CARPATHIAN J. MATH. 34 (2018), No. 1, 65 - 75
Online version at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Best proximity point theorems for G-proximal weak contractions in complete metric spaces endowed with graphs

CHALONGCHAI KLANARONG and SUTHEP SUANTAI

ABSTRACT. In this paper, the existence of best proximity point theorems for two new types of nonlinear non-self mappings in a complete metric space endowed with a directed graph are established. Our main results extend and generalize many known results in the literatures. As a special case of the main results, the best proximity point theorems on partially ordered sets are obtained.

1. INTRODUCTION

Banach contraction principle is the most important and useful tool for proving the existence problems of various equations in Science, Applied Science, Economics, Physics and Engineering. This principle was given by the famous Polish mathematician, Stefan Banach [4], in 1922.

Theorem 1.1 (Banach Contraction Principle). *Let (X,d) be a complete metric space. Let a self-mapping $T : X \to X$ be a contraction, that is, if there exists $a \in [0,1)$ such that*

$$d(Tx,Ty) \leq ad(x,y), \text{ for all } x,y \in X, \tag{1.1}$$

then T has a unique fixed point in X.

This principle has been extended and generalized in various directions by many authors (see [4,7,13–15,24–27,29]).

In 2004, Berinde [7] introduced and studied the concept of weak contraction mapping in the context of a complete metric space, later named by himself as almost contraction, see [8–11]. Let (X,d) be a metric space. A mapping $T : X \to X$ is called *weak contraction* if there exist $a \in (0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq ad(x,y) + Ld(x,Ty), \text{ for all } x,y \in X. \tag{1.2}$$

In the same paper, he also introduced (θ, L)-weak contraction by replacing a in (1.2) by $\theta(d(x,y))$ where $\theta : [0,\infty) \to [0,1)$ satisfying $\limsup_{r \to t^+} \theta(r) < 1$ for all $t \in (0,\infty)$. We note that his main results extended and generalized the Banach contraction principle and others, see [7] and references therein.

Now, let A and B be two nonempty subsets of X, and $T : A \to B$ a non-self mapping. A point $x \in A$ is called a *best proximity point* of T if $d(x,Tx) = D(A,B)$, where $D(A,B) = \inf\{d(x,y) \mid (x,y) \in A \times B\}$. It is clear that the fixed point equation $Tx = x$ may has no solution if A and B are disjoint. In fact, if A and B are two nonempty closed subsets of X and $D(A,B) = 0$, then the best proximity point will be a fixed point of T. The following best approximation theorem was established by Ky Fan [20].

Received: 24.04.2017. In revised form: 23.01.2018. Accepted: 30.01.2018
2010 *Mathematics Subject Classification.* 47H10, 46N10, 30L99.
Key words and phrases. *Best proximity point, partially ordered set, ordered proximal contraction, proximally order-preserving, proximally edge-preserving, G-proximal weak contraction.*
Corresponding author: Suthep Suantai; suthep.s@cmu.ac.th

Theorem 1.2 ([20]). *Let X be a normed linear space, A be a nonempty compact convex subset of X and $T : A \to X$ be a continuous mapping. Then there exists $u \in A$ such that $\|u - Tu\| = D(Tu, A)$ where $D(Tu, A) := \inf\{\|Tu - a\| : a \in A\}$.*

It is known that a point $u \in A$ in Theorem 1.2 is called a *best approximant point* of T in A.

Many interesting results about best proximity points can be found in [1,3,16–19,21,22, 28].

Fixed point theory in partially ordered metric spaces has been studied by many authors (see [2,6,12,25–27]).

In 2004, Ran and Reurings [27] introduced and studied the existence of fixed points in the context of a partially ordered metric space.

In 2005, Nieto and Lopez [25,26] proved some fixed point theorems for monotone mappings in partially ordered metric spaces. They also applied the main result to prove the existence of solutions of some ODE problems.

In 2013, Basha [5] studied the best proximity point theorems and introduced the concept of ordered proximal contraction and proximity order-preserving for non-self mappings in a partially ordered metric space. Let $(X, d, \preceq)$ be a partially ordered metric space, A and B be two nonempty subsets of X.

