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Suzuki ψF-contractions and some fixed point results

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ABSTRACT. The purpose of this paper is to combine and extend some recent fixed point results of Suzuki, T., [*A new type of fixed point theorem in metric spaces*, Nonlinear Anal., **71** (2009), 5313–5317] and Secelean, N. A. & Wardowski, D., [*ψF-contractions: not necessarily nonexpansive Picard operators*, Results Math., **70** (2016), 415–431]. The continuity and the completeness conditions are replaced by orbitally continuity and orbitally completeness respectively. It is given an illustrative example of a Picard operator on a non complete metric space which is neither nonexpansive nor expansive and has a unique continuity point.

1. INTRODUCTION AND PRELIMINARIES

The Banach-Picard-Caccioppoli fixed point theorem, generally known as the Banach contraction principle, appeared in an explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution to an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. This principle states that, if T is a self mapping on a complete metric space (X, d) for which there exists a constant $\lambda \in (0,1)$ such that $\mathrm{d}(Tx,Ty) \leq \lambda \mathrm{d}(x,y)$, for all $x, y \in X$, then T has a unique fixed point which is approximated by the sequence $(T^n x)$, for each $x \in X$, where T^n means the n-th composition of T. The Banach contraction principle has been generalized in many ways over the years. For example, B. E. Rhoades [15] analyzed a wide variety of contractive self-mappings on a complete metric space which have a unique fixed point which can be obtained using Picard iteration and compared these classes of maps. The literature contains many works in fixed point theory which reflects an intense concern in many directions (see e.g. [18] and references therein).

Nevertheless, many of new fixed point theorems are significant and cover more and more general families of mappings, thereby they increase the new possibilities of applying the metric fixed point theory. Numerous direct applications of some fixed point results can be found in various modern scientific and technical areas, from Fractals Theory (see very recently [9]) to Graph Theory (see e.g. [1] and references therein and, also, recently, [4, 11]).

As usual, in this paper the symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all real numbers, all positive real numbers and all positive integers, respectively. We will also write $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

If $\nu, \lambda \in \overline{\mathbb{R}}_+$, by "$\nu > \lambda$" we understand $\nu > \lambda$ if $\lambda \in \mathbb{R}_+$ and $\nu = \infty$ otherwise.

Throughout this paper (X, d) will be a given metric space and T a self mapping on X. If $\varnothing \neq M \subset X$, we use the notation $\operatorname{diam} M = \sup_{x,y \in M} \mathrm{d}(x,y)$ for the diameter of M.

Definition 1.1. [18, 3.1.1] We say that T is:
(i) *contractive* if $\mathrm{d}(Tx,Ty) < \mathrm{d}(x,y)$, for all $x, y \in X$, $x \neq y$;

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(ii) *nonexpansive* if $\mathrm{d}(Tx,Ty) \le \mathrm{d}(x,y)$, for all $x,y \in X$; (iii) *expansive* if $\mathrm{d}(Tx,Ty) > \mathrm{d}(x,y)$, for all $x,y \in X$, $x \ne y$.

T is called *Picard operator* (abbreviated P. O.) if there is a unique $\xi \in X$ such that $T\xi = \xi$ and $T^n x \underset{n}{\longrightarrow} \xi$ for every $x \in X$. In this setting ξ is called the *fixed point* of T and $(T^n x)$ the sequence of *successively approximation* or also *Picard iteration* of T (the concepts of Picard operator and Picard approximation was introduced by I. A. Rus, see [17, 16]).

In 1962 M. Edelstein [5] proved the following version of the Banach contraction principle.

Theorem 1.1. [5, Rem. 3.1] *If (X, d) is a compact metric space and T is a contractive self-map on X, then T is a P.O.*

In 2008, T. Suzuki provided the following generalized version of the above theorem.

Theorem 1.2. [24, Th. 3] *Assume that (X, d) is a compact metric space and, for every $x, y \in X$, $x \ne y$, one has*

$$\frac{1}{2}\mathrm{d}(x,Tx) < \mathrm{d}(x,y) \;\Rightarrow\; \mathrm{d}(Tx,Ty) < \mathrm{d}(x,y).$$

Then T is a P.O.

In 2012 D. Wardowski [29] considered the class $\mathcal{F}$ of functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following three properties:

(F1) F is increasing,

(F2) $F(t) \to -\infty$ if and only if $t \searrow 0$ and

(F3) $\lim_{t\to 0} t^\lambda F(t) = 0$ for some $\lambda \in (0,1)$.

