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Existence of a unique positive solution for a singular fractional boundary value problem

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ABSTRACT. In the present work, we discuss the existence of a unique positive solution of a boundary value problem for a nonlinear fractional order equation with singularity. Precisely, order of equation $D_{0+}^{\alpha}u(t) = f(t,u(t))$ belongs to $(3,4]$ and f has a singularity at $t=0$ and as a boundary conditions we use $u(0)=u(1)=u'(0)=u'(1)=0$. Using a fixed point theorem, we prove the existence of unique positive solution of the considered problem.

1. INTRODUCTION

In this paper, we study the existence and uniqueness of positive solution for the following singular fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t)),\ 0<t<1 \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases} \tag{1.1}$$

where $\alpha \in (3,4]$, and D_{0+}^{α} denotes the Riemann-Liouville fractional derivative. Moreover, $f:(0,1]\times[0,\infty)\to[0,\infty)$ with $\lim\limits_{t\to 0+} f(t,-)=\infty$ (i.e. f is singular at $t=0$).

Similar problem was investigated in [2], in case when $\alpha\in(1,2]$ and with boundary conditions $u(0)=u(1)=0$. We note as well work [3], where the following problem

$$\begin{cases} D^{\alpha}u + f(t,u,u',D^{\mu}u) = 0,\ 0<t<1 \\ u(0)=u'(0)=u'(1)=0, \end{cases}$$

was under consideration. Here $\alpha\in(2,3)$, $\mu\in(0,1)$ and function $f(t,x,y,z)$ is singular at the value of 0 of its arguments x,y,z.

We would like notice some related recent works [1, 5, 7], which consider higher order fractional nonlinear equations for the subject of the existence of positive solutions.

2. PRELIMINARIES

We need the following lemma, which appear in [6].

Lemma 2.1. (Lemma 2.3 of [6]) *Given* $h\in C[0,1]$ *and* $3<\alpha\le 4$, *a unique solution of*

$$\begin{cases} D_{0+}^{\alpha}u(t) = h(t),\ 0<t<1 \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases}$$

is

$$u(t)=\int_0^1 G(t,s)h(s)ds,$$

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where

$$G(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}\left[(s-t)+(\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0\le s\le t\le 1\\ \dfrac{(1-s)^{\alpha-2}t^{\alpha-2}\left[(s-t)+(\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0\le t\le s\le 1\end{cases}$$

Lemma 2.2. (Lemma 2.4 of [6]) *The function $G(t,s)$ appearing in Lemma 2.1 satisfies:*

(a) $G(t,s)>0$ *for* $t,s\in(0,1)$*;*
(b) $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$.

For our study, we need a fixed point theorem. This theorem uses the following class of functions $\mathfrak{F}$.

By $\mathfrak{F}$ we denote the class of functions $\varphi:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

(a) φ is strictly increasing;
(b) For each sequence $(t_n)\subset(0,\infty)$

$$\lim_{n\to\infty}t_n=0\ \Leftrightarrow\ \lim_{n\to\infty}\varphi(t_n)=-\infty;$$

(c) There exists $k\in(0,1)$ such that $\lim\limits_{t\to0^+}t^k\varphi(t)=0$.

Examples of functions belonging to $\mathfrak{F}$ are $\varphi(t)=-\frac{1}{\sqrt{t}}$, $\varphi(t)=\ln t$, $\varphi(t)=\ln t+t$, $\varphi(t)=\ln(t^2+t)$.

The result about fixed point which we use is the following and it appears in [4]:

Theorem 2.1. *Let (X,d) be a complete metric space and $T:X\to X$ a mapping such that there exist $\tau>0$ and $\varphi\in\mathfrak{F}$ satisfying for any $x,y\in X$ with $d(Tx,Ty)>0$,*

$$\tau+\varphi\left(d(Tx,Ty)\right)\le\varphi(d(x,y)).$$

Then T has a unique fixed point.

3. Main result

Our starting point of this section is the following lemma.

Lemma 3.3. *Let $0<\sigma<1$, $3<\alpha<4$ and $F:(0,1]\to\mathbb{R}$ is continuous function with $\lim\limits_{t\to0^+}F(t)=\infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0,1]$. Then the function defined by*

$$H(t)=\int_0^1 G(t,s)F(s)ds$$

is continuous on $[0,1]$, where $G(t,s)$ is the Green function appearing in Lemma 2.1.