Definition 1.1 ([5]). A mapping $T : A \to B$ is said to be an *ordered contraction* if there exists $a \in [0, 1)$ such that

$$x \preceq y \Longrightarrow d(Tx, Ty) \leq ad(x, y)$$

for all $x, y \in A$.

Definition 1.2 ([5]). A mapping $T : A \to B$ is said to be an *ordered proximal contraction* if there exists $a \in [0, 1)$ such that

$$\left.\begin{array}{l} x \preceq y \\ d(u_1, Tx) = D(A, B) \\ d(u_2, Ty) = D(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq ad(x, y)$$

for all $x, y, u_1, u_2 \in A$.

Definition 1.3 ([5]). A mapping $T : A \to B$ is said to be *order-preserving* if

$$x \preceq y \Longrightarrow Tx \preceq Ty$$

for all $x, y \in A$.

Definition 1.4 ([5]). A mapping $T : A \to B$ is said to be *order-reversing* if

$$x \preceq y \Longrightarrow Tx \succeq Ty$$

for all $x, y \in A$.

Definition 1.5 ([5]). A mapping $T : A \to B$ is said to be *monotone* if it is order-preserving or order-reversing.

Definition 1.6 ([5]). A mapping $T : A \to B$ is said to be *proximally order-preserving* if

$$\left.\begin{array}{l} x \preceq y \\ d(u_1, Tx) = D(A, B) \\ d(u_2, Ty) = D(A, B) \end{array}\right\} \Longrightarrow u_1 \preceq u_2$$

for all $x, y, u_1, u_2 \in A$.

Definition 1.7 ([5])**.** A mapping $T : A \to B$ is said to be *proximally order-reversing* if

$$\left.\begin{array}{l} x \preceq y \\ d(u_1, Tx) = D(A, B) \\ d(u_2, Ty) = D(A, B) \end{array}\right\} \Longrightarrow u_1 \succeq u_2$$

for all $x, y, u_1, u_2 \in A$.

The following theorem is a best proximity point theorem for continuous, monotone, ordered proximal contractions which was established by [5].

Theorem 1.3 ([5])**.** *Let $(X, \preceq)$ be a partially ordered set such that every pair of elements in X has a lower bound and an upper bound. Let d be a metric on X such that (X, d) is a complete metric space. Furthermore, let A and B be non-void closed subsets of the metric space (X, d) such that A_0 and B_0 are non-void. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) T is a continuous, proximally monotone, ordered proximal contraction.
(b) $T(A_0) \subseteq B_0$.
(c) g is a surjective isometry, its inverse is a monotone mapping and A_0 is contained in $g(A_0)$.
(d) There exist elements x_0 and $x_1 \in A_0$ such that $d(gx_1, Tx_0) = D(A, B)$ and $[x_0 \preceq x_1$ or $x_0 \succeq x_1]$.

Then there exists a unique element $x^* \in A_0$ such that

$$d(gx^*, Tx^*) = D(A, B)$$

Further, for any arbitrary $x_0' \in A_0$, the sequence $\{x_n'\}$ in A_0, defined by

$$d(gx_{n+1}', Tx_n') = D(A, B),$$

converges to the element x^*.

Motivated and inspired by the works mentioned above, we aim to introduce a new class of single valued non-self mappings which is more general than that of Berinde [7] and Basha [5] and to prove best proximity point theorems for this type of mappings in complete metric spaces endowed with directed graphs. We also apply our main results to obtain the existence result of the best proximal point for a single valued mapping in a partially ordered metric space.