He defined a new kind of contractive self-mapping on (X, d) as follows: for some $F \in \mathcal{F}$, T is an *F-contraction* if there is $\tau > 0$ such that

$$F\big(\mathrm{d}(Tx,Ty)\big) + \tau \le F\big(\mathrm{d}(x,y)\big),\ \forall\, x,y \in X,\ Tx \ne Ty$$

and showed that every F-contraction on a complete metric space is a continuous P. O. In this way, Wardowski generalized the Banach contraction principle in a different manner from the well-known results in the literature.

In the last years, there is a current effort of many authors to extend the theory of F-contractions in order to obtain new classes of Picard mappings by relaxing the conditions (F1), (F2), (F3). It should be mentioned here that Turinici in [25] and N. A. Secelean in [19] proved the existence of fixed point of F-contractions without (F3).

Recently, N. A. Secelean and D. Wardowski [20] extended these results by introducing a new class of P. O. which strictly includes the family of F-contractions. More precisely, for every $\mu \in \overline{\mathbb{R}}_+$, the family Φ_μ of all increasing functions $\psi : (-\infty,\mu) \to (-\infty,\mu)$ such that $\psi^n(t) \to -\infty$, for every $t \in (-\infty,\mu)$, is considered. It is proved that, if $\psi \in \Phi_\mu$, then $\psi(t) < t$, for all $t \in (-\infty,\mu)$ and $\mu \in \overline{\mathbb{R}}_+$ and, conversely, if $\psi : (-\infty,\mu) \to (-\infty,\mu)$ is an increasing and continuous from the right function such that $\psi(t) < t$, for every $t \in (-\infty,\mu)$, then $\psi \in \Phi_\mu$.

Definition 1.2. [20] We say that $T : X \to X$ is a *ψF-contraction*, where $F : (0,\nu) \to \mathbb{R}$, $F \in \mathcal{F}$, $\psi \in \Phi_\mu$, $\mu = \sup_{0<t<\nu} F(t)$, $\nu > \operatorname{diam} X$, provided that

$$\forall\, x,y \in X,\ \big[Tx \ne Ty \;\Rightarrow\; F\big(\mathrm{d}(Tx,Ty)\big) \le \psi\big(F\big(\mathrm{d}(x,y)\big)\big)\big].$$

If F satisfies only (F1) and (F2), then T is called a *weak ψF-contraction*.

Theorem 1.3. [20, Th. 3.3] *Let $T\colon X \to X$ be a weak ψF-contraction, where $F : (0,\nu) \to \mathbb{R}$, $\nu > \operatorname{diam} X$ and $\psi \in \Phi_\mu$ be continuous, $\mu > \sup F$. Assume that the set $\Delta(F)$ of discontinuities of F is at most countable. If (X, d) is complete, then T is a P.O.*

H. Piri & P. Kumam in [14] and N. Hussain & P. Salimi in [6] combined the concepts of Suzuki [24] and F-contractions and obtained other generalizations. Thus (see [14, Def. 1.10]), a self-mapping T on (X, d) is said to be an *F-Suzuki contraction* if there exists $\tau > 0$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}\mathrm{d}(x, Tx) < \mathrm{d}(x, y) \ \Rightarrow\ \tau + F\big(\mathrm{d}(Tx, Ty)\big) \leq F\big(\mathrm{d}(x, y)\big),$$

where $F : (0, \infty) \to \mathbb{R}$ is continuous and satisfies (F1) and (F2).

Theorem 1.4. [14, Th. 2.2] *If T is an F-Suzuki contraction and (X, d) is complete, then T is a P. O.*

Very recently, H. Piri & P. Kumam [13] extended this concept and obtained *generalized F-Suzuki contraction* on b-metric spaces which, if the space is complete, is a P. O.

In the topic of F-contractions, we can find many other preoccupations to improve this theory (see e.g. [3, 7, 10, 19, 21, 26, 28]) and to use it in various applications such as functional and integral equations (see [8, 12, 23]), fractals theory (see [22]), multistage decision processes (see [27]) and others.

In the present paper, we provide some new sufficient properties for the mapping T to be a Picard operator by extending and weakening the F-contraction type conditions. Some important known results published in this topic can be obtained as special cases of the above. Example 2.3 states that our results are applicable and they are real improvements of some of those already published. Thus, while the most of known Picard operators are either non-expansive (even continuous) or expansive mappings on a complete metric space, our example proved that the result of the paper allows finding a Picard operator which does not satisfy any of the above conditions.

2. Main results

According to Ćirić [2], T is said to be *orbitally continuous at a point* $x_0 \in X$ if $\lim_k T^{n_k} x_0 = u \in X$ implies $Tu = \lim_k T^{n_k+1} x_0$. We say that T is *orbitally continuous* if it is orbitally continuous at every $x \in X$. The space X is *T-orbitally complete* if every Cauchy sequence of the form $(T^{n_k} x)_k$ converges in X.

