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On an Aitken-Steffensen-Newton type method

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ABSTRACT. We consider an Aitken-Steffensen type method in which the nodes are controlled by Newton and two-step Newton iterations. We prove a local convergence result showing the q-convergence order 7 of the iterations. Under certain supplementary conditions, we obtain monotone convergence of the iterations, providing an alternative to the usual ball attraction theorems.

Numerical examples show that this method may, in some cases, have larger (possibly sided) convergence domains than other methods with similar convergence orders.

1. INTRODUCTION

We are interested in solving the nonlinear equation

$$f(x) = 0 \tag{1.1}$$

where $f : [a, b] \to \mathbb{R}$ is given. We consider two auxiliary functions $g_1, g_2 : [a, b] \to [a, b]$ such that the above equation is equivalent to the following ones

$$x - g_i(x) = 0, \quad i = 1, 2. \tag{1.2}$$

We assume

A) equation (1.1) has a solution $x^* \in]a, b[$.

Obviously, x^* is a fixed point of g_1 and g_2.

A well-known method for approximating x^* is the chord method:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_{n-1}, x_n; f]}, \quad n = 0, 1, ..., x_0, x_1 \in [a, b],$$

where $[x, y; f]$ denotes the first order divided difference of f at the nodes x, y.

If one of the nodes in the above formula is controlled by g_1, one obtains the Steffensen method, given by (see [1], [2], [4], [5], [8], [10], [11], [18], [22], [24], [25])

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f]}, \quad n = 0, 1, ..., x_0 \in [a, b].$$

The Aitken method is obtained if in the above relation $f(x)$ is replaced by $h(x) = x - g_2(x)$ (see [1], [14], [16]).

A more general Aitken-type method was studied in [16], [17], [21]:

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(x_n); f]}, \quad n = 0, 1, ..., x_0 \in [a, b].$$

Some even more general methods, called Aitken-Steffensen, have been studied in [17], [20]:

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(g_1(x_n)); f]}, \quad n = 0, 1, ..., x_0 \in [a, b].$$

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Considering the inverse interpolatory polynomial of degree 2, one can obtain methods similar to the above ones, studied in [19]–[20]. In this paper we consider again this polynomial. We assume that $J = f([a,b])$ and that $f : [a,b] \to J$ is bijective, so there exists the inverse of f, $f^{-1} : J \to [a,b]$. Obviously, $x^* = f^{-1}(0)$. Let $a_i \in [a,b]$ and $b_i = f(a_i)$, $i = 1,2,3$.

Denote by P the interpolation polynomial for f^{-1}, determined by the nodes b_1, b_2, b_3 and values a_1, a_2, a_3 (see, e.g., [19]–[20]):

$$P(y) = a_1 + [b_1, b_2; f^{-1}](y - b_1) + [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2). \tag{1.3}$$

The solution x^* can be approximated by

$$P(0) = a_1 - [b_1, b_2; f^{-1}]b_1 + [b_1, b_2, b_3; f^{-1}]b_1 b_2, \tag{1.4}$$

and taking into account that the divided differences of order 1 and 2 satisfy respectively the following relations:

$$\begin{aligned} \left[b_i, b_j; f^{-1}\right] &= \frac{1}{[a_i, a_j; f]}, \quad i,j = \overline{1,3}, i \neq j, \\ [b_1, b_2, b_3; f^{-1}] &= -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_2, a_3; f][a_1, a_3; f]}, \end{aligned}$$

relation (1.4) becomes

$$P(0) = a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f] f(a_1) f(a_2)}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}.$$

Given an approximation $x_n \in [a,b]$ for x^*, we take $a_1 = x_n$, $a_2 = g_1(x_n)$, $a_3 = g_2(x_n)$, and we obtain the next approximation by

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f]} - \frac{[x_n, g_1(x_n), g_2(x_n); f] f(x_n) f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(x_n); f][g_1(x_n), g_2(x_n); f]}. \tag{1.5}$$

In [19] we have studied the case when in (1.5) is taken $g_1(x) = x - \lambda f(x)$, $\lambda \in \mathbb{R}$ a fixed parameter, and $g_2(x) = g_1(g_1(x))$. We have obtained conditions assuring the monotone convergence of the sequences $(x_n)_{n \geq 0}$ and $(g_1(x_n))_{n \geq 0}$.

In this paper we consider the case when g_1 is given by the the Newton iteration

$$g_1(x) = x - \frac{f(x)}{f'(x)},$$

while g_2 is given by the two-step Newton iteration: $g_2(x) = g_1(g_1(x))$.