2. Preliminaries

In this section, we present and introduce two classes of non-self mappings, called proximally edge-preserving and proximally weak edge-preserving in metric spaces endowed with directed graphs. We start with recalling some notions of graph theory. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that the set of its vertices $V(G) = X$ and the set of its edges $E(G) \subseteq X \times X$. We usually assume that the graph G has no parallel edges. The conversion of a graph G, is denoted by G^{-1}, that is,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Let A and B be nonempty subsets of X, We put

$$D(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = D(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = D(A, B) \text{ for some } x \in A\}.$$

Definition 2.8. Let X be a nonempty set, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and let $T : X \to X$ be a self mapping on X. Then T is said to be *edge-preserving* if for any $x, y \in X$,

$$(x, y) \in E(G) \text{ implies } (Tx, Ty) \in E(G).$$

Definition 2.9. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and $E(G)$. Let $T : A \to B$ be a non-self mapping. Then T is said to be *proximally edge-preserving* if it satisfies the condition that

$$\left.\begin{array}{l}(x, y) \in E(G)\\ d(u_1, Tx) = D(A, B)\\ d(u_2, Ty) = D(A, B)\end{array}\right\} \Longrightarrow (u_1, u_2) \in E(G)$$

where $u_1, u_2, x, y \in A$.

Example 2.1. Let $X = \mathbb{R}$ with the usual metric. Let $A = \{-2\} \cup [2, 3]$ and $B = [-1, 1]$. It is easy to see that $D(A, B) = 1,\ A_0 = \{-2, 2\}$ and $B_0 = \{-1, 1\}$. Let $G = (\mathbb{R}, E(G))$ be a directed graph defined by

$$E(G) = \{(-x, x), (-2x, 2x + 1) : x \in \mathbb{R}\}$$

and let $T : A \to B$ be defined by

$$Tx = \begin{cases} -1 & \text{if } x = -2;\\ x - 2 & \text{if } x \in [2, 3].\end{cases}$$

Then T is proximally edge-preserving.

Example 2.2. Let $X = \mathbb{R}^2$ with the Euclidean metric d on X. Let $A = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1 \text{ and } y \ge 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1 \text{ and } y \le -1\}$. Obviously, $D(A, B) = 2$ and then $A_0 = \{(x, 1) \in \mathbb{R} : -1 \le x \le 1\}$, $B_0 = \{(x, -1) \in \mathbb{R} : -1 \le x \le 1\}$. Let $G = (\mathbb{R}^2, E(G))$ be a directed graph defined by

$$E(G) = \{((x, y), (x', y')) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x - x'| \le \frac{1}{2} \text{ and } |y - y'| \le \frac{1}{2}\},$$

and let $T : A \to B$ be defined by

$$T(x, y) = (x, -y), \text{ for all } (x, y) \in A.$$

Let $(x, y) \in E(G)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is easy to see that if $x \ne (x_1, 1)$ or $y \ne (y_1, 1)$, then there are no $u_1, u_2 \in A$ such that $d(u_1, Tx) = D(A, B)$ and $d(u_2, Ty) = D(A, B)$.
Now, if $x = (x_1, 1)$ and $y = (y_1, 1)$, then $Tx = (x_1, -1)$ and $Ty = (y_1, -1)$. We see that $u_1 = (x_1, 1)$ and $u_2 = (y_1, 1)$ are in A such that $d(u_1, Tx) = d(u_2, Ty) = D(A, B)$, and by the definition of $E(G)$ we have $(u_1, u_2) \in E(G)$. Therefore, T is proximally edge-preserving.

Definition 2.10. Let (X, d) be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $T : A \to B$ be a non-self mapping. Then T is said to be a *G-proximal weak contraction* if there exist $L \ge 0$ and a function $\varphi : [0, \infty) \to [0, 1)$ with $\limsup_{r \to t^+} \varphi(r) < 1$ for all $t \in [0, \infty)$ such that

$$\left.\begin{array}{l}(x, y) \in E(G)\\ d(u_1, Tx) = D(A, B)\\ d(u_2, Ty) = D(A, B)\end{array}\right\} \Longrightarrow d(u_1, u_2) \le \varphi(d(x, y))d(x, y) + Ld(y, u_1) \tag{2.3}$$

where $x, y, u_1, u_2 \in A$.

