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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Coincidence points for multivalued weak Γ-contraction mappings on metric spaces

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ABSTRACT. We present some new coincidence fixed point theorems for generalized multi-valued weak Γ-contraction mappings. Our outcomes extend several recent results in the framework of complete metric spaces endowed with a graph. Two illustrative examples are included and some consequences are derived.

1. INTRODUCTION

Fixed point theory plays a key role in nonlinear analysis, particularly in the solvability of integral equations with many applications in applied mathematics. In this theory, Banach's Contraction Principle, dating back to 1922, is one of the most important tool and stands as the starting point of what is now called metric fixed point theory.

Given a metric space (M, d), we shall denote by $\mathcal{CB}(M)$ the family of nonempty closed bounded subsets of M, by $\mathcal{C}(M)$ the family of nonempty closed subsets of M, and by $\mathcal{K}(M)$ the family of nonempty compact subsets of M. For $X, Y \in \mathcal{C}(M)$, define the Hausdorff-Pompeiu distance:

$$H(X,Y) = \max\{\sup_{m\in X} d(m,Y), \sup_{n\in Y} d(n,X)\},$$

where $d(m,Y) = \inf\{d(m,n) : n \in Y\}$. H is a metric on $\mathcal{CB}(M)$. A multi-valued mapping $T : M \to \mathcal{CB}(M)$ is called a contraction if there exists $k \in [0,1)$ such that for all $m, n \in M$

$$H(T(m), T(n)) \leq k\, d(m,n).$$

A point $m \in M$ is a fixed point of a multi-valued mapping T if $m \in T(m)$. In 2007, Berinde and Berinde [4] introduced the notion of multi-valued weak contraction which generalizes Mizoguchi and Takahashi definition given in [13]. Regarding the structure of the metric space, we cite Jachymski [12] who have introduced the concept of a contraction single-valued mapping in a metric space endowed with a graph Γ. Such type of a contraction T is called a Γ-contraction. More precisely T preserves the edges of the graph Γ:

$$(m,n) \in E(\Gamma) \implies (T(m), T(n)) \in E(\Gamma)$$
$$\text{and } \exists\, k \in [0,1) : (m,n) \in E(\Gamma) \implies d(T(m), T(n)) \leq k\, d(m,n).$$

Jachymski has proved that a Γ-contraction mapping on a complete metric space that satisfies certain properties has a fixed point if and only if there exists $m \in M$ such that $(m, T(m)) \in E(\Gamma)$, providing an important step in the connection between fixed point theory and graph theory. Recall that a graph Γ is an ordered pair (V, E), where V is a set and $E \subset V \times V$ is a binary relation on V. Elements of E are called edges and are denoted by $E(\Gamma)$ while elements of V, denoted $V(\Gamma)$, are called vertices. If the direction

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is imposed in E (the edges are directed), then we have a directed graph (digraph). In our setting, no two vertices are connected by more than one edge and hence we can identify Γ with its pair $(V(\Gamma), E(\Gamma))$. If (M,d) is a metric space, we assume that $M = V(\Gamma)$ and the diagonal of $M \times M$ is contained in $E(\Gamma)$. More details on graph theory can be found in the monograph [2] edited by Alfuraidan and Ansari (see Chapter 7, pp. 287–363) where a special attention is given to the connection between graph theory and fixed point theory. Recently, Hanjing and Suanti [11] suggested a definition of a weak Γ-contraction with respect to a single-valued function g. Then they have established the existence of some coincidence point and fixed point theorems. Recall that a point $m \in M$ is a coincidence point of the hybrid pair (g,T) if $g(m) \in T(m)$. If g is the identity map on M, we recover the definition of a fixed point of T. We will use $Fix(T)$ and $Coin(g,T)$ to represent the set of fixed points of T and the set of coincidence points of g and T, respectively. By $Coin(g,T) \cap Fix(T) \neq \emptyset$, it is meant that a fixed point of T exists and is a coincidence point of (g,T). Alfuraidan and Khamsi [3] recently proposed the following definition which generalizes Hanjing-Suanti Definition.

Definition 1.1. Let (M,d) be a metric space and Γ be a digraph. Let $T : M \to \mathcal{C}(M)$ and $g : M \to M$ be two mappings. T is called a weak Γ-contraction with respect to g if for any $m, n \in M$ such that $m \neq n$ and $(m,n) \in E(\Gamma)$, we have
(i) if $a \in T(m)$, there exists $b \in T(n)$ such that $(a,b) \in E(\Gamma)$ and
(ii) $d(a,b) \leq \gamma(d(m,n))\, d(m,n) + h(g(n))\, d(g(n), T(m))$,
where $\gamma : (0,+\infty) \to [0,1)$ satisfies $\limsup\limits_{s \to t^+} \gamma(s) < 1$, for all $t \in [0,+\infty)$ and $h : M \longrightarrow [0,+\infty)$.