From (1.5) we therefore obtain the following iterations:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{[x_n, y_n; f]} - \frac{[x_n, y_n, z_n; f] f(x_n) f(y_n)}{[x_n, y_n; f][x_n, z_n; f][y_n, z_n; f]}, \quad n = 0,1,\ldots, x_0 \in [a,b] \end{aligned} \tag{1.6}$$

which we call Aitken-Steffensen-Newton iterations.

At this point we notice that the above formula has six equivalent writings, since the interpolation polynomial P is the same, no matter the order of the three interpolation nodes. We shall use this fact in the proof of Theorem 3.2.

In Section 2 we provide a local convergence result for these iterates, while in Section 3 we obtain monotone convergence of the iterates under certain supplementary conditions

(Fourier type, convexity) on f. The last section contains some numerical examples, which confirm the theory.

2. Local convergence of the iterations

We shall denote by I_{a_1,a_2,a_3} the smallest open interval determined by the numbers $a_1, a_2, a_3 \in [a,b]$ and by E_f the following expression

$$E_f(x) := 3f''(x)^2 - f'(x)f'''(x).$$

We obtain the following local convergence result.

Theorem 2.1. *Assume that hypotheses A) and*

B) $f \in C^3[a,b]$, and $f'(x) \neq 0, \forall x \in [a,b]$,

hold. Then the Aitken-Steffensen-Newton method converges locally to x^*, i.e., for any x_0 sufficiently close to x^*, the iterations $(x_n)_{n\geq 0}$ given by (1.6) are well defined and converge to x^*. Moreover, the following estimates hold:

$$x^* - x_{n+1} = \frac{E_f(\xi_n)f''(\theta_n)^3 f''(\mu_n)f'(\alpha_n)f'(\beta_n)f'(\gamma_n)}{96f'(\xi_n)^5 f'(x_n)^3 f'(y_n)}(x^* - x_n)^7, \tag{2.7}$$

with some $\xi_n \in I_{x^*,x_n,y_n,z_n}, \theta_n, \alpha_n \in I_{x^*,x_n}, \mu_n, \beta_n \in I_{x^*,y_n}, \gamma_n \in I_{x^*,z_n}$; this shows that the method attains convergence with q-order at least 7, with asymptotic constant given by

$$\mathcal{C} = \frac{E_f(x^*)f''(x^*)^4}{96f'(x^*)^6}.$$

Proof. We suppose for the moment that the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ remain in $[a,b]$.

The remainder of the interpolation polynomial P in (1.3) is known that satisfies

$$f^{-1}(y) - P(y) = [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3),$$

so we obtain

$$x^* - x_{n+1} = -[0, f(x_n), f(y_n), f(z_n); f^{-1}]f(x_n)f(y_n)f(z_n), \quad n = 0, 1, \ldots \tag{2.8}$$

The mean value formula for divided differences yields:

$$[0, f(x_n), f(y_n), f(z_n); f^{-1}] = \frac{\left(f^{-1}(\eta_n)\right)'''}{6},$$

where $\eta_n \in I_{0,f(x_n),f(y_n),f(z_n)}$. Since f has derivatives up to order 3 on $[a,b]$ and $f'(x) \neq 0$, $\forall x \in [a,b]$, then f^{-1} is three times derivable and (see, e.g., [19])

$$(f^{-1}(\eta_n))''' = \frac{3f''(\xi_n)^2 - f'(\xi_n)f'''(\xi_n)}{f'(\xi_n)^5} = \frac{E_f(\xi_n)}{f'(\xi_n)^5}, \tag{2.9}$$

where $\xi_n = f^{-1}(\eta_n) \in [a,b]$. One can show that in fact $\xi_n \in I_{x^*,x_n,y_n,z_n}$.

Combining the above relations, (2.8) leads to

$$x^* - x_{n+1} = -\frac{E_f(\xi_n)}{6f'(\xi_n)^5}f(x_n)f(y_n)f(z_n), \quad n = 0, 1, \ldots \tag{2.10}$$

The Lagrange Theorem ensures the existence of $\alpha_n \in I_{x^*,x_n}, \beta_n \in I_{x^*,y_n}, \gamma_n \in I_{x^*,z_n}$ such that

$$f(x_n) = f'(\alpha_n)(x_n - x^*), \quad f(y_n) = f'(\beta_n)(y_n - x^*), \quad f(z_n) = f'(\gamma_n)(z_n - x^*),$$

and therefore the error (2.10) becomes

$$x^* - x_{n+1} = \frac{E_f(\xi_n)}{6f'(\xi_n)^5}f'(\alpha_n)f'(\beta_n)f'(\gamma_n)(x^* - x_n)(x^* - y_n)(x^* - z_n). \tag{2.11}$$