For a self-mapping, we see that the G-proximal weak contraction reduces to the following contraction:

$$(x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx). \tag{2.4}$$

If $(X, d, \preceq)$ is a partially ordered metric space and $G = (V(G), E(G))$ is a directed graph generated by $\preceq$, i.e., $E(G) = \{(x, y) : x \preceq y\}$, then the G-proximal weak contraction reduces to the following ordered proximal weak contraction which is more general than that of Basha [5]:

$$\left.\begin{array}{l} x \preceq y \\ d(u_1, Tx) = D(A, B) \\ d(u_2, Ty) = D(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq \varphi(d(x, y))d(x, y) + Ld(y, u_1). \tag{2.5}$$

Example 2.3. Let X, G, A, B and T be as in Example 2.2. Let $x, y, u_1, u_2 \in A$, if $(x, y) \in E(G)$, $d(u_1, Tx) = d(u_2, Ty) = D(A, B)$, then x, y, u_1, u_2 are in the following forms $x = (x_1, 1) = u_1$ and $y = (y_1, 1) = u_2$. Choose $L > 0$ and let $k \in (0, 1)$ such that $k + L \geq 1$. Let $\varphi(t) = k$, for all $t \in [0, \infty)$. We get

$$\begin{aligned} d(u_1, u_2) = |x_1 - x_2| &\leq k|x_1 - x_2| + L|x_1 - x_2| \\ &= \varphi(d(x, y))d(x, y) + L|x_1 - x_2|. \end{aligned}$$

Hence T is the G-proximal weak contraction. Moreover, it is clear that T does not satisfy the inequality (2.3) with $L = 0$.

From Definition 2.8, 2.9, 2.10, in the sense of a partially ordered set, if $G = (V(G), E(G))$ is a directed graph such that $V(G) = X$ and $E(G) = \{(x, y) : x \preceq y\}$, then it is easy to observe that the following statements hold.

(1) Edge-preserving reduces to order-preserving.
(2) Proximally edge-preserving reduces to proximally order-preserving.
(3) If $L = 0$ and we take $\varphi(t) = k$ where $k \in [0, 1)$ for all $t \in [0, \infty)$, then the G-proximal weak contraction (2.3) reduces to an ordered proximal contraction.

Definition 2.11. Let (X, d) be a metric space, A and B be two nonempty subsets of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Let $T : A \to B$ be a non-self mapping. Then T is said to be *proximally weak edge-preserving* if for every $x \in A$, there exists $y \in A$ such that $d(y, Tx) = D(A, B)$ and $(x, y) \in E(G)$.

Example 2.4. Let $X = \mathbb{R}^2$ with the Euclidean metric d on $\mathbb{R}^2$. Let $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, y) : y \in [0, \infty)\}$. Obviously, $D(A, B) = 1$. Let $G = (\mathbb{R}^2, E(G))$ be a directed graph defined by

$$E(G) = \{((0, x), (0, y)) : x, y \in \mathbb{R} \text{ and } |x - y| \leq \frac{1}{2}\},$$

and let $T : A \to B$ be defined by

$$T(0, x) = (1, \frac{x}{2}), \text{ for all } (0, x) \in A.$$

Clearly, for $(0, x) \in A$, we have $(0, \frac{x}{2}) \in A$ such that $d((0, \frac{x}{2}), T(0, x)) = d((0, \frac{x}{2}), (1, \frac{x}{2})) = 1 = D(A, B)$ and $((0, x), (0, \frac{x}{2})) \in E(G)$. Therefore, T is proximally weak edge-preserving.

3. Main results

We first prove a best proximity point theorem for proximally edge-preserving, continuous and a G-proximal weak contraction in a complete metric space endowed with a directed graph.

Theorem 3.4. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$, and let A and B be nonempty closed subsets of X such that $A_0 \neq \emptyset$. If $T : A \to B$ is a mapping satisfying the following properties:*

(i) *T is proximally edge-preserving, continuous and G-proximal weak contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = D(A, B) \text{ and } (x_0, x_1) \in E(G),$$

then there exists an element $x \in A$ such that $d(x, Tx) = D(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \in \mathbb{N},$$

converges to the element x.