Then they have established the following coincidence point theorem:

Theorem 1.1. [3, Theorem 2.1] *Let (M,d) be a complete metric space such that the triplet (M,d,Γ) satisfies the following property ($\mathcal{P}$).*

($\mathcal{P}$): *for every sequence (m_i) in M, if $m_i \to m$ and $(m_i, m_{i+1}) \in E(\Gamma)$ for all $i \in \mathbb{N}$, there exists a subsequence (m_{i_k}) of (m_i) with $(m_{i_k}, m) \in E(\Gamma)$, for $k \in \mathbb{N}$.*

Let $g : M \to M$ be a continuous self-mapping and $T : M \to \mathcal{C}(M)$ be a weak Γ-contraction mapping with respect to g. Suppose that $g(n) \in T(m)$, for all $(m,n) \in E(\Gamma)$ with $n \in T(m)$ and there is $m_0 \in M$ such that $(m_0, n) \in E(\Gamma)$ for some $n \in T(m_0)$. Then $Coin(g,T) \cap Fix(T) \neq \emptyset$.

2. MAIN RESULTS

We first present a slight extension of Definition 1.1, namely:

Definition 2.2. Let (M,d) be a metric space endowed with a digraph Γ. Let $T : M \to \mathcal{C}(M)$ and $g : M \to M$ be two mappings. T is called a weak Γ-contraction with respect to g if for every $m, n \in M$ such that $m \neq n$ and $(m,n) \in E(\Gamma)$, we have
(i) if $a \in T(m)$, there exists $b \in T(n)$ such that $(a,b) \in E(\Gamma)$ and
(ii) $d(a,b) \leq \zeta(d(m,T(m)), d(m,n))\ \ d(m,n) + h(g(n))\, d(g(n), T(m))$,
where $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi function and $h : M \longrightarrow [0,+\infty)$.

Notice that in Definition 2.2, we can replace in (ii) $\zeta(d(m,T(m)), d(m,n))$ by $\zeta(d(n,T(n)), d(m,n))$. Now we consider the following class of functions.

Definition 2.3. Let $\Psi = \{\psi \mid \psi : [0,+\infty) \to [0,+\infty)\}$ be the class of nondecreasing functions that satisfy the following two conditions:
(i) for every sequence $(t_n) \subset \mathbb{R}^+$, $\psi(t_n) \to 0$ if and only if $t_n \to 0$;
(ii) for every sequence $t_1, t_2 \in \mathbb{R}^+$, $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$.

Note that we have dropped the condition that either ψ is continuous or $\psi(t) \le t,\ \forall\, t > 0$ commonly used by many authors. With this class of functions, we now present a new generalization of Mizoguchi-Takahashi Γ-contraction.

Definition 2.4. Let (M, d) be a metric space and Γ be a digraph. Let $T : M \to \mathcal{C}(M)$ and $g : M \to M$ be two mappings. Then T is called a weak (ψ, Γ)-contraction with respect to g if for every $m, n \in M$ such that $m \neq n$ and $(m, n) \in E(\Gamma)$, we have
(i) if $a \in T(m)$, there exists $b \in T(n)$ such that $(a, b) \in E(\Gamma)$ and
(ii) $\psi(d(a, b)) \le \gamma(d(m, n))\ \psi(\mathcal{M}(m, n)) + h(g(n))\ d(g(n), T(m))$,
where

$$\mathcal{M}(m, n) = \max\left\{d(m, n), d(T(m), m), d(T(n), n), \frac{d(n, T(m)) + d(m, T(n))}{2}\right\},$$

$\psi \in \Psi$, $\gamma : (0, +\infty) \to [0, 1)$ satisfies $\limsup\limits_{s \to t^+} \gamma(s) < 1$, for all $t \ge 0$, and h is non-negative on M.

Making use of these definitions, we will establish several coincidence point theorems extending Theorem 1.1. This is the aim of Section 2 where some consequence are derived. Two examples as applications are also supplied. Section 2 ends with some remarks showing how the obtained theorems improve some results from the very recent literature.

The proof of the following theorem is similar to that of Theorem 1.1 and so is omitted. Here the definition of weak Γ-contraction is that given in Definition 2.2. We will show (see Remark 2.2) that this theorem encompasses Edelstein's, Mizoguchi-Takahasi, and Berinde-Berinde Theorems.

Theorem 2.2. *Let (M, d) be a complete metric space endowed with a digraph Γ and assume that (M, d, Γ) satisfies property ($\mathcal{P}$). Let $g : M \to M$ be a continuous self-mapping and $T : M \to \mathcal{C}(M)$ be a weak Γ-contraction mapping with respect to g. Suppose that $g(n) \in T(m)$, for all $(m, n) \in E(\Gamma)$ with $n \in T(m)$ and $E(\Gamma) \cap Graph(T) \neq \emptyset$. Then $Coin(g, T) \cap Fix(T) \neq \emptyset$.*

This is illustrated by the following example inspired from [1, Example 3.1].