Proof. We consider three cases:

- Case 1: $t_0=0$.
  It is clear that $H(0)=0$. Since $t^\sigma F(t)$ is continuous on $[0,1]$, we can find a constant $M>0$ such that

$$|t^\sigma F(t)|\le M\quad\text{for any}\quad t\in[0,1].$$

Moreover, we have

$$
\begin{aligned}
|H(t)-H(0)| = |H(t)| &= \left|\int_0^1 G(t,s)F(s)ds\right| = \left|\int_0^1 G(t,s)s^{-\sigma}s^{\sigma}F(s)ds\right| \\
&= \left|\int_0^t \frac{(t-s)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right. \\
&\quad \left. + \int_t^1 \frac{(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right| \\
&= \left|\int_0^1 \frac{(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right. \\
&\quad \left. + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right| \\
&\leq \frac{Mt^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}|(s-t)+(\alpha-2)(1-t)s|s^{-\sigma}ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\sigma}ds \leq \frac{M(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}s^{-\sigma}ds \\
&\quad + \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} s^{-\sigma}ds.
\end{aligned}
$$

Considering definition of Euler's beta-function, we derive

$$
|H(t)-H(0)| \leq \frac{M(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}B(1-\sigma,\alpha-1) + \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)}B(1-\sigma,\alpha).
$$

From this we deduce that $|H(t)-H(0)| \to 0$ when $t \to 0$.

This proves that H is continuous at $t_0 = 0$.

- Case 2: $t_0 \in (0,1)$.

  We take $t_n \to t_0$ and we have to prove that $H(t_n) \to H(t_0)$. Without loss of generality, we consider $t_n > t_0$. Then, we have

$$
\begin{aligned}
&|H(t_n)-H(t_0)| = \\
&= \left|\int_0^{t_n} \frac{(t_n-s)^{\alpha-1}+(1-s)^{\alpha-2}t_n^{\alpha-2}[(s-t_n)+(\alpha-2)(1-t_n)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right. \\
&\quad + \int_{t_n}^1 \frac{(1-s)^{\alpha-2}t_n^{\alpha-2}[(s-t_n)+(\alpha-2)(1-t_n)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds \\
&\quad - \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}+(1-s)^{\alpha-2}t_0^{\alpha-2}[(s-t_0)+(\alpha-2)(1-t_0)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds \\
&\quad \left. - \int_{t_0}^1 \frac{(1-s)^{\alpha-2}t_0^{\alpha-2}[(s-t_0)+(\alpha-2)(1-t_0)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right| \\
&= \left|\int_0^1 \frac{(1-s)^{\alpha-2}t_n^{\alpha-2}[(s-t_n)+(\alpha-2)(1-t_n)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_n}\frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\\
&-\int_0^1\frac{(1-s)^{\alpha-2}t_0^{\alpha-2}[(s-t_0)+(\alpha-2)(1-t_0)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\\
&-\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\Bigg|\\
=&\Bigg|\int_0^1\frac{(1-s)^{\alpha-2}\left(t_n^{\alpha-2}-t_0^{\alpha-2}\right)[(s-t_n)+(\alpha-2)(1-t_n)s]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\\
+&\int_0^1\frac{(1-s)^{\alpha-2}t_0^{\alpha-2}\left[(s-t_n)+(\alpha-2)(1-t_n)s-[(s-t_0)+(\alpha-2)(1-t_0)s]\right]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\\
&+\int_0^{t_0}\frac{\left[(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right]}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds+\int_{t_0}^{t_n}\frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^{\sigma}F(s)ds\Bigg|\\
\leq&\frac{M\left|t_n^{\alpha-2}-t_0^{\alpha-2}\right|}{\Gamma(\alpha)}(\alpha-1)\int_0^1(1-s)^{\alpha-2}s^{-\sigma}ds+\frac{Mt_0^{\alpha-2}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}|t_n-t_0|(\alpha-1)s^{-\sigma}ds\\
&+\frac{M}{\Gamma(\alpha)}\int_0^{t_0}\left|(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right|s^{-\sigma}ds\frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n}(t_n-s)^{\alpha-1}s^{-\sigma}ds\\
\leq&\frac{M}{\Gamma(\alpha)}\left(t_n^{\alpha-2}-t_0^{\alpha-2}\right)(\alpha-1)B(1-\sigma,\alpha-1)+\frac{M(t_n-t_0)}{\Gamma(\alpha)}(\alpha-1)B(1-\sigma,\alpha-1)\\
&+\frac{M}{\Gamma(\alpha)}I_n^1+\frac{M}{\Gamma(\alpha)}I_n^2,
\end{aligned}
$$

where

$$
I_n^1=\int_0^{t_0}\left[(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right]s^{-\sigma}ds,\ \ I_n^2=\int_{t_0}^{t_n}(t_n-s)^{\alpha-1}s^{-\sigma}ds.
$$

In the sequel, we will prove that $I_n^1\to 0$ when $n\to\infty$. In fact, as

$$
\left[(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right]s^{-\sigma}\leq\left[|t_n-s|^{\alpha-1}-|t_0-s|^{\alpha-1}\right]s^{-\sigma}\leq 2s^{-\sigma}
$$

and $\int_0^1 2s^{-\sigma}ds=\frac{2}{1-\sigma}<\infty$. By Lebesque's dominated convergence theorem $I_n^1\to 0$ when $n\to\infty$.

