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Multiple existence of solutions for a coupled system involving the distributional Henstock-Kurzweil integral

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ABSTRACT. This paper deals with a coupled system in the sense of distributions (generalized functions). Our main goal is to get the basic multiple existence results via some degree theory arguments. Differently from the literatures, the proof is based on the concept of a general integral named distributional Henstock-Kurzweil integral, which includes the Lebesgue and Henstock-Kurzweil integrals as special cases. Finally, an example is given to illustrate that the presented abstract theory contains some previous results as special cases.

1. INTRODUCTION

The purpose of this paper is to establish the existence of non-zero solutions for a coupled system

$$\begin{cases} -u'' = f_1 + h_1(x,u,v), & x \in (0,1), \\ -v'' = f_2 + h_2(x,u,v), & x \in (0,1), \\ u(0) = u(1) = 0, \quad v(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $u, v : [0,1] \to \mathbb{R}$ and the derivatives are understood in the sense of L. Schwartz's distributions, f_i is a distribution and $h_i : [0,1] \times C[0,1] \times C[0,1] \to \mathbb{R}$ is a function, $i = 1, 2$.

In ODE, the problem of the existence of solutions for a coupled system of the type

$$\begin{cases} -\frac{d^2u}{dx^2} = h_1(x,u,v), & x \in (0,1), \\ -\frac{d^2v}{dx^2} = h_2(x,u,v), & x \in (0,1), \end{cases} \tag{1.2}$$

has received an extensive attention by researchers, see, for example, Liu and Sun [11], Cheng and Zhong [3], Cheng and Zhang [4], Cheng [5], Yang [21], Asif and Khan [2], Infante and Pietramala [9]. Likewise, many papers are concerned with the equivalent integral system which is known as the nonlinear Hammerstein system

$$\begin{cases} u(x) = \displaystyle\int_0^1 k_1(x,y)h_1(y,u(y),v(y))dy, & x \in [0,1], \\ v(x) = \displaystyle\int_0^1 k_2(x,y)h_2(y,u(y),v(y))dy, & x \in [0,1], \end{cases} \tag{1.3}$$

where $k_i(x,y)$ $(i = 1,2)$ is the kernel function, see, e. g., [16, 11, 4, 19, 20] and references therein. In almost all the aforementioned papers, f_i is required to be continuous, and the solutions are in $C^2[0,1]$.

However, continuous functions do not generally have pointwise derivatives, for example, the Riemann function

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi x}{n^2} \tag{1.4}$$

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does not have pointwise derivative, see details in [8]. Since the distributional derivative includes the pointwise one, several papers started to study the distributional differential equations, we refer the reader to [12, 13, 14, 18]. Recently, a new integral based on the concept of distributions was studied in [1, 17, 22]. Precisely, a distribution is called distributionally Henstock-Kurzweil integrable if it is a distributional derivative of a continuous function. With this definition, the distributional Henstock-Kurzweil (D_{HK} for short) integral contains the Henstock-Kurzweil (HK for short), Lebesgue and Riemann integrals as special cases. Further, many basic properties of usual integrals still hold in this case, see details in [1, 10, 15, 17, 22].

Motivated by the above discussion, in this paper, we aim to study the existence results of (1.1). Firstly, we assume that f and h satisfy the following assumptions:

(D_1) for every $u, v \in C[0,1]$, $h_i(., u(.), v(.))$ is HK integrable on $[0,1]$;

(D_2) for all $x \in [0,1]$, $h_i(x,u,v)$ $(i=1,2)$ is continuous with respect to (u,v);

(D_3) there exist constant $R_0 > 0$ and HK integrable functions $l_i(x), q_i(x) \geq 0$ $(i=1,2)$ on $[0,1]$ such that

$$-l_i(|u|+|v|) - q_i \leq h_i(., u, v) \leq l_i(|u|+|v|) + q_i, \quad \text{for every } u, v \in B_{R_0},\ i=1,2,$$

on $[0,1]$, where $B_{R_0} = \{u \in C^1[0,1] \mid \|u\| \leq R_0\}$;

(D_4) $f_i \in D_{HK}$, $i = 1,2$.