Proof. From (ii), there exist elements $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = D(A, B) \text{ and } (x_0, x_1) \in E(G).$$

Since $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = D(A, B).$$

Since T is proximally edge-preserving, we have $(x_1, x_2) \in E(G)$. Continuing this process, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = D(A, B) \text{ with } (x_n, x_{n+1}) \in E(G), \text{ for all } n \geq 0. \tag{3.6}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $d(x_{n_0+1}, Tx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = D(A, B)$. Hence x_{n_0} is a best proximity point of T.

Suppose that $x_n \neq x_{n+1}$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Next, let us show that $\{x_n\}$ is a Cauchy sequence and its limit is the best proximity point of T.

As T is the G-proximal weak contraction and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$,

$$d(x_n, Tx_{n-1}) = D(A, B),$$

$$d(x_{n+1}, Tx_n) = D(A, B),$$

we have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1}))d(x_n, x_{n-1}) + Ld(x_n, x_n) \leq d(x_n, x_{n-1}).$$

By the above inequality, we get the sequence $\{d(x_{n+1}, x_n)\}$ is a non-increasing and bounded below. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{3.7}$$

Let $\varepsilon > 0$. Since $\limsup_{s\to r^+} \varphi(s) < 1$, there exists $a \in [0, 1)$ such that $\varphi(s) \leq a$ for all $s \in (r, r + \varepsilon)$.

By (3.7), there exists $N \in \mathbb{N}$ such that

$$r \leq d(x_{n+1}, x_n) < r + \varepsilon, \text{ for all } n \geq N.$$

Thus for all $n > N$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \varphi(d(x_n, x_{n-1}))d(x_n, x_{n-1}) + Ld(x_n, x_n) \\ &= \varphi(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \leq ad(x_n, x_{n-1}) \\ &\leq \varphi(d(x_{n-1}, x_{n-2}))d(x_{n-1}, x_{n-2}) + Ld(x_{n-1}, x_{n-1}) \\ &= \varphi(d(x_{n-1}, x_{n-2}))d(x_{n-1}, x_{n-2}) \leq a^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq a^{n-N} d(x_{N+1}, x_N). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_{n+1}, x_n) &= \sum_{n=1}^{N} d(x_{n+1}, x_n) + \sum_{n=N+1}^{\infty} d(x_{n+1}, x_n) \\ &\leq \sum_{n=1}^{N} d(x_{n+1}, x_n) + \sum_{n=N+1}^{\infty} a^{n-N} d(x_{N+1}, x_N) \\ &\leq \sum_{n=1}^{N} d(x_{n+1}, x_n) + d(x_{N+1}, x_N) \sum_{n=N+1}^{\infty} a^{n-N} < \infty. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in A.
Since A is complete, there exists $x \in A$ such that $x_n \to x$. By the continuity of T, it implies that $Tx_n \to Tx$. Hence the continuity of the metric function d implies that $d(x_{n+1}, Tx_n)$ converges to $d(x, Tx)$. By (3.6), we get

$$d(x, Tx) = D(A, B).$$

Hence $x \in A$ is a best proximity point of T. This completes the proof. □

We give example to illustrating Theorem 3.4

Example 3.5. *Let $X = \mathbb{R}^2$ with the Euclidean metric d on X. Let $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1$ and $y \geq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1$ and $y \leq -1\}$. It is easy to see that $D(A, B) = 2$ and then $A_0 = \{(x, 1) \in \mathbb{R} : -1 \leq x \leq 1\}$, $B_0 = \{(x, -1) \in \mathbb{R} : -1 \leq x \leq 1\}$. Let $G = (\mathbb{R}^2, E(G))$ be a directed graph defined by*

$$E(G) = \{((x, y), (x', y')) \in \mathbb{R}^2 \ : \ |x - x'| \leq \frac{1}{2} \text{ and } |y - y'| \leq \frac{1}{2}\},$$

and let $T : A \to B$ be defined by

$$T(x, y) = (x, -y), \text{ for all } (x, y) \in A.$$

By Example 2.2 and 2.3, we have T is both proximally edge-preserving and the G-proximal weak contraction. Clearly, T is continuous, $T(A_0) \subseteq B_0$, $(0, 1) \in A_0$, $d((0, 1), T(0, 1)) = D(A, B)$ and $((0, 1), (0, 1)) \in E(G)$. Hence this example satisfies all conditions of Theorem 3.4 and we see that every $x \in A_0$ is a best proximity point of T. But we see that this example does not satisfy Theorem 1.3, in the case $L = 0$.