From the first and second relation in (1.6) it follows that there exists $\theta_n \in I_{x^*,x_n}$ and $\mu_n \in I_{x^*,y_n}$ such that

$$x^* - y_n = -\frac{f''(\theta_n)}{2f'(x_n)}(x^* - x_n)^2, \quad x^* - z_n = -\frac{f''(\mu_n)}{2f'(y_n)}(x^* - y_n)^2, \quad n = 0, 1, \ldots$$

These relations imply (2.7), as well as the rest of the statements.

It can be easily proved that the elements of all sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, $(z_n)_{n\geq 0}$, $(\mu_n)_{n\geq 0}$, $(\xi_n)_{n\geq 0}$, $(\theta_n)_{n\geq 0}$ are well defined if x_0 is chosen sufficiently close to x^*. □

The q-convergence order p of the method (see, e.g., [14] for definitions and properties) is at least 7, and since at each iteration step one computes $d = 5$ function evaluations $(f(x_n),\ f'(x_n),\ f(y_n),\ f'(y_n),\ f(z_n))$, the efficiency index of the method (see, e.g., [15] for definitions) is $E = p^{\frac{1}{d}} = 7^{\frac{1}{5}} \approx 1.47$. This value is greater than $2^{\frac{1}{2}} \approx 1.41$ and $3^{\frac{1}{3}} \approx 1.44$ which correspond to the Newton method, resp. the generalized Steffensen method (see [19]).

3. Monotone convergence

In order to study the monotone convergence of the method, we consider the Fourier condition:

C) the initial approximation $x_0 \in [a, b]$ verifies $f(x_0) \cdot f''(x_0) > 0$.

We obtain the following results.

Theorem 3.2. *If f satisfies assumptions A)–C) and, moreover*

i_1. $f'(x) > 0, \forall x \in [a, b]$;
ii_1. $f''(x) \geq 0, \forall x \in [a, b]$;
iii_1. $E_f(x) \geq 0, \forall x \in [a, b]$,
(1) then the elements of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ generated by (1.6) remain in $[a, b]$ and satisfy:
j_1. $x_n > y_n > z_n > x_{n+1} > x^*, n = 0, 1, \ldots,$
jj_1. $\lim x_n = \lim y_n = \lim z_n = x^*$.

Moreover, as soon as the iterates $(x_n)_{n\geq 0}$ become sufficiently close to the solution, they obey also the conclusions of Theorem 2.1.

Proof. By i_1) and B) it follows that the solution x^* is unique in $]a, b[$. Let $x_n \in [a, b]$ be an approximation for x^*, which satisfies C). From ii_1) we have $f(x_n) > 0$ and, by i_1), $x_n > x^*$. Relations A), i_1), ii_1) attract, by first relation in (1.6), that $y_n < x_n$ and $y_n > x^*$. A similar reasoning leads to $x^* < z_n < y_n$ and $f(z_n) > 0$. The last relation in (1.6), together with (2.8), (2.9), i_1) and iii_1) imply that $x_{n+1} > x^*$. The inequality $x_{n+1} < z_n$ is obtained from the equivalent writing of the interpolation polynomial, i.e.

$$x_{n+1} = z_n - \frac{f(z_n)}{[z_n, y_n; f]} - \frac{[z_n, y_n, x_n; f] f(z_n) f(y_n)}{[z_n, y_n; f][z_n, x_n; f][y_n, x_n; f]}.$$

By induction, we show j_1). It remains to show jj_1), which follows by passing to limit in the first relation in (1.6), and taking into account j_1). □

Remark 3.1. We notice that the above result allows a possibly larger convergence domain of the iterates, compared to conditions required by Theorem 2.1 (as is the case when we consider the Newton method and the Fourier condition). The same observation applies to the subsequent results.

Theorem 3.3. *If f obeys assumptions A)–C), iii_1) and, moreover,*

i_2. $f'(x) < 0, \forall x \in [a, b]$;

ii_2. $f''(x) \le 0, \forall x \in [a,b]$,
(1) then the same conclusions hold as in Theorem 3.2.

Proof. Instead of (1.1) we consider $h(x) = 0$, with $h = -f$, and we take into account that $E_f = E_{-f}$. □

In the case when f' and f'' have different signs, we obtain the following results.