Definition 1.1. (u,v) is said to be a solution of the differential system (1.1) on $[0,1]$ if (u,v) satisfies the following integral system

$$\left\{ \begin{array}{ll} u(x) = \displaystyle\int_0^1 k_1(x,y)(f_1(y) + h_1(y,u(y),v(y)))dy, & x \in [0,1], \\ v(x) = \displaystyle\int_0^1 k_2(x,y)(f_2(y) + h_2(y,u(y),v(y)))dy, & x \in [0,1], \end{array} \right. \tag{1.5}$$

where

$$k_1(x,y) = \left\{ \begin{array}{ll} x(1-y), & 0 \leq x \leq y \leq 1, \\ y(1-x), & 0 \leq y < x \leq 1, \end{array} \right. \qquad k_2(x,y) = \left\{ \begin{array}{ll} x, & 0 \leq x \leq y \leq 1, \\ y, & 0 \leq y < x \leq 1. \end{array} \right. \tag{1.6}$$

Let

$$F_i(x) = \int_0^x f_i(y)dy, \quad L_i(x) = \int_0^x l_i(y)dy, \quad Q_i(x) = \int_0^x q_i(y)dy, \quad x \in [0,1]. \tag{1.7}$$

Obviously, F_i, L_i, Q_i $(i=1,2)$ are all continuous on $[0,1]$.

The main results of this paper are as follows.

Theorem 1.1. *Let the hypotheses* $(D_1) - (D_4)$ *be fulfilled and* $\max\{\|L_1\|, \|L_2\|\} < \frac{1}{2}$. *Then the system (1.1) has at least one nontrivial continuous solution.*

Theorem 1.2. *If the assumptions in Theorem 1.1 are satisfied and if there exists* $R > 2R_0 + 1$ *such that for* $\|u\| + \|v\| = R$, *either*

$$h_1(., u, v) \geq \pi^2(u + 2\|F_1\|) \tag{1.8}$$

or

$$h_2(., u, v) \geq \frac{\pi^2}{4}(v + 2\|F_2\|) \tag{1.9}$$

holds on $[0,1]$. *Then the system (1.1) has at least two nontrivial continuous solutions.*

2. Main results

In this section, we first present two methods of computation for topological degree (see details in [7, 11, 16]), which will be used to prove Theorems 1.1 and 1.2.

We denote by Ω a bounded open set in a Banach space X and by $\overline{\Omega}$ its closure. θ denotes the zero element in X. Let $A : \overline{\Omega} \to X$ be a *condensing operator* (see definition in [7]). Actually, a completely continuous operator is condensing.

Lemma 2.1 ([7, Lemma 2.5.1]). *Let Ω be a bounded open set in real Banach space X with $\theta \in \Omega$, and $A : \overline{\Omega} \to X$ be condensing. If*

$$\xi x \neq Ax, \ \text{for every } x \in \partial\Omega, \ \xi \geq 1,$$

then

$$\deg(I - A, \Omega, \theta) = 1.$$

A nonempty convex closed set $P \subset X$ is called a *cone* **i)** If $x \in P$, $\lambda \geq 0$ then $\lambda x \in P$; **ii)** If $x \in P$, $-x \in P$ then $x = \theta$. A nonempty convex closed set $W \subset X$ is called a *wedge* **i)** if $x \in W$, $\lambda \geq 0$ then $\lambda x \in W$; **ii)** there exists $y \in W$, such that $-y \notin W$. Obviously, if P is a cone in X then it is also a wedge in X. For $u \in X$, let $W(u) = \{x \in X \mid x + u \in W\}$.

Let Y_j be a Banach space and P_j a cone in Y_j ($j = 1, 2, \ldots, n$). For convenience, the partial ordering in each Y_j, which is induced by P_j, is expressed by $\leq$.

Lemma 2.2 ([16, Corollary 1]). *Let $A : \overline{\Omega} \to X$ be a condensing operator which has no fixed point on $\partial\Omega$. Suppose that there exist linear operators $T : W \to W$ and $N_j : W \to P_j$ ($j = 1, 2, \ldots, n$), such that*

(I) *for each j, $1 \leq j \leq n$, $N_j T x = N_j x$, for each $x \in W$;*
(II) *$N_j u_0 \neq \theta$, for some $u_0 \in W \setminus \{\theta\}$, $j = 1, 2, \ldots, n$;*
(III) *$A(\partial\Omega) \subset W(u^*)$ for some $u^* \in W$;*
(IV) *for any $x \in \partial\Omega \cap W(u^*)$, there exists j_0 ($1 \leq j_0 \leq n$) which is independent of x , such that $N_{j_0} Ax \geq N_{j_0} Tx$.*

Then

$$\deg(I - A, \Omega, \theta) = 0.$$

Let $X = C^1[0,1] \times C^1[0,1]$ with norm $\|(u,v)\|_X = \|u\| + \|v\|$, and let $A_i : B_{R_0} \to C^1[0,1]$ $(i = 1, 2)$, $A : B_{R_0} \times B_{R_0} \to X$ satisfy

$$\begin{aligned} A_i(u,v)(x) &= \int_0^1 k_i(x,y) f_i(y, u(y), v(y)) dy, \quad i = 1, 2, \\ A(u,v)(x) &= (A_1(u,v)(x), A_2(u,v)(x)). \end{aligned} \tag{2.10}$$

We are now in the position to prove our main results.