Clearly, if T is continuous, then it is orbitally continuous. In the following we give a simple example which states that the converse is not true.

Example 2.1. The Dirichlet map $T : \mathbb{R} \to \mathbb{R}$, given by

$$Tx = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is orbitally continuous but it is nowhere continuous.

For some $x_0 \in X$, we will denote by $\mathcal{O}(x_0) = \{x_0, Tx_0, \ldots, T^n x_0, \ldots\}$ the orbit of T.

In order to proof the next theorem we need the following result that is a slight improvement of [25, Prop. 3].

Proposition 2.1. *Let (x_n) be a sequence of elements from X and Δ be a subset of $(0, \nu)$, $\nu \in \overline{\mathbb{R}}_+$, such that $(0, \nu) \setminus \Delta$ is dense in $(0, \nu)$. If $\mathrm{d}(x_n, x_{n+1}) \underset{n}{\longrightarrow} 0$ and (x_n) is not Cauchy, then there exist $\eta \in (0, \nu) \setminus \Delta$, $n_0 \in \mathbb{N}$ and the sequences of natural numbers $(m_k), (n_k)$ such that*

(1) $\forall\, k \in \mathbb{N},\ k \leq m_k < n_k$, $\mathrm{d}(x_{m_k}, x_{n_k}) > \eta$,
(2) $\forall\, k \geq n_0,\ n_k - m_k \geq 2,\ \mathrm{d}(x_{m_k}, x_{n_k - 1}) \leq \eta$,
(3) $\mathrm{d}(x_{m_k}, x_{n_k}) \searrow \eta$, $k \to \infty$,
(4) $\mathrm{d}(x_{m_k+p}, x_{n_k+q}) \to \eta$, $k \to \infty$, $p, q \in \{0, 1\}$.

For what follows we will slightly refine Definition 1.2.

For some $\nu \in \overline{\mathbb{R}}_+$ let denote by $\mathcal{F}_2^\nu$ the family of all functions $F : (0,\nu) \to \mathbb{R}$ which satisfy (F2). Similarly, for a given $\mu \in \overline{\mathbb{R}}_+$, Ψ_μ stands for the class of all mappings $\psi : (-\infty, \mu) \to \mathbb{R}$ satisfying $\psi(t) < t$, for all $t \in (-\infty, \mu)$.

Definition 2.3. We say that a mapping $T : X \to X$ is a *ψF-contraction on a set $M \subset X$*, where $F \in \mathcal{F}_2^\nu$, $\psi \in \Psi_\mu$, $\mu \geq \sup_{0<t<\nu} F(t)$, $\nu > \max\{\operatorname{diam} M, \operatorname{diam} T(M)\}$, whenever

$$\forall\, x, y \in M,\ \big[Tx \neq Ty \ \Rightarrow\ F\big(\mathrm{d}(Tx,Ty)\big) \leq \psi\big(F\big(\mathrm{d}(x,y)\big)\big)\big]. \tag{2.1}$$

If $M = X$ we say for short that T is a *ψF-contraction*.

Theorem 2.5. *Let $x_0 \in X$ and T be a ψF-contraction on $\mathcal{O}(x_0)$, where $F \in \mathcal{F}_2^\nu$, $\psi \in \Psi_\mu$, $\nu > \operatorname{diam} \mathcal{O}(x_0)$, $\mu > \sup F$. Assume that F is continuous on a dense set $A \subset (0,\nu)$ and ψ is upper semicontinuous. If T is orbitally continuous at x_0 and (X, d) is T-orbitally complete, then T has a fixed point ξ and $\xi = \lim_n T^n x_0$.*

Proof. Set $x_n = T^n x_0$ and $\gamma_n = d(x_{n-1}, x_n)$, $n \in \mathbb{N}$. Clearly, it is enough to consider the case when $\gamma_n > 0$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we obtain

$$F(\gamma_n) \leq \psi(F(\gamma_{n-1})) < F(\gamma_{n-1}) \tag{2.2}$$

hence the sequence $\big(F(\gamma_n)\big)$ is decreasing. Let $\lambda = \lim_n F(\gamma_n)$.

If $-\infty < \lambda$, then, from (2.2) one has

$$\lambda \leq \limsup_{t \searrow \lambda} \psi(t) \leq \psi(\lambda)$$

which is a contradiction. So $F(\gamma_n) \to -\infty$, which, by (F2), gives $\gamma_n \to 0$.