Theorem 3.4. *If f satisfies $A)$–$C)$, $iii_1)$ and, moreover,*
i_3. $f'(x) < 0, \forall x \in [a,b]$;
ii_3. $f''(x) \ge 0, \forall x \in [a,b]$,
(1) then the elements of $(x_n)_{n\ge 0}$, $(y_n)_{n\ge 0}$, $(z_n)_{n\ge 0}$ remain in $[a,b]$ and obey
j_3. $x_n < y_n < z_n < x_{n+1} < x^*$, $n = 0, 1, ...$;
jj_3. $\lim x_n = \lim y_n = \lim z_n = x^*$.
Moreover, as soon as the iterates $(x_n)_{n\ge 0}$ become sufficiently close to the solution, they obey also the conclusions of Theorem 2.1.

The proof is similar to the proof of Theorem 3.2.

Theorem 3.5. *If f satisfies $A)$–$C)$, $iii_1)$ and*
i_4. $f'(x) > 0, \forall x \in [a,b]$;
ii_4. $f''(x) \le 0, \forall x \in [a,b]$,
(1) then the same conclusions hold as in Theorem 3.4.

The proof is obtained as in the proof of Theorem 3.3.

4. Numerical examples

We present some examples, solved using Matlab in double precision, and we compare the studied method to other methods. In order to obtain smaller tables, we used the `format short` command in Matlab, and for better legibility we used the `\numprint` LaTeX command and package. It is worth mentioning that such choice may lead to results that appear integers, while they are not (e.g., the value of y_4 in Table 4 is shown to be 2, while $f(y_4)$ should be 0); the explanation resides in the rounding made in the conversion process.

We shall consider the Aitken-Newton method introduced in [16]:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{[z_n, y_n; f]} - \frac{[z_n, y_n, y_n; f] f(z_n) f(y_n)}{[y_n, z_n; f]^2 f'(y_n)}, \quad n = 0, 1, \ldots \end{aligned} \tag{4.12}$$

It has the q-convergence order 8, the efficiency index $\sqrt[5]{8} \approx 1.51$, and similar monotone convergence of the iterates as obtained in Theorems 3.2–3.5. However, the numerical examples performed in double precision arithmetic show a slight better convergence of this method over the Aitken-Steffensen-Newton method studied in this paper.

Example 4.1. Consider the following equation (see, e.g., [6])

$$f(x) = e^x \sin x + \ln(x^2 + 1), \qquad x^* = 0.$$

The largest interval to study the monotone convergence of our method by Theorems 3.2–3.5 is $[a,b] := [x^*, 1.54\ldots]$, since f'' vanishes at b (being positive on $[a,b]$). The Fourier condition D) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$, while the derivatives f', f'' are positive on this interval. The conclusions of Theorem 3.2 apply.

The Aitken-Newton-Steffensen method leads to the following results, presented in Table 1.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	1.54	5.877 8	0.512 33	1.051 3	0.171 52	0.231 6
1	0.066 475	0.075 401	0.007 091 5	0.007 192 2	$9.802\,8 \cdot 10^{-05}$	$9.804\,7 \cdot 10^{-05}$
2	$2.934\,8 \cdot 10^{-07}$	$2.934\,8 \cdot 10^{-07}$	$1.722\,4 \cdot 10^{-13}$	$1.722\,4 \cdot 10^{-13}$	$8.898\,4 \cdot 10^{-26}$	$8.898\,4 \cdot 10^{-26}$

TABLE 1. Aitken-Newton-Steffensen iterates, $f(x) = e^x \sin x + \ln(x^2 + 1)$.

The Aitken-Newton method 4.12 leads to the results presented in Table 2.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	1.54	5.877 8	0.512 33	1.051 3	0.171 52	0.231 6
1	0.048 016	0.052 662	0.003 916 6	0.003 947 3	$3.024\,5 \cdot 10^{-05}$	$3.024\,6 \cdot 10^{-05}$
2	$3.482\,1 \cdot 10^{-09}$	$3.482\,1 \cdot 10^{-09}$	$3.637\,5 \cdot 10^{-17}$	$3.637\,5 \cdot 10^{-17}$	0	0

TABLE 2. Aitken-Newton iterates, $f(x) = e^x \sin x + \ln(x^2 + 1)$.

The convergence may be not very fast for initial approximations away from the solution.