Proof of Theorem 1.1. Let $R_0 > \max\left\{\frac{2\|F_1\|+\|Q_1\|}{1-2\|L_1\|}, \frac{2\|F_2\|+\|Q_2\|}{1-2\|L_2\|}\right\}$ be fixed. In order to explore Lemma 2.1, we first prove that the operator $A : B_{R_0} \times B_{R_0} \to X$ is completely continuous.

Obviously, $\|k_i(x,y)\| \leq 1$ on $[0,1] \times [0,1]$. By virtue of Assumption (D_3), for any $(u,v) \in B_{R_0} \times B_{R_0}$, one has

$$\begin{aligned} &\|A_1(u,v)(x)\| \leq \left\|\int_0^1 k_1(x,y) f_1(y) dy\right\| + \left\|\int_0^1 k_1(x,y) h_1(y, u(y), v(y)) dy\right\| \\ \leq &\left\|x \int_0^1 F_1(y) dy - \int_0^x F_1(y) dy\right\| + \left\|\int_0^1 k_1(x,y)(l_1(y)(|u(y)| + |v(y)|) + q_1(y)) dy\right\| \\ \leq &2\|F_1\| + 2R_0\|L_1\| + \|Q_1\| < R_0. \end{aligned}$$

Similarly, $\|A_1(u,v)(x)\| \leq R_0$. Therefore, $A(B_{R_0} \times B_{R_0}) \subset B_{R_0} \times B_{R_0}$ and

$$\|A(u,v)\|_X < 2R_0. \tag{2.11}$$

Furthermore, for all $x_1,\ x_2 \in [0,1]$, by (D_3), one has

$$\begin{aligned} |A_1(u,v)(x_1) - A_1(u,v)(x_2)| = \Big| &(x_1 - x_2) \int_0^1 F_1(y)dy - \int_{x_2}^{x_1} F_1(y)dy \\ &+(x_1 - x_2) \int_0^1 \int_0^t h_1(y,u(y),v(y))dydt - \int_{x_2}^{x_1} \int_0^t h_1(y,u(y),v(y))dydt \Big| \\ \leq &2(\|F_1\| + 2R_0\|L_1\| + \|Q_1\|)|x_1 - x_2|. \end{aligned} \tag{2.12}$$

Similarly,

$$|A_2(u,v)(x_1) - A_2(u,v)(x_2)| \leq 2(\|F_2\| + 2R_0\|L_2\| + \|Q_2\|)|x_1 - x_2|. \tag{2.13}$$

Since F_i, L_i, Q_i $(i = 1,2)$ are continuous and so are bounded on [0,1], it follows from (2.12) and (2.13) that $A_i(u,v)$ $(i = 1,2)$ is equiuniformly continuous on $[0,1]$ for all $(u,v) \in B_{R_0} \times B_{R_0}$.

We now show that $A : B_{R_0} \times B_{R_0} \to X$ is continuous.

Let $(u,v) \in B_{R_0} \times B_{R_0}$. Without loss of generality, we assume that $u_j \to u$, $v_j \to v$ in B_{R_0} as $j \to \infty$. According to the condition (D_2), $h_i(\cdot,u_j,v_j) \to h_i(\cdot,u,v)$ as $j \to \infty,\ i = 1,2$. Therefore, by (2.10) and the convergence Theorem 1 of [22], one has

$$\lim_{j\to\infty} \int_0^1 k_i(x,y)h_i(y,u_j(y),v_j(y))dy = \int_0^1 k_i(x,y)h_i(y,u(y),v(y))dy, \quad i = 1,2,$$

uniformly on $[0,1]$. Therefore, $\lim_{j\to\infty} A_i(u_j,v_j)(.) = A_i(u,v)(.)$ $(i = 1,2)$ on $[0,1]$, which implies that A_i is continuous and hence A is continuous on $[0,1]$. Therefore, according to the Arzelà-Ascoli theorem, $A : B_{R_0} \times B_{R_0} \to B_{R_0} \times B_{R_0}$ is completely continuous.