Now, assume that (x_n) is not a Cauchy sequence. Taking $\Delta = (0,\nu) \setminus A$ in Proposition 3.2 it follows that there exist $\eta \in A$ and the sequences (m_k), (n_k) such that

$$\mathrm{d}(x_{m_k}, x_{n_k}) \searrow \eta,\ \mathrm{d}(x_{m_k+1}, x_{n_k+1}) \to \eta, k \to \infty.$$

Since $\eta > 0$, one can find $K \in \mathbb{N}$ such that $\mathrm{d}(x_{m_k+1}, x_{n_k+1}) > 0$ for all $k \geq K$. Therefore we get

$$F\big(\mathrm{d}(x_{m_k+1}, x_{n_k+1})\big) \leq \psi\big(F(\mathrm{d}(x_{m_k}, x_{n_k}))\big),\ \forall\, k \geq K.$$

Letting $k \to \infty$ and using the facts that F is continuous at η and ψ is upper semicontinuous, one obtains

$$F(\eta) \leq \limsup_{t \to F(\eta)} \psi(t) \leq \psi(F(\eta)),$$

which contradicts the hypothesis. Therefore (x_n) is Cauchy, hence, X being T-orbitally complete, is convergent. Let $\xi \in X$ be its limit. Then, using the orbitally continuity of T at x_0, one has $T\xi = \xi$. □

Corollary 2.1. *Let $T\colon X \to X$ be a ψF-contraction, where $F \in \mathcal{F}_2^\nu$, $\psi \in \Psi_\mu$, $\nu > \operatorname{diam} X$, $\mu > \sup F$. Assume that F is continuous on a dense set $A \subset (0,\nu)$ and ψ is upper semicontinuous. If (X, d) is T-orbitally complete, then T is a P. O.*

Proof. From (2.1), we have

$$F\big(\mathrm{d}(Tx,Ty)\big) \leq \psi(F\big(\mathrm{d}(x,y)\big)) < F\big(\mathrm{d}(x,y)\big),\ \forall\, x, y \in X,\ Tx \neq Ty,$$

which proves that T is continuous and has at most one fixed point. The conclusion now follows from Theorem 2.5. □

Remark 2.1. If $(J_i)_{i \in \Im}$ is a countable partition of $(0,\nu)$ consisting of intervals, one can consider in the previous theorem a function $F \in \mathcal{F}_2^\nu$ which is monotonic on each J_i, $i \in \Im$.

Example 2.2. The function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(t) = \begin{cases} t(\sin t + 2) - 1, & \text{if } t < 0, \\ t \sin t - 1, & \text{if } t \geq 0 \end{cases}$$

is continuous, non-monotonic and $\psi(t) < t$ for all $t \in \mathbb{R}$, so it satisfies the conditions of Theorem 2.1.

Remark 2.2. Corollary 2.1 is a generalization of [14, Th. 2.1] where the function F is supposed to be continuous and increasing and $\psi(t) = t - \tau$. This result also improves Theorem 1.3 where ψ is continuous and increasing.

In the following we describe a generalization of ψF-contraction.

Definition 2.4. A function $T : X \to X$ is said to be a *Suzuki ψF-contraction on a set $M \subset X$*, where $F \in \mathcal{F}_2^\nu$, $\psi \in \Psi_\mu$, $\mu \geq \sup_{0<t<\nu} F(t)$, $\nu > \max\{\operatorname{diam} M, \operatorname{diam} T(M)\}$ if, for every $x, y \in M$ with $Tx \neq Ty$,

$$\frac{1}{2}\mathrm{d}(x, Tx) < \mathrm{d}(x, y) \Rightarrow F\big(\mathrm{d}(Tx, Ty)\big) \leq \psi\big(F(\mathrm{d}(x, y))\big). \tag{2.3}$$

If $M = X$, we say shortly that T is a *Suzuki ψF-contraction.*

Remark 2.3. If T is a Suzuki ψF-contraction on $M \subset X$ which has a fixed point $\xi \in M$, then $T_{|M}$ (the restriction to M of T) is continuous at ξ.

Proof. Let (x_n) be a sequence from M such that $x_n \to \xi$, $x_n \neq \xi$, for all n. Then, for every $n \geq 1$, one has $\frac{1}{2}\mathrm{d}(\xi, T\xi) < \mathrm{d}(\xi, x_n)$ and

$$T(\xi) \neq T(x_n) \;\Rightarrow\; F\big(\mathrm{d}(Tx_n, T\xi)\big) \leq \psi\big(F(\mathrm{d}(x_n, \xi))\big) < F(\mathrm{d}(x_n, \xi)).$$

Since $F(\mathrm{d}(x_n, \xi)) \underset{n}{\longrightarrow} -\infty$, it follows that $F\big(\mathrm{d}(Tx_n, T\xi)\big) \underset{n}{\longrightarrow} -\infty$ so $\mathrm{d}(Tx_n, T\xi) \to 0$. ☐

Lemma 2.1. *Let $x_0 \in X$ and assume that T is a Suzuki ψF-contraction on $\mathcal{O}(x_0)$. For each $n \geq 1$ we set $x_n = Tx_{n-1}$. If the function ψ is upper semicontinuous from the right, then $\mathrm{d}(x_n, Tx_n) \underset{n}{\longrightarrow} 0$.*

Proof. If there exists $N \in \mathbb{N}$ such that $\mathrm{d}(x_N, Tx_N) = 0$, then $\mathrm{d}(x_n, Tx_n) = 0$, for every $n \geq N$, and the conclusion is obvious.