If we take $A = B = X$ in Theorem 3.4, then we obtain the following corollary.

Corollary 3.1. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. If $T : X \to X$ is a mapping satisfying the following properties:*

(i) T is an edge-preserving mapping and continuous;
(ii) there exist $x_0 \in X$ such that $(x_0, T(x_0)) \in E(G)$;

(iii) *there exists* $L \geq 0$ *and a function* $\varphi : [0, \infty) \to [0, 1)$ *satisfies* $\limsup_{r \to t^+} \varphi(r) < 1$, *for all* $t \in [0, \infty)$ *such that*

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$,

then T *has a fixed point.*

The following property is useful for our main results.

Property (A) [23] For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$, for $n \in \mathbb{N}$.

Similarly, we can establish the following result which is an analogue of Theorem 3.4.

Theorem 3.5. *Let* X, G, A, B *be as in Theorem 3.4 such that* X *has property (A). Suppose that* A_0 *is closed in* X *and let* $T : A \to B$ *be a mapping satisfying the following properties:*

(i) T *is proximally edge-preserving and G-proximal weak contraction such that* $T(A_0) \subseteq B_0$*;*

(ii) *there exist* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = D(A, B) \text{ and } (x_0, x_1) \in E(G),$$

then there exists an element $x \in A$ *such that* $d(x, Tx) = D(A, B)$*. Further, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \in \mathbb{N},$$

converges to the element x*.*

Proof. Following the proof of Theorem 3.4, there exists a sequence $\{x_n\}$ in A satisfying the following condition

$$d(x_{n+1}, Tx_n) = D(A, B) \text{ with } (x_n, x_{n+1}) \in E(G), \text{ for all } n \geq 0.$$

and $\{x_n\}$ converges to x in A. Note that the sequence $\{x_n\}$ in A_0 and A_0 is closed. Therefore, $x \in A_0$. Since $T(A_0) \subseteq B_0$, we get $Tx \in B_0$. Hence there exists $z \in A$ such that

$$d(z, Tx) = D(A, B). \tag{3.8}$$

Since $(x_n, x_{n+1}) \in E(G)$ and $x_n \to x$ as $n \to \infty$, by the assumption, we get that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$, for all $n \in \mathbb{N}$. Hence

$$d(x_{n_k+1}, z) \leq \varphi(d(x_{n_k}, x))d(x_{n_k}, x) + Ld(x, x_{n_{k+1}}).$$

Since $x_n \to x$ as $n \to \infty$, we have $d(x_{n_k+1}, z) \to 0$ as $k \to \infty$. This means that $x = z$. By the equality (3.8), we obtain $d(x, Tx) = D(A, B)$, that is, $x \in A$ is a best proximity point of T. □

Theorem 3.6. *Let* (X, d) *be a complete metric space,* $G = (V(G), E(G))$ *a directed graph such that* $V(G) = X$*, and let* A *and* B *be nonempty closed subsets of* X*. If* $T : A \to B$ *is proximally weak edge-preserving, continuous and G-proximal weak contraction, then there exists an element* $x \in A$ *such that* $d(x, Tx) = D(A, B)$*. Further, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \in \mathbb{N},$$

converges to the element $x \in A$*.*

Proof. Since $A_0 \neq \emptyset$, there exist $x_0 \in A$. From T is proximally weak edge-preserving, there exists $x_1 \in A$ such that

$$d(x_1, Tx_0) = D(A, B) \text{ and } (x_0, x_1) \in E(G).$$

Since $x_1 \in A$, again by T is proximally weak edge-preserving, there exists $x_1 \in A$ such that

$$d(x_2, Tx_1) = D(A, B) \text{ and } (x_1, x_2) \in E(G).$$

Continuing this process, we obtain a sequence $\{x_n\}$ in A such that

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \geq 0 \text{ with } (x_n, x_{n+1}) \in E(G). \tag{3.9}$$

Now, following the proof of Theorem 3.4, we get the required result. □

If we take $A = B = X$ in Theorem 3.6, then we obtain the following corollary.