Finally, let $\Omega_0 = \{(u,v) \in X \mid \|(u,v)\|_X < 2R_0\}$, $\partial\Omega_0 = \{(u,v) \in X \mid \|(u,v)\|_X = 2R_0\}$. We now verify that for any $(u,v) \in \partial\Omega_0$, $\lambda \geq 1$,

$$A(u,v) \neq \lambda(u,v). \tag{2.14}$$

If not, there exist $(u_0,v_0) \in \partial\Omega_0$, $\lambda_0 \geq 1$ such that

$$A(u_0,v_0) = \lambda_0(u_0,v_0). \tag{2.15}$$

By (2.11) and (2.15), $2R_0 > \|A(u_0,v_0)\|_X = \lambda_0\|(u_0,v_0)\|_X = 2R_0\lambda_0$, which leads to a contradiction. Thus, (2.14) holds. So far, all the conditions in Lemma 2.1 are satisfied. Therefore,

$$\deg(I - A, \Omega_0, \theta) = 1. \tag{2.16}$$

Thus, by the solution property of topological degree (see [7, Theorem A.3.1]), A has at least one fixed point in Ω_0. The proof is therefore complete. □

If $\|L_1\| = \|L_2\| = 0$, then (D_3) is reduced to

(D_3') there exist constant $R_0 > 0$ and HK integrable function $q_i(x) \geq 0$ $(i = 1,2)$ on $[0,1]$ such that

$$-q_i \leq h_i(.,u,v) \leq q_i, \quad \text{for every } u,v \in B_{R_0},\ i = 1,2,$$

on $[0,1]$, where $B_{R_0} = \{u \in C^1[0,1] \mid \|u\| \leq R_0\}$.

Moreover, we have a direct consequence of Theorem 1.1.

Corollary 2.1. *Let the hypotheses (D_1), (D_2), (D_3') and (D_4) be fulfilled. Then the system (1.5) and, hence, the system (1.1) has at least one nontrivial continuous solution.*

Remark 2.1. The condition (D_3') together with (D_1) and (D_2) was first proposed by Chew and Flordeliza in 1991 [6], to deal with first order Cauchy problems.

To prove Theorem 1.2, we first let

$$K_i u = \int_0^1 k_i(x,y)u(y)dy, \quad i = 1,2.$$

One can calculate the spectral radius $r(K_i)$ of K_i $(i = 1,2)$ by a routine method, and has $r(K_1) = \frac{1}{\pi^2}$, $r(K_2) = \frac{4}{\pi^2}$. Let

$$g_1(x) = \sin \pi x, \quad g_2(x) = \sin \frac{\pi}{2} x, \quad x \in [0,1]. \tag{2.17}$$

It is easy to see that $g_i(x) \geq 0$, $g_i(x) \not\equiv 0$ on $[0,1]$, and satisfies

$$\int_0^1 k_i(x,y)g_i(y)dy = r(K_i)g_i(x), \quad x \in [0,1],\ i = 1,2. \tag{2.18}$$

Moreover, let

$$\alpha_1(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}, \\ 1-x, & \frac{1}{2} < x \leq 1, \end{cases} \qquad \alpha_2(x) = x, \quad 0 \leq x \leq 1, \tag{2.19}$$

one has

$$k_i(x,y) \geq \alpha_i(x) k_i(\tau, y), \qquad x, y, \tau \in [0,1],\ i = 1,2, \tag{2.20}$$

and hence, by (2.18),

$$g_i(x) \geq \delta_i k_i(\tau, x), \qquad x, \tau \in [0,1],\ i = 1,2, \tag{2.21}$$

where $\delta_i = r^{-1}(K_i) \int_0^1 \alpha_i(y) g_i(y) dy > 0,\ i = 1,2$.

Let $Y_1 = Y_2 = \mathbb{R}$, $P_1 = P_2 = [0,+\infty)$, $W = W_1 \times W_2$, where

$$W_i = \left\{ u \in C^1[0,1] \ \middle|\ u(x) \geq 0,\ \int_0^1 g_i(x)u(x)dx \geq r(K_i)\delta_i \|u\| \right\}, \quad i = 1,2.$$

Obviously, W is a cone in X.

We now prove the second main result.