Now, we will prove that $I_n^2\to 0$ when $n\to\infty$. In fact, since

$$
I_n^2=\int_{t_0}^{t_n}(t_n-s)^{\alpha-1}s^{-\sigma}ds\leq\int_{t_0}^{t_n}s^{-\sigma}ds=\frac{1}{1-\sigma}\left(t_n^{1-\sigma}-t_0^{1-\sigma}\right)
$$

and as $t_n\to t_0$, we obtain the desired result.

Finally, taking into account above obtained estimates, we infer that $|H(t_n)-H(t_0)|\to 0$ when $n\to\infty$.

- Case 3: $t_0=1$.

  It is clear that $H(1)=0$ and following the same argument that in Case No 1, we can prove that continuity of H at $t_0=1$.

□

Lemma 3.4. *Suppose that $0<\sigma<1$. Then there exists*

$$
N=\max_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}ds.
$$

Proof. Considering representation of the function $G(t,s)$ and evaluations of Lemma 3.3, we derive

$$\int_0^1 G(t,s)s^{-\sigma}ds = \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-\sigma}B(1-\sigma,\alpha) - t^{\alpha-1}\left(B(1-\sigma,\alpha-1)\right.\right.$$
$$\left.\left.+(\alpha-2)B(2-\sigma,\alpha-1)\right) + (\alpha-1)t^{\alpha-2}B(2-\sigma,\alpha-1)\right].$$

Taking

$$B(1-\sigma,\alpha) = \frac{\alpha-1}{\alpha-\sigma}B(1-\sigma,\alpha-1);\ B(2-\sigma,\alpha-1) = \frac{1-\sigma}{\alpha-\sigma}B(1-\sigma,\alpha-1),$$

into account we infer

$$\int_0^1 G(t,s)s^{-\sigma}ds = \frac{B(1-\sigma,\alpha-1)}{\Gamma(\alpha)}\left[\frac{\alpha-1}{\alpha-\sigma}t^{\alpha-\sigma} - \left(1+\frac{(\alpha-2)(1-\sigma)}{\alpha-\sigma}\right)t^{\alpha-1}\right.$$
$$\left.+\frac{(\alpha-1)(1-\sigma)}{\alpha-\sigma}t^{\alpha-2}\right].$$

Denoting $L(t) = \int_0^1 G(t,s)s^{-\sigma}ds$, from the last equality one can easily derive that $L(0) = 0$, $L(1) = 0$. Since $G(t,s) \geq 0$, then $L(t) \geq 0$ and as $L(t)$ is continuous on $[0,1]$, it has a maximum. □

Theorem 3.2. *Let* $0 < \sigma < 1$, $3 < \alpha \leq 4$, $f : (0,1] \times [0,\infty) \to [0,\infty)$ *be continuous and* $\lim_{t\to 0^+} f(t,\cdot) = \infty$, $t^\sigma f(t,y)$ *be continuous function on* $[0,1]\times[0,\infty)$. *Assume that there exist constants* $0 < \lambda \leq \frac{1}{N}$, *and* $\tau > 0$ *such that for* $x, y \in [0,\infty)$ *and* $t \in [0,1]$

$$t^\sigma |f(t,x) - f(t,y)| \leq \frac{\lambda|x-y|}{\left(1+\tau\sqrt{|x-y|}\right)^2}.$$

Then Problem (1.1) *has a unique non-negative solution.*

Proof. Consider the cone $P = \{u \in C[0,1] : u \geq 0\}$. Notice that P is a closed subset of $C[0,1]$ and therefore, (P,d) is a complete metric space where

$$d(x,y) = \sup\{|x(t)-y(t)| : t \in [0,1]\} \quad \text{for} \quad x,y \in P.$$

Now, for $u \in P$ we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds = \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,u(s))ds.$$

In virtue of Lemma 3.3, for $u \in P$, $Tu \in C[0,1]$ and, since $G(t,s)$ and $t^\sigma f(t,y)$ are non-negative functions, $Tu \geq 0$ for $u \in P$. Therefore, T applies P into itself.