It is worth noting that the method converges for $-0.3 \le x_0 < x_1^*$ too (local convergence near $x^* = 0$ assured by Theorem 2.1), despite the Fourier condition does not hold. For $x_0 = -0.3$ one obtains $y_0 = -2.4\ldots < 0$, then $z_0 = -0.14\ldots < 0$, $x_1 = 0.37\ldots$ and the rest of the iterates remain positive, converging monotonically to x_1^*. For $x_0 = -0.4$, the method converges to another solution of the equation, $x_2^* = -0.603\ldots$

The optimal method introduced by Cordero, Torregrosa and Vassileva in [9] has a smaller convergence domain to the right of x_1^*, since the iterates converge for $x_0 = 1.48$ ($x_4 = 1.374\,1 \cdot 10^{-32}$), while for $x_0 = 1.49$ the iterates jump over x_1^* and converge to x_2^*; in fact, the initial approximation 1.442 does not lead to convergence (see [16]).

The Kou-Wang method (formula (25) in [12]) converges for $x_0 = 1.48$ ($x_4 = 0$) and diverges for $x_0 = 1.49$ (see [16]).

Example 4.2. Consider the following equation (see, e.g., [23])

$$f(x) = (x-2)(x^{10} + x + 1)e^{-x-1}, \qquad x^* = 2.$$

The largest interval to study the monotone convergence of our method by Theorems 3.2–3.5 is $[a, b] := [x^*, 7.9\ldots]$, since f'' vanishes at b (being positive on $[a, b]$).

The Fourier condition D) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$, while both the derivatives f', f'' are positive on this interval. The conclusions of Theorem 3.2 apply.

It is interesting to note that in [23, Rem. 6] Petković observed that the methods studied there have a small convergence domain to the left of the solution: the choice of $x_0 = 1.8$ caused a bad convergence behavior of those iterates. We believe that this behavior may be explained by the fact that the derivative of f vanishes at $x = 1.78\ldots$

The Aitken-Newton method leads to the results presented in Table 3. The iterates converge even for $x_0 > 7.9$ (and to the left of the solution as well, but for x_0 higher than 1.81). Of course the convergence may be not very fast when the initial approximations are away from the solution.

In Table 4 we present the iterates generated by the Aitken-Newton method 4.12.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	7.9	761 907.133 4	5.602 8	148 982.786	4.661 5	44 837.664 1
1	4.207	20 996.709 9	3.660 6	6 787.212 6	3.232 1	2 226.165 8
2	2.978 3	1 005.759 1	2.682 4	331.268 7	2.443 9	107.821 4
3	2.303 8	47.056 6	2.153	14.005 4	2.054 7	3.465 5
4	2.017 1	0.934 7	2.0011	0.055 388	2	0.000 235 97
5	2	$1.0223 \cdot 10^{-07}$	2	0		

TABLE 3. Aitken-Newton iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	7.9	761 907.133 4	5.602 8	148 982.786	4.661 5	44 837.664 1
1	4.081 8	16 594.415 5	3.563 7	5 385.369 6	3.154 8	1 769.547 3
2	2.856 8	655.665	2.584 1	215.334 2	2.365 8	69.424 9
3	2.212 5	24.072 7	2.090 9	6.608 7	2.023 2	1.300 4
4	2.002 6	0.132 54	2	0.001 326 4	2	$1.3712 \cdot 10^{-07}$
5	2	$-1.1353 \cdot 10^{-14}$	2	0		

TABLE 4. Aitken-Newton iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

The optimal method introduced by Cordero, Torregrosa and Vassileva in [9] converges to x^* for $x_0 = 6.46$ and it does not for $x_0 = 6.47$, as shown in [16] .

The optimal method introduced by Liu and Wang (formula (18) in [13]) converges to $x^* = 2$ for $x_0 = 2.359$ (it needs 5 iterates) but for $x_0 = 2.36$ it converges to another solution, $x_1^* = 1512.626\ldots$. The results are presented in [16].

Among the optimal methods in [23] (the methods with convergence orders higher than 8 were corrected in a subsequent Corrigendum paper), the modified Ostrowski and Maheshwari methods behave very well for this example (we have studied the convergence only to the right of the solution). The modified Euler-like method has a small domain of convergence (in $\mathbb{R}$), since it converges to x^* for $x_0 = 2.15$, while for $x_0 = 2.16$ it generates square roots of negative numbers. Matlab has the feature of implicitly dealing with complex numbers, and the iterates finally converge (in $\mathbb{C}$) to the solution (see [16]).

Conclusions. The sufficient conditions for guaranteed convergence of the method studied by us may theoretically lead to larger convergence domains (especially sided convergence intervals) than from estimates of attraction balls, while a few examples shown that these domains are larger than those corresponding to some optimal methods of order 8. The performances of the studied method are comparable to those of the Aitken-Steffensen-Newton method studied in [16].

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