Proof of Theorem 1.2. Let

$$T(u,v) = (r^{-1}(K_1)K_1 u, r^{-1}(K_2)K_2 v),$$
$$N_1(u,v) = \int_0^1 g_1(x)u(x)dx, \quad N_2(u,v) = \int_0^1 g_2(x)v(x)dx,$$

and $u^* = u_0 = (u_1, u_2)$, where

$$u_i(x) = 2\|F_i\| + 1 + \int_0^1 k_i(x,y)\Big(l_i(y)(|u(y)| + |v(y)|) + q_i(y) \Big) dy, \quad i = 1,2.$$

We now verify that the linear operator T maps W into W. In fact, for any given $(u,v) \in W$, one has $u \in W_1$. Obviously, $r^{-1}(K_1)(K_1 u)(x) \geq 0$, by (2.18) and (2.20), one has

$$\int_0^1 g_1(x) r^{-1}(K_1)(K_1 u)(x)dx = \int_0^1 g_1(y)u(y)dy \geq \delta_1 (K_1 u)(\tau), \quad \tau \in [0,1]. \tag{2.22}$$

Therefore, $\int_0^1 g_1(x) r^{-1}(K_1)(K_1 u)(x)dx \geq \delta_1 r(K_1) \|r^{-1}(K_1)K_1 u\|$, hence, $r^{-1}(K_1)K_1 u \in W_1$. Similarly, $r^{-1}(K_2)K_2 v \in W_2$, which implies $T(u,v) \in W$, i.e., $T(W) \subset W$.

In view of Lemma 2.2, we now divide the proof into four steps.

Step 1. For any $(u,v) \in W$, by (2.18), (2.22)

$$N_1T(u,v) = \int_0^1 g_1(x)r^{-1}(K_1)(K_1u)(x)dx = \int_0^1 g_1(y)u(y)dy = N_1(u,v). \tag{2.23}$$

Similarly, one has $N_2T(u,v) = N_2(u,v)$. Thus condition (I) in Lemma 2.2 is satisfied.

Step 2. It is easy to see that the linear operator N_i maps W into P_i $(i = 1, 2)$, $u_0 \in W \setminus \{\theta\}$. By virtue of (2.18), one has

$$N_iu_0 = \int_0^1 g_i(x)u_i(x)dx \geq \int_0^1 g_i(y)dy > 0.$$

Consequently, condition (II) in Lemma 2.2 is satisfied.

Step 3. For any $(u,v) \in X$, $A(u,v) + u^* = (A_1(u,v) + u_1, A_2(u,v) + u_2)$. One has

$$A_1(u,v)(x) + u_1(x) \geq 2\|F_1\| + 1 + x\int_0^1 F_1(y)dy - \int_0^x F_1(y)dy \geq 0. \tag{2.24}$$

Let $\mu(y) = f_1(y) + h_1(y,u(y),v(y)) + l_1(y)(|u(y)| + |v(y)|) + q_1(y)$. Then, by (2.18) and (2.20), one has

$$\begin{aligned}\int_0^1 g_1(x)(A_1(u,v)(x) + u_1(x))dx &= r(K_1)\int_0^1 g_1(y)\mu(y)dy + (2\|F_1\| + 1)\int_0^1 g_1(x)dx \\ &\geq r(K_1)\delta_1(A_1(u,v)(\tau) + u_1(\tau)), \quad \tau \in [0,1].\end{aligned}$$

This and (2.24) imply that

$$\int_0^1 g_1(x)(A_1(u,v)(x) + u_1(x))dx \geq r(K_1)\delta_1\|A_1(u,v) + u_1\|.$$

Hence, $A_1(u,v) + u_1 \in W_1$. Similarly, one has $A_2(u,v) + u_2 \in W_2$, thus $A(u,v) + u^* \in W$. This implies that condition (III) in Lemma 2.2 holds.

Step 4. Let $\Omega = \{(u,v) \in X \mid \|u\| + \|v\| < R\}$, $R > R_0 + 1$. Without loss of generality, we may assume that A has no fixed point on $\partial\Omega$.

For any $(u,v) \in \partial\Omega \cap W(u^*)$, then $\|u\| + \|v\| = R$ and $u + u_1 \in W_1$, $v + u_2 \in W_2$. It follows from (2.23), (D_3), (2.18) and (1.8) (similarly, if (1.9) holds) that

$$\begin{aligned}&N_1A(u,v) - N_1T(u,v) = N_1A(u,v) - N_1(u,v) \\ \geq &\int_0^1 g_1(x)\left(x\int_0^1 F_1(y)dy - \int_0^x F_1(y)dy + r(K_1)h(x,u(x),v(x)) - u\right)dx \\ \geq &0.\end{aligned}$$

Thus condition (IV) in Lemma 2.2 holds.