Corollary 3.2. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. If $T : X \to X$ is a mapping satisfying the following properties:*

- (i) *T is continuous and $(x, Tx) \in E(G)$, for all $x \in X$;*
- (ii) *there exist $L \geq 0$ and a function $\varphi : [0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} \varphi(r) < 1$, for all $t \in [0, \infty)$ such that*

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$ with $(x, y) \in E(G)$,

Then T has a fixed point, i.e., there exists $x \in X$ such that $Tx = x$.

Remark 3.1. In Corollary 3.2, if we set $G = (V(G), E(G))$ as a complete graph on X, i.e., $E(G) = X \times X$, then we obtain the result of Berinde [7].

4. Some best proximity theorems in partially ordered metric spaces

Over years ago, there have been many important developments for the existence of best proximity point theorems in a complete metric space endowed with a partially order. In this section, by applying the main results obtained in this paper, we can deduce various best proximity point theorems in a complete metric space endowed with a partially order.

Theorem 4.7. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let A and B be nonempty closed subsets of X such that $A_0 \neq \emptyset$. If $T : A \to B$ is a mapping satisfying the following properties:*

- (i) *T is continuous and proximally order-preserving such that $T(A_0) \subseteq B_0$;*
- (ii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = D(A, B)$ and $x_0 \preceq x_1$;*
- (iii) *there exist $L \geq 0$ a function $\varphi : [0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} \varphi(r) < 1$, for all $t \in [0, \infty)$ such that*

$$\left.\begin{array}{l} x \preceq y \\ d(u_1, Tx) = D(A, B) \\ d(u_2, Ty) = D(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq \varphi(d(x, y))d(x, y) + Ld(y, u_1) \tag{4.10}$$

for all $x, y, u_1, u_2 \in A$,

then there exists an element $x \in A$ such that $d(x, Tx) = D(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \in \mathbb{N},$$

converges to the element x.

Proof. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : x \preceq y\}.$$

Then T is proximally edge-preserving. By (iii), it is easy to see that T is a G-proximal weak contraction. Hence all assumptions of Theorem 3.4 are satisfied. Next, we see that

(ii) of Theorem 3.4 is also satisfied. Therefore, all conditions of Theorem 3.4 are satisfied. As a result of this, we obtain this corollary directly by Theorem 3.4. □

If $L = 0$, the following result is obtained directly by Theorem 4.7 and it is noted that this result was given by Basha [5].

Corollary 4.3. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let A and B be nonempty closed subsets of X such that $A_0 \neq \emptyset$. If $T : A \to B$ is a mapping satisfying the following properties:*

(i) *T is continuous and proximally order-preserving, and an ordered proximal contraction such that $T(A_0) \subseteq B_0$;*

(ii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = D(A, B)$ and $x_0 \preceq x_1$,*

then there exists an element $x \in A$ such that $d(x, Tx) = D(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = D(A, B), \text{ for all } n \in \mathbb{N},$$

converges to the element x.

If T is a self-mapping on X, then the following result is obtained directly by Corollary 4.3.

Corollary 4.4. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$. If $T : X \to X$ is a mapping satisfying the following properties:*

(i) *T is order-preserving, continuous which satisfies the inequality* (2.4)*;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$,*

then T has a fixed point.

Acknowledgement. The authors would like to thank the referees for many comments and suggestions to improve the exposition of this paper and the Thailand Research Fund under the project RTA5780007 and Chiang Mai University, Chiang Mai, Thailand for the financial support.