Next, we check that assumptions of Theorem 2.1 are satisfied. In fact, for $u, v \in P$ with $d(Tu,Tv) > 0$, we have

$$d(Tu,Tv) = \max_{t\in[0,1]}|(Tu)(t)-(Tv)(t)| = \max_{t\in[0,1]}\left|\int_0^1 G(t,s)s^{-\sigma}s^\sigma\left(f(s,u(s))-f(s,v(s))\right)ds\right|$$
$$\leq \max_{t\in[0,1]}\int_0^1 G(t,s)s^{-\sigma}s^\sigma|f(s,u(s))-f(s,v(s))|ds$$
$$\leq \max_{t\in[0,1]}\int_0^1 G(t,s)s^{-\sigma}\frac{\lambda|u(s)-v(s)|}{\left(1+\tau\sqrt{|u(s)-v(s)|}\right)^2}ds \leq \max_{t\in[0,1]}\int_0^1 G(t,s)s^{-\sigma}\frac{\lambda d(u,v)}{\left(1+\tau\sqrt{d(u,v)}\right)^2}ds$$
$$= \frac{\lambda d(u,v)}{\left(1+\tau\sqrt{d(u,v)}\right)^2}\max_{t\in[0,1]}\int_0^1 G(t,s)s^{-\sigma}ds = \frac{\lambda d(u,v)}{\left(1+\tau\sqrt{d(u,v)}\right)^2}N \leq \frac{d(u,v)}{\left(1+\tau\sqrt{d(u,v)}\right)^2},$$

where we have used that $\lambda \leq \frac{1}{N}$ and the non-decreasing character of the function $\beta(t) = \dfrac{t}{(1+\tau\sqrt{t})^2}$. Therefore,

$$d(Tu, Tv) \leq \frac{d(u,v)}{\left(1+\tau\sqrt{d(u,v)}\right)^2}.$$

This gives us

$$\sqrt{d(Tu,Tv)} \leq \frac{\sqrt{d(u,v)}}{1+\tau\sqrt{d(u,v)}} \qquad \text{or} \qquad \tau - \frac{1}{\sqrt{d(Tu,Tv)}} \leq -\frac{1}{\sqrt{d(u,v)}}$$

and the contractivity condition of the Theorem 2.1 is satisfied with the function $\varphi(t) = -\frac{1}{\sqrt{t}}$ which belongs to the class $\mathfrak{F}$.

Consequently, by Theorem 2.1, the operator T has a unique fixed point in P. This means that Problem (1.1)) has a unique non-negative solution in $C[0,1]$. This finishes the proof. □

An interesting question from a practical point of view is that the solution of Problem (1.1) is positive. A sufficient condition for that solution is positive, appears in the following result:

Theorem 3.3. *Under assumptions of Theorem 3.2, if the function $t^\sigma f(t,y)$ is non-decreasing respect to the variable y, then the solution of Problem* (1.1) *given by Theorem 3.2 is positive.*

Proof. In contrary case, we find $t^* \in (0,1)$ such that $u(t^*)$=0. Since $u(t)$ is a fixed point of the operator T (see Theorem 3.2) this means that

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds \quad \text{for} \quad 0<t<1.$$

Particularly,

$$0 = u(t^*) = \int_0^1 G(t^*,s)f(s,u(s))ds.$$

Since that G and f are non-negative functions, we infer that

$$G(t^*,s)f(s,u(s)) = 0 \;\; a.e.\,(s)$$

On the other hand, as $\lim_{t\to 0^+} f(t,0) = \infty$ for given $M>0$ there exists $\delta>0$ such that for $s\in(0,\delta)$ $f(s,0)>M$. Since $t^\sigma f(t,y)$ is increasing and $u(t)\geq 0$,

$$s^\sigma f(s,u(s)) \geq s^\sigma f(s,0) \geq s^\sigma M \quad \text{for} \quad s\in(0,\delta)$$

and, therefore, $f(s,u(s)) \geq M$ for $s\in(0,\delta)$ and $f(s,u(s)) \neq 0$ $a.e.\,(s)$. But this is a contradiction since $G(t^*,s)$ is a function of rational type in the variable s and, consequently, $G(t^*,s)\neq 0$ $a.e.\,(s)$. Therefore, $u(t)>0$ for $t\in(0,1)$. □

In the sequel, we present an example illustrating our results.

Example 3.1. Consider the following singular fractional boundary value problem

$$\begin{cases} D_{0+}^{7/2}u(t) = \dfrac{\lambda u(t)}{\sqrt{t}\left(1+5\sqrt{u(t)}\right)^2}, & 0<t<1 \\ u(0)=u(1)=u'(0)=u'(1)=0, \end{cases} \tag{3.2}$$

where $\lambda > 0$.