Assume that $\mathrm{d}(x_n, Tx_n) > 0$, for all $n \geq 1$. Then $\frac{1}{2}\mathrm{d}(x_n, Tx_n) < \mathrm{d}(x_n, Tx_n)$ hence, by (2.3), we obtain

$$F\big(\mathrm{d}(x_{n+1}, Tx_{n+1})\big) \leq \psi\big(F(\mathrm{d}(x_n, x_{n+1}))\big) < F\big(\mathrm{d}(x_n, x_{n+1})\big), \;\; \forall\, n \geq 1, \tag{2.4}$$

that is the sequence $\big(F(\mathrm{d}(x_n, Tx_n))\big)$ is decreasing. Let $\lambda \in [-\infty, \mu)$ be its limit.

If $\lambda > -\infty$, then, from (2.4), one has

$$\lambda = \lim_n \big(F(\mathrm{d}(x_n, Tx_n))\big) \leq \limsup_{t \searrow \lambda} \psi(t) \leq \psi(\lambda) < \lambda$$

which is impossible. It follows that $\lambda = -\infty$. From (F2) one deduces $\mathrm{d}(x_n, Tx_n) \to 0$. ☐

Theorem 2.6. *Let $x_0 \in X$ and $T : X \to X$ be a Suzuki ψF-contraction on $\overline{\mathcal{O}(x_0)}$. Assume that F is increasing and ψ is upper semicontinuous from the right. If (X, d) is T-orbitally complete, then T has a fixed point ξ and $\xi = \lim_n T^n x_0$.*

Proof. First of all notice that, by monotonicity of F, the set Δ of its discontinuities is at most countable hence $(0, \nu) \setminus \Delta$ is dense in $(0, \nu)$.

Let (x_n) be the iterative sequence $x_n = Tx_{n-1}$, for $n = 1, 2, \ldots$ We first prove that (x_n) is Cauchy. If there exists $n \geq 1$ with $x_n = Tx_n$, the assertion is clear. Assume by contradiction that (x_n) is not Cauchy and $\mathrm{d}(x_n, Tx_n) > 0$, for all n. Then, by Lemma 2.1, $\mathrm{d}(x_n, x_{n+1}) \to 0$.

According to Proposition 2.1, there are $\eta > 0$, $\eta \notin \Delta$ and the sequences of positive integers (m_k), (n_k) such that

$$\mathrm{d}(x_{m_k}, x_{n_k}) \searrow \eta, \;\; \mathrm{d}(x_{m_k+1}, x_{n_k+1}) \to \eta, \text{ when } k \to \infty.$$

Hence there is $K_1 \in \mathbb{N}$ such that $\mathrm{d}(x_{m_k+1}, x_{n_k+1}) > 0$, for all $k \geq K_1$. By Lemma 2.1, one can find $K \in \mathbb{N}$, $K \geq K_1$, such that

$$0 < \mathrm{d}(x_{m_k}, Tx_{m_k}) < 2\eta \;\Rightarrow\; \frac{1}{2}\mathrm{d}(x_{m_k}, Tx_{m_k}) < \eta \leq \mathrm{d}(x_{m_k}, x_{n_k}),\; \forall\, k \geq K.$$

So

$$F\big(\mathrm{d}(x_{m_k+1}, x_{n_k+1})\big) \leq \psi\big(F(\mathrm{d}(x_{m_k}, x_{n_k}))\big),\; \forall\, k \geq K.$$

By continuity of F at η, it follows that

$$F(\eta) \leq \limsup_{t \searrow F(\eta)} \psi(t) \leq \psi\big(F(\eta)\big) < F(\eta)$$

which is a contradiction. Consequently (x_n) is a Cauchy sequence, so, by the T-orbitally completeness of X, it is convergent.

Set $\xi = \lim_n x_n$. It remains to show that ξ is a fixed point of T.