References

[1] Abkar, A. and Gabeleh, M., *Generalized cyclic contractions in partially ordered metric spaces*, Optim Lett., In Press DOI DOI 10.1007/s11590-011-0379-y

[2] Agarwal, R. P., El-Gebeily, M. A. and O'Regan, D., *Generalized contractions in partially ordered metric spaces*, Appl. Anal., **87** (2008), 1–8

[3] Al-Thagafi, M. A., and Shahzad, N., *Convergence and existence results for best proximity points*, Nonlinear Analysis, **70** (2009), 3665–3671

[4] Banach, S., *Sur les operations dans les ensembles abstraits et leur application aux equations itegrales*, Fundam. Math., **3** (1922), 133–181

[5] Basha, S. S., *Best proximity point theorems on partially ordered sets*, Optim. Lett., **7** (2013), 1035–1043

[6] Beg, I. and Butt, A. R., *Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal., **71** (2009), 3699–3704

[7] Berinde, V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **1** (2004), No. 9, 43–53

[8] Berinde, V., *Approximating common fixed points of noncommuting almost contractions in metric spaces*, Fixed Point Theory, **11** (2010), No 2, 179–188

[9] Berinde, V., *Approximating fixed points of implicit almost contractions*, Hacet. J. Math. Stat., **41** (2012), No. 1, 93–102

[10] Berinde, V. and Păcurar, M., *Fixed points and continuity of almost contractions*, Fixed Point Theory, **9** (2008), No. 1, 23–34

[11] Berinde, V. and Păcurar, M., *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory, **14** (2013), No. 2, 301–311

[12] Bhaskar, T. G. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. TMA, **65** (2006), 1379–1393

[13] Bojor, F., *Fixed point theorems for Reich type contraction on metric spaces with a graph*, Nonlinear Anal., **75** (2012), 3895–3901

[14] Branciari, A., *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, J. Math. Anal. Appl., **29** (2002), 531–536

[15] Chatterjea, S. K., *Fixed point theorems*, C. R. Acad. Bulgare Sci., **25** (1972), 727–730

[16] Derafshpour, M., Rezapour, S. and Shahzad, N., *Best Proximity Points of cyclic ϕ-contractions in ordered metric spaces*, Topological Methods in Nonlinear Analysis, **37** (2011), 193–202

[17] Di Bari, C., Suzuki, T. and Vetro, C., *Best proximity points for cyclic Meir- Keeler contractions*, Nonlinear Anal., **69** (2008), 3790–3794

[18] Eldred, A. and Veeramani, P., *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006

[19] Espinola, R. and Fernandez-Leon, A., *On best proximity points in metric and Banach spaces*, Canad. J. Math., **63** (2011), 533–550

[20] Fan, K., *Extensions of two fixed point theorems of F. E. Browder*, Math. Z., **122** (1969), 234–240

[21] Gabeleh, M., *Global optimal solutions of non-self mappings*, Politehn. Univ. Bucharest Sci. Bull., Ser. A, Appl. Math. Phys., **75** (2013), No. 3, 67–74

[22] Giannessi, F., Maugeri, A. and Pardalos, P. M., *Equilibrium Problems: nonsmooth optimization and variational inequality models*, Nonconvex Optim. Appl., **58** (2004), DOI 10.1007/b101835

[23] Jachymski, J., *The contraction principle for mappings on a metric with a graph*, Proc. Am. Math. Soc., **139** (2008), 1359–1373

[24] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc., **10** (1968), 71–76

[25] Nieto, J. J. and Rodriguez-Lopez, R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order **22** (2005), 223–239

[26] Nieto, J. J. and Rodriguez-Lopez, R., *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation*, Acta Math. Sin. (Engl.Ser.), **23** (2007), No. 12, 2205–2212

[27] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. math. Soc., **132** (2004), 1435–1443

[28] Suzuki, T., Kikkawa, M. and Vetro, C., *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918–2926

[29] Zamfirescu, T., *Fixed point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292–298

CENTER OF EXCELLENCE IN MATHEMATICS AND APPLIED MATHEMATICS
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CHIANG MAI UNIVERSITY
CHIANG MAI 50200 THAILAND
E-mail address: chalongchai1001@gmail.com
E-mail address: suthep.s@cmu.ac.th