Notice that Problem(3.2) is a particular case of Problem (1.1) with $\alpha = 7/2$, and $f(t,u) = \dfrac{\lambda u(t)}{\sqrt{t}\left(1+5\sqrt{u(t)}\right)^2}$.

It is clear that $f\colon (0,1]\times[0,\infty)\to[0,\infty)$ is continuous and for $\sigma = 1/2$, $t^\sigma f(t,u) = \sqrt{t}f(t,u) = \dfrac{\lambda u(t)}{\left(1+5\sqrt{u(t)}\right)^2}$ is a continuous function on $[0,1]\times[0,\infty)$.

On the other hand, for $t\in[0,1]$ and without loss of generality, for $x,y\in[0,\infty)$ with $x\geq y$, we have

$$\sqrt{t}|f(t,x)-f(t,y)| = \lambda\left|\frac{x}{(1+5\sqrt{x})^2}-\frac{y}{\left(1+5\sqrt{y}\right)^2}\right| =$$

$$= \lambda\left(\frac{x}{\left(1+5\sqrt{x}\right)^2}-\frac{y}{\left(1+5\sqrt{y}\right)^2}\right) \leq \lambda\left(\frac{x}{\left(1+5\sqrt{x}\right)^2}-\frac{y}{\left(1+5\sqrt{x}\right)^2}\right)$$

$$= \frac{\lambda(x-y)}{\left(1+5\sqrt{x}\right)^2} \leq \frac{\lambda(x-y)}{\left(1+5\sqrt{x-y}\right)^2}.$$

where we have used the nondecreasing character of the function $\varphi\colon[0,\infty)\to[0,\infty)$ defined by $\varphi(u) = \dfrac{u}{\left(1+5\sqrt{u}\right)^2}$ (since $\varphi'(u)\geq 0$) and the fact that $x\geq y$.

In our case,

$$N = \max_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-1/2}ds =$$

$$= \max_{0\leq t\leq 1}\frac{B(1/2,5/2)}{\Gamma(7/2)}\left[\frac{5}{6}t^3-\frac{5}{4}t^{5/2}+\frac{5}{12}t^{3/2}\right] = \max_{0\leq t\leq 1}\frac{\sqrt{\pi}}{5}\left[\frac{5}{6}t^3-\frac{5}{4}t^{5/2}+\frac{5}{12}t^{3/2}\right]$$

$$= \max_{0\leq t\leq 1}\sqrt{\pi}\left[\frac{1}{6}t^3-\frac{1}{4}t^{5/2}+\frac{1}{12}t^{3/2}\right] \leq \max_{0\leq t\leq 1}\sqrt{\pi}\left[\frac{1}{6}t^3+\frac{1}{12}t^{3/2}\right] \leq \frac{\sqrt{\pi}}{4} \leq 0.45.$$

Therefore,

$$2.\widehat{2} = \frac{1}{0.45} \leq \frac{1}{N}.$$

Now, by using Theorem 3.2, if $\lambda\leq 2.\widehat{2}$ then Problem(3.2) has a unique nonnegative solution. Moreover, since $\sqrt{t}f(t,u) = \dfrac{\lambda u}{\left(1+5\sqrt{u}\right)^2}$ is nondecreasing respect to the variable u, Theorem 3.3 says us that the solution is positive.

Notice that Example 3.1 cannot be treated by the Banach's contraction principle since, in this case, the operator T is defined on the cone P as

$$(Tu)(t) = \int_0^1 \lambda G(t,s)\frac{u(s)}{\sqrt{s}(1+5\sqrt{u(s)})^2}ds \quad \text{for} \quad t\in[0,1]$$

and, for $\lambda\leq 2.\widehat{2}$ we have

$$d(Tu,Tv) \leq \frac{d(u,v)}{(1+5\sqrt{d(u,v)})^2}, \quad \text{for} \quad u,v\in P$$

and consequently,

$$\frac{d(Tu,Tv)}{d(u,v)} \leq \frac{1}{\left(1+5\sqrt{d(u,v)}\right)^2}, \quad \text{for} \quad u,v \in P \text{ with } u \neq v.$$

Therefore, when $d(u,v) \to 0$ (for example, por $u \in P$ fixed and $v = u + \frac{1}{n} \in P,\ n \in \mathbb{N}$), $\frac{d(Tu,Tv)}{d(u,v)} \to 1$, and, this proves that the Banach's contraction principle cannot be applied since $\frac{d(Tu,Tv)}{d(u,v)}$ is not bounded by a constant less than one.

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