For this purpose, we will first prove that there exists a subsequence (x_{n_k}) of (x_n) such that

$$\frac{1}{2}\mathrm{d}(x_{n_k}, Tx_{n_k}) < \mathrm{d}(x_{n_k}, \xi),\; \forall\, k \geq 1. \tag{2.5}$$

Indeed, on the contrary, one can find $N \in \mathbb{N}$ such that

$$\mathrm{d}(x_n, \xi) \leq \frac{1}{2}\mathrm{d}(x_n, Tx_n),\; \forall\, n \geq N. \tag{2.6}$$

Then, for some $n \geq N$, we get

$$\mathrm{d}(x_n, \xi) \leq \frac{1}{2}\mathrm{d}(x_n, Tx_n) \leq \frac{1}{2}\mathrm{d}(x_n, \xi) + \frac{1}{2}\mathrm{d}(\xi, Tx_n)$$

hence, using again (2.6),

$$\mathrm{d}(x_n, \xi) \leq \mathrm{d}(\xi, Tx_m) = \mathrm{d}(\xi, x_{n+1}) \leq \frac{1}{2}\mathrm{d}(x_{n+1}, Tx_{n+1}). \tag{2.7}$$

Next, $\frac{1}{2}\mathrm{d}(x_n, Tx_n) < \mathrm{d}(x_n, Tx_n)$ implies

$$F\big(\mathrm{d}(x_{n+1}, Tx_{n+1})\big) \leq \psi\big(F(\mathrm{d}(x_n, Tx_n))\big) < F(\mathrm{d}(x_n, Tx_n))$$

so, F being increasing, $\mathrm{d}(x_{n+1}, Tx_{n+1}) < \mathrm{d}(x_n, Tx_n)$.

Therefore, by (2.7) and (2.6), one obtains

$$\begin{aligned}\mathrm{d}(x_{n+1}, Tx_{n+1}) &< \mathrm{d}(x_n, Tx_n) \leq \mathrm{d}(x_n, \xi) + \mathrm{d}(\xi, Tx_n)\\ &\leq \frac{1}{2}\mathrm{d}(x_{n+1}, Tx_{n+1}) + \frac{1}{2}\mathrm{d}(x_{n+1}, Tx_{n+1}) \leq \mathrm{d}(x_{n+1}, Tx_{n+1})\end{aligned}$$

which is a contradiction.

Now, since $\xi \in \overline{\mathcal{O}(x_0)}$, from (2.5) one deduces

$$F\big(\mathrm{d}(Tx_{n_k}, T\xi)\big) \leq \psi\big(F\big(\mathrm{d}(x_{n_k}, \xi)\big)\big) < F\big(\mathrm{d}(x_{n_k}, \xi)\big),\; \forall\, k \geq 1.$$

Since $\mathrm{d}(x_{n_k}, \xi) \underset{k}{\longrightarrow} 0$, we also find $\mathrm{d}(Tx_{n_k}, T\xi) \underset{k}{\longrightarrow} 0$ and so $x_{n_k} \to T\xi$. It follows that $T\xi = \xi$. □

Corollary 2.2. *Let $T : X \to X$ be a Suzuki ψF-contraction and suppose that F is increasing and ψ is upper semicontinuous from the right. If (X, d) is T-orbitally complete, then T is a P.O.*

Proof. According to the previous theorem, it remains to prove the uniqueness of the fixed point of T. Assume that $\xi_1 \neq \xi_2$ are two fixed points of T. Then $T\xi_1 \neq T\xi_2$ and $\frac{1}{2}\mathrm{d}(\xi_1, T\xi_1) < \mathrm{d}(\xi_1, \xi_2)$. So

$$F\big(\mathrm{d}(\xi_1, \xi_2)\big) = F\big(\mathrm{d}(T\xi_1, T\xi_2)\big) \leq \psi\big(F\big(\mathrm{d}(\xi_1, \xi_2)\big)\big) < F\big(\mathrm{d}(\xi_1, \xi_2)\big).$$

This contradiction completes the proof. □

Remark 2.4. Corollary 2.2 is a generalization of [6, Th. 3.2] and Theorem 1.2.

In the following we give a version of Theorem 2.6 and Corollary 2.2 where the increasing condition of F has been replaced with some continuity hypothesis for F and T.

Theorem 2.7. *Let $T : X \to X$ be a Suzuki ψF-contraction on $M \subset X$, where F is continuous on a dense subset of $(0, \eta)$ and ψ is upper semicontinuous. Suppose that there exists $x_0 \in X$ such that T is orbitally continuous at x_0. If (X, d) is T-orbitally complete, then:*
a) if $M = \mathcal{O}(x_0)$, then T has a fixed point ξ and $\xi = \lim_n T^n x_0$;
b) if $M = X$, then T is a P.O.

Proof. *a)* From the first part of the proof of Theorem 2.6 we deduce that $x_n = T^n x_0 \to \xi$. By the orbitally continuity of T at x_0, one obtains $x_n \to T\xi$ so $T\xi = \xi$.

b) Let any $y_0 \in X$ and set $y_n = T^n y_0$, $n \geq 1$. We will show that $y_n \to \xi$.