According to the arguments above and Lemma 2.2, one has

$$\deg(I - A, \Omega, \theta) = 0. \tag{2.25}$$

It follows from (2.25), (2.16) and the additivity property of topological degree (see [7, Theorem A.3.1]) that

$$\deg(I - A, \Omega \setminus \overline{\Omega_0}, \theta) = -1. \tag{2.26}$$

Thus, A has at least one fixed point in $\Omega \setminus \overline{\Omega_0}$. This and Theorem 1.1 imply that A has at least two fixed points in Ω. The proof is therefore complete. □

3. Illustrative example

Now, we give an example to illustrate that our results are more general.

Example 3.1. Consider

$$
\left\{\begin{array}{ll} -u'' = \frac{1}{\pi^2}u + \mathcal{R}', & x \in (0,1), \\ -v'' = \frac{4}{\pi^2}(u+|v|) + \left(\frac{\pi^4-16}{4\pi^2}|v| + \frac{\pi^4+48}{8\pi^2}\right)\frac{\|u\|+\|v\|-10}{60} + \frac{3}{2\pi^2}\mathcal{R}', & x \in (0,1), \\ u(0)=u(1)=0, \quad v(0)=v'(1)=0, & \end{array}\right. \tag{3.27}
$$

where $\mathcal{R}'$ is the distributional derivative of $\mathcal{R}$ as given in (1.4).

Obviously, (3.27) can be regarded as a special case of (1.1), where

$$
\begin{array}{ll} f_1 = \mathcal{R}', & h_1(x,u,v) = \frac{1}{\pi^2}u, \\ f_2 = \frac{3}{2\pi^2}\mathcal{R}', & h_2(x,u,v) = \frac{4}{\pi^2}(u+|v|) + \left(\frac{\pi^4-16}{4\pi^2}|v| + \frac{\pi^4+48}{8\pi^2}\right)\frac{\|u\|+\|v\|-10}{60}. \end{array}
$$

Let $l_1(x) = \frac{1}{\pi^2}$, $l_2(x) = \frac{4}{\pi^2}$, $q_1(x) = 0$, $q_2(x) = \frac{\pi^4+48}{48\pi^2}$ on [0,1], then

$$
\|F_1\| \le \frac{\pi^2}{6}, \quad \|L_1\| = \frac{1}{\pi^2}, \quad \|Q_1\| = 0, \quad \|F_2\| \le \frac{1}{4}, \quad \|L_2\| = \frac{4}{\pi^2}, \quad \|Q_2\| = \frac{\pi^4+48}{48\pi^2}.
$$

Hence,

$$
\frac{2\|F_1\|+\|Q_1\|}{1-2\|L_1\|} \le \frac{\pi^4}{3\pi^2-6} \approx 4.1260, \qquad \frac{2\|F_2\|+\|Q_2\|}{1-2\|L_2\|} \le \frac{\pi^4+24\pi^2+48}{48(\pi^2-8)} \approx 4.2595.
$$

So, for $\|u\| < 5$ and $\|v\| < 5$, one has

$$
|h_1(x,u,v)| \le \frac{1}{\pi^2}(|u|+|v|), \qquad |h_2(x,u,v)| \le \frac{4}{\pi^2}(|u|+|v|) + \frac{\pi^4+48}{48\pi^2}.
$$

On the other hand, if $\|u\| + \|v\| = 70 > 11$, then

$$
h_2(x,u,v) = \frac{\pi^2}{4}\left(|v| + \frac{1}{2}\right) + \frac{4}{\pi^2}\left(u + \frac{3}{2}\right).
$$

Further,

$$
\|u\| \le \sum_{n=1}^{+\infty} \frac{4\pi^3}{n^4\pi^4 - 1} < \frac{3}{2}.
$$

Thus, $h_2(x,u,v) \ge \frac{\pi^2}{4}\left(v + 2\|F_2\|\right)$. Hence, (3.27) satisfies all the assumptions in Theorem 1.2. Therefore, (3.27) has at least two solutions.

Remark 3.2. Since $\mathcal{R}'$ given in (1.4) is D_{HK} integrable but neither Henstock-Kurzweil nor Lebesgue integrable, (3.27) cannot be transformed into an integral system by using each one of the last two integrals. This implies that the D_{HK} integral is more powerful, and that Theorems 1.1 and 1.2 are more general.

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