of $C[0,\infty)$ which contains the polynomials, and $L_n : E \to C[0,\infty)$ be a sequence of l.p.o such that $L_n e_i = e_i, i = 0, 1$. If $f \in E$ and $f'' \in C_\varrho[0,\infty)$, then we have for $x \in [0,\infty)$ that*

$$\left| L_n(f;x) - f(x) - \frac{1}{2} f''(x)\, \mu_{n,2}^L(x) \right| \leq 16\left(1+x^2\right) \Omega\left(f''; \left(\frac{\mu_{n,6}^L(x)}{\mu_{n,2}^L(x)} \right)^{1/4} \right) \mu_{n,2}^L(x),$$

where $\mu_{n,m}^L(x), m \in \mathbb{N}$ is the moment of order m of L_n.

Corollary 5.1. *The above result may be stated without the assumptions $L_n e_i = e_i, i = 0, 1$, as*

$$\left| L_n(f;x) - f(x)\, L_n(e_0;x) - f'(x)\, L_n(e_1 - x;x) - \frac{1}{2} f''(x)\, L_n\left((e_1 - x)^2; x \right) \right|$$
$$\leq \quad 16\left(1+x^2\right) \Omega\left(f''; \left(\frac{\mu_{n,6}^L(x)}{\mu_{n,2}^L(x)} \right)^{1/4} \right) \mu_{n,2}^L(x).$$

Theorem 5.5. *Let $c > 0$, $n \in \mathbb{N}$, $\alpha = \alpha(n) \to \infty$, as $n \to \infty$. If f', f'' exist at any point $x \in [0,\infty)$ and $f'' \in C_\varrho[0,\infty)$, then it follows for $x \in [0,\infty)$ that*

$$\left| n\left[\mathcal{H}_{n,c}^{(\alpha)}(f;x) - f(x) \right] - (cx+1) f'(x) - \frac{1}{2}\left[x(2+cx+lx) \right] f''(x) \right|$$
$$\leq \left| \frac{xc^2 + c}{n - c} \right| |f'(x)| + \left| \frac{n}{2}\left(\frac{x^2n^2 + \left(cx^2 + 2x\right)\alpha(2c+n) + 2\alpha}{(n-2c)^2\alpha} \right) - \frac{1}{2}\left[x(2+cx+lx) \right] \right| |f''(x)|$$
$$+16\left(1+x^2\right) \Omega\left(f''; \left(\frac{\beta_{n,\alpha,6}^c(x)}{\beta_{n,\alpha,2}^c(x)} \right)^{1/4} \right) n\beta_{n,\alpha,2}^c(x).$$

Proof. According to Corollary 5.1 we immediately have the desired result. □

Corollary 5.2. *Let $c > 0$, $n \in \mathbb{N}$, $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and f', f'' exist at any point $x \in [0,\infty)$, $f'' \in C_\varrho[0,\infty)$, if we take limit in above theorem as $n \to \infty$, then we have Voronovskaya theorem for the operators $\mathcal{H}_{n,c}^{(\alpha)}$ as*

$$\lim_{n\to\infty} n\left[\mathcal{H}_{n,c}^{(\alpha)}(f;x) - f(x) \right] = (cx+1) f'(x) + \frac{1}{2}\left[x(2+cx+lx) \right] f''(x).$$

6. Rate of convergence

In this section we discuss the approximation of functions with a derivative of bounded variation.

Let $DBV[0,\infty)$ be the class of all functions $f \in B_\varrho[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. The function $f \in DBV[0,\infty)$ has the following representation$f(x) = \int_0^x g(t) + f(0)$, where g is a function of bounded variation on each finite subinterval of $[0,\infty)$.

In order to study the convergence of the operators $\mathcal{H}_{n,c}^{(\alpha)}$ for functions having a derivative of bounded variation, we rewrite the operators (1.2) as follows

$$\mathcal{H}_{n,c}^{(\alpha)}(f;x) = \int_0^\infty \mathcal{P}_{n,c}^{(\alpha)}(x,t) f(t)\,dt, \tag{6.11}$$

$$\mathcal{P}_{n,c}^{(\alpha)}(x,t) = \sum_{k=0}^\infty m_{n,k}^{(\alpha)}(x)\, b_{n,k}^c(t).$$

Lemma 6.5. *Let $\alpha=\alpha(n)\to\infty$, as $n\to\infty$ and $\lim\limits_{n\to\infty}\dfrac{n}{\alpha(n)}=l\in\mathbb{R}$. For all $x\in(0,\infty)$ and sufficiently large n, we have*

i) $\vartheta_{n,c}^{(\alpha)}(x,t)=\displaystyle\int_0^t \mathcal{P}_{n,c}^{(\alpha)}(x,u)du\leq \frac{M(c,l)}{(x-t)^2}\frac{1+x^2}{n}$, $0\leq t<x$,

ii) $1-\vartheta_{n,c}^{(\alpha)}(x,t)=\displaystyle\int_t^\infty \mathcal{P}_{n,c}^{(\alpha)}(x,u)du\leq \frac{M(c,l)}{(t-x)^2}\frac{1+x^2}{n}$, $x\leq t<\infty$,

where $M(c,l)$ is a positive constant depending on c and l..

Proof. For sufficiently large n it follows Remark 2.1 that

$$\mathcal{H}_{n,c}^{(\alpha)}((u-x)^2;x)<M(c,l)\frac{1+x^2}{n}. \tag{6.12}$$

Applying Lemma 2.3, we have

$$\begin{aligned}\vartheta_{n,c}^{(\alpha)}(x,t)&=\int_0^t \mathcal{P}_{n,c}^{(\alpha)}(x,u)du\leq\int_0^t\left(\frac{x-u}{x-t}\right)^2\mathcal{P}_{n,c}^{(\alpha)}(x,u)du\\&\leq\frac{1}{(x-t)^2}\mathcal{H}_{n,c}^{(\alpha)}\left((u-x)^2;x\right)\leq\frac{M(c,l)}{(x-t)^2}\frac{1+x^2}{n}.\end{aligned}$$

The proof of ii) is similar hence the details are omitted. □

The proof of following result is omitted. One can refer to the papers [5, 6, 21] for a method of the proof in the light of Lemma 6.5 .

Theorem 6.6. *Let $f\in DBV[0,\infty)$, $\alpha=\alpha(n)\to\infty$, as $n\to\infty$ and $\lim\limits_{n\to\infty}\dfrac{n}{\alpha(n)}=l\in\mathbb{R}$. Then, for every $x\in(0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}|\mathcal{H}_{n,c}^{(\alpha)}(f;x)-f(x)|\ \leq\ &\frac{(cx+1)}{(n-c)}\left|\frac{f'(x+)+f'(x-))}{2}\right|+\sqrt{M(c,l)\frac{1+x^2}{n}}\left|\frac{f'(x+)-f'(x-))}{2}\right|\\&+M(c,l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x}f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}f'_x\right)\\&+\left(4N_f+\frac{N_f+|f(x)|}{x^2}\right)M(c,l)\frac{1+x^2}{n}+|f'(x+)|\sqrt{M(c,l)\frac{1+x^2}{n}}\\&+M(c,l)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}f'_x\\&+M(c,l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}f'_x,\end{aligned}$$

where $M(c,l)$ is a positive constant depending on c and l, $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t)=\begin{cases}f'(t)-f'(x-), & 0\leq t<x,\\ 0, & t=x,\\ f'(t)-f'(x+), & x<t<\infty.\end{cases} \tag{6.13}$$

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