If $y_n = \xi$ for some $n \in \mathbb{N}$, then $y_{n+1} = T\xi = \xi$ hence $y_n \to \xi$. Suppose that $y_n \neq \xi$ for every $n \geq 1$. Then $Ty_n \neq T\xi$ and $0 = \frac{1}{2}\mathrm{d}(\xi, T\xi) < \mathrm{d}(\xi, y_n)$, for all n. Therefore

$$F\big(\mathrm{d}(\xi, y_{n+1})\big) = F\big(\mathrm{d}(T\xi, Ty_n)\big) \leq \psi\big(F\big(\mathrm{d}(\xi, y_n)\big)\big) < F\big(\mathrm{d}(\xi, y_n)\big)$$

meaning that the sequence $\big(F\big(\mathrm{d}(\xi, y_n)\big)\big)$ is decreasing. Let $\lambda \in [-\infty, \mu)$ be its limit. If $\lambda > -\infty$, then

$$\lambda \leq \limsup_{t \searrow \lambda} \psi(t) \leq \psi(\lambda) < \lambda$$

which is impossible. It follows that $\lambda = -\infty$ and, from (F2), one deduces $\mathrm{d}(\xi, y_n) \to 0$.

The uniqueness of ξ follows in the same manner as in the proof of Corollary 2.2. □

Remark 2.5. 1. Theorem 2.7 is a generalization of Theorem 2.5 and Corollary 2.1. Moreover, from Theorem 2.1 one can easily obtain, as a particular case, [6, Cor. 4.1].

2. Corollary 2.2 and Theorem 2.7 are generalizations of [14, Th. 2.2] where the function F is supposed to be continuous and increasing and $\psi(t) = t - \tau$.

Example 2.3. Consider the set $X = [1, \frac{3}{2})$ endowed with the Euclidean metric and the operator $T : X \to X$ defined by

$$Tx = \begin{cases} \sqrt{x}, & \text{if } x \in [1, \frac{3}{2}) \setminus \mathbb{Q}; \\ 1.1\sqrt{x} - 0.1, & \text{if } x \in [1, \frac{3}{2}) \cap \mathbb{Q}. \end{cases}$$

If $F : (0, \frac{1}{2}) \to \mathbb{R}$, $F(t) = -\frac{e^t}{t}$ and $\psi(t) = t - e^t$, $t \in \big(-\infty, -2\sqrt{e}\big)$, then:

(1) the metric space X is not complete;
(2) T has a unique continuity point;
(3) T is orbitally continuous;
(4) T is a Suzuki ψF-contraction, hence it is a P.O.;
(5) T is not a ψF-contraction;
(6) T is neither nonexpansive nor expansive.

Proof. (1) Since X is not closed in the real line, the assertion is obvious.

(2) T is a classical Dirichlet type function. It has only $x = 1$ as continuity point.

(3) Using some elementary arguments from number theory, it can easily prove that, for every $x \geq 1$, there exists $n \in \mathbb{N}$ such that $T^n x \in \mathbb{R} \setminus \mathbb{Q}$. Hence $T^n x \to 1$, so $T(T^n x) \to 1 = T(1)$, that is T is orbitally continuous.

(4) Let $x, y \in X$ satisfy $x < y$ and $\frac{1}{2}|x - Tx| < y - x$. We have to prove that

$$\frac{e^{y-x}}{y-x} + e^{-\frac{e^{y-x}}{y-x}} \leq \frac{e^{|Ty-Tx|}}{|Ty-Tx|}. \tag{2.8}$$

First we will show that

$$\frac{e^{y-x}}{y-x} + e^{-\frac{e^{y-x}}{y-x}} < \frac{e^{1.6(\sqrt{y}-\sqrt{x})}}{1.6(\sqrt{y}-\sqrt{x})}. \tag{2.9}$$

For this, we observe that

$$1.6(\sqrt{y}-\sqrt{x}) < \frac{4}{5}(y-x). \tag{2.10}$$

Since the function $t \mapsto \frac{e^t}{t}$ is decreasing on $(0, 1/2)$, one has, using (2.10),

$$\frac{5e^{y-x}}{4(y-x)e^{\frac{1}{5}(y-x)}} = \frac{5e^{\frac{4}{5}(y-x)}}{4(y-x)} < \frac{e^{1.6(\sqrt{y}-\sqrt{x})}}{1.6(\sqrt{y}-\sqrt{x})}. \tag{2.11}$$

Next, denoting $s = \frac{e^{y-x}}{y-x}$, we have $s > 2\sqrt{e}$ and

$$s + e^{-s} < \frac{5s}{4e^{\frac{1}{10}}} < \frac{5e^{y-x}}{4(y-x)e^{\frac{1}{3}(y-x)}} = \frac{5s}{4e^{\frac{1}{3}}}. \tag{2.12}$$

Indeed, the first inequality follows from

$$se^s > 2\sqrt{e}\, e^{2\sqrt{e}} > \frac{5-4e^{\frac{1}{10}}}{4e^{\frac{1}{10}}}.$$

In order to prove (2.8), we need to consider the following four cases.

Case I. $x, y \in [1, \frac{3}{2}) \setminus \mathbb{Q}$. Then $|Tx - Ty| = \sqrt{y} - \sqrt{x}$.
Since $\sqrt{y} - \sqrt{x} < 1.6\big(\sqrt{y} - \sqrt{x}\big)$, we deduce

$$\frac{e^{1.6(\sqrt{y}-\sqrt{x})}}{1.6(\sqrt{y}-\sqrt{x})} < \frac{e^{\sqrt{y}-\sqrt{x}}}{\sqrt{y}-\sqrt{x}}$$

and (2.8) comes from (2.9).

Case II. $x, y \in [1, \frac{3}{2}) \cap \mathbb{Q}$. Then $|Tx - Ty| = 1.1\big(\sqrt{y} - \sqrt{x}\big)$ and the relation follows as before.

Case III. $x \in [1, \frac{3}{2}) \setminus \mathbb{Q}$, $y \in [1, \frac{3}{2}) \cap \mathbb{Q}$. Now $|Tx - Ty| = 1.1\sqrt{y} - \sqrt{x} - 0.1$ and

$$\frac{1}{2}\big|x - Tx\big| < y - x \;\Leftrightarrow\; \frac{1}{2}\big(x - \sqrt{x}\big) < y - x \;\Leftrightarrow\; y > \frac{3}{2}x - \frac{1}{2}\sqrt{x}. \tag{2.13}$$

We will prove that

$$1.1\sqrt{y} - \sqrt{x} - 0.1 < 1.6\big(\sqrt{y} - \sqrt{x}\big). \tag{2.14}$$

Indeed, one has

$$1.1\sqrt{y} - \sqrt{x} - 0.1 < 1.6\big(\sqrt{y} - \sqrt{x}\big) \;\Leftrightarrow\; 0.5\sqrt{y} > 0.6\sqrt{x} - 0.1$$

and, from (2.13),

$$0.5\sqrt{y} > \frac{1}{2}\sqrt{\frac{3}{2}x - \frac{1}{2}\sqrt{x}}.$$

By some elementary calculus we get

$$\sqrt{\frac{3}{2}x - \frac{1}{2}\sqrt{x}} > 1.2\sqrt{x} - 0.2 \Leftrightarrow 3x - \sqrt{x} - 2 > 0, \ \forall\, x \geq 1.$$

From (2.14) and (2.9) one obtains (2.8).

Case IV. $x \in [1, \frac{3}{2}) \cap \mathbb{Q}$, $y \in [1, \frac{3}{2}) \setminus \mathbb{Q}$.

$$\frac{1}{2}|x - Tx| < y - x \Leftrightarrow \frac{1}{2}(x - 1.1\sqrt{x} + 0.1) < y - x \Leftrightarrow y > 1.5x - 0.55\sqrt{x} + 0.05.$$

Therefore $|Tx - Ty| = \sqrt{y} - 1.1\sqrt{x} + 0.1 < 1.6\big(\sqrt{y} - \sqrt{x}\big)$, so

$$\frac{e^{1.6(\sqrt{y}-\sqrt{x})}}{1.6(\sqrt{y} - \sqrt{x})} < \frac{e^{\sqrt{y}-1.1\sqrt{x}+0.1}}{\sqrt{y} - 1.1\sqrt{x} + 0.1}.$$

The last assertion follows from either of Theorem 2.6 or Theorem 2.7.

(5), (6) For $x = \sqrt{6}/2$, $y = 1.4$, we have

$$|Tx - Ty| = 1.1\sqrt{1.4} - \sqrt{\sqrt{6}/2} - 0.1 > 0.082 > 0.01526 > y - x,$$

$$F\big(|Tx - Ty|\big) > -55 > -65 > \psi\big(F(y - x)\big).$$

Hence T is not nonexpansive and is not ψF-contraction.

Finally, taking $x = \sqrt{1.2}$, $y = 1.4$, we get

$$F\big(|Tx - Ty|\big) < -7.5 < -4.5 < \psi\big(F(y - x)\big) \text{ and}$$

$$|Tx - Ty| = 1.1\sqrt{1.4} - \sqrt[4]{1.2} - 0.1 < 0.155 < 0.030 < y - x.$$

$\square$

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