Example 2.1. *Step 1 (Setting):* Consider the space $M = \left\{\frac{1}{2^i},\ i \in \mathbb{N}\right\} \cup \{0\}$ with the standard metric $d(m, n) = |m - n|$, for $m, n \in M$. Let

$$E(\Gamma) = \Delta \cup \left\{\left(\frac{1}{2^i}, 0\right),\ i \in \mathbb{N}\right\},$$

define the function $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(m) = \begin{cases} \frac{m}{2}, & \text{if} \quad m \in \left[0, \frac{1}{2}\right), \\ \frac{1}{4}, & \text{if} \quad m = \frac{1}{2}, \\ \frac{2}{3}, & \text{if} \quad m > \frac{1}{2}, \end{cases}$$

and let $\zeta(u, v) = 1 - \dfrac{\varphi(v)}{v}$, for all $u, v > 0$. For any bounded sequence $(u_i) \subset (0, +\infty)$ and any non-increasing sequence $(v_i) \subset (0, +\infty)$, we have

$$\limsup_{i \to +\infty} \zeta(u_i, v_i) = \limsup_{i \to +\infty} \left(1 - \frac{\varphi(v_i)}{v_i}\right) < 1.$$

Let the mapping $T : M \to \mathcal{C}(M)$ be defined by

$$T(m) = \begin{cases} \{0,1\}, & \text{if} \quad m = 0, \\ \left\{0, \dfrac{1}{2^{i+1}}\right\}, & \text{if} \quad m = \dfrac{1}{2^i}, i \neq 2 \\ \left\{\dfrac{1}{2^2}, \dfrac{1}{2^3}\right\}, & \text{if} \quad m = \dfrac{1}{2^2}. \end{cases}$$

Let $g : M \to M$ be defined by

$$g(m) = \begin{cases} 1, & \text{if} \quad m \in \{0,1\}, \\ \dfrac{1}{2^{i+1}}, & \text{if} \quad m = \dfrac{1}{2^i}. \end{cases}$$

Let $h : M \to [0,\infty)$ be defined by

$$h(m) = \begin{cases} 1, & \text{if} \quad \in \{0,1\}, \\ 2^{i+1}, & \text{if} \quad m = \dfrac{1}{2^{i+3}}. \end{cases}$$

Step 2: $T : M \to \mathcal{C}(M)$ is a weak Γ-contraction with respect to g. Indeed, let $m,n \in M$ be such that $(m,n) \in E(\Gamma)$. Two cases are discussed separately. If $m = n$, then we are done. If $(m,n) = \left(\frac{1}{2^i}, 0\right)$, $i \neq 2$, then $T\left(\frac{1}{2^i}\right) = \left\{0, \frac{1}{2^{i+1}}\right\}$ and $T(0) = \{0,1\}$. Let $a \in T(m)$. If $a = 0$, then clearly the two conditions of Definition 2.2 are satisfied by taking $b = 0$. If $a = \frac{1}{2^{i+1}}$, then $(a,0) \in E(\Gamma)$ and $\zeta\left(d\left(\frac{1}{2^i}, T\left(\frac{1}{2^i}\right)\right), d\left(\frac{1}{2^i}, 0\right)\right) d\left(\frac{1}{2^i}, 0\right) + h(g(0))\, d\left(g(0), T\left(\frac{1}{2^i}\right)\right) \geq \frac{1}{2^{i+1}} \geq d(a,0) = \frac{1}{2^{i+1}}$. If $(m,n) = \left(\frac{1}{2^2}, 0\right)$, then $T\left(\frac{1}{2^2}\right) = \left\{\frac{1}{2^2}, \frac{1}{2^3}\right\}$ and $T(0) = \{0,1\}$. Let $a \in T(m)$. If $a = \frac{1}{2^s}$ for some $s \in \{2,3\}$, then $(a,0) \in E(\Gamma)$ and

$$\begin{aligned} \zeta\left(d\left(\frac{1}{2^2}, T\left(\frac{1}{2^2}\right)\right), d\left(\frac{1}{2^2}, 0\right)\right) d\left(\frac{1}{2^2}, 0\right) + h(g(0))\, d\left(g(0), T\left(\frac{1}{2^2}\right)\right) &= \frac{1}{2^3} + 1 - \frac{1}{2^2} \\ &\geq d(a,0) = \frac{1}{2^s}. \end{aligned}$$

Therefore, $T : M \to \mathcal{C}(M)$ is a weak Γ-contraction with respect to g. In addition, it is easy to check that if $(m,n) \in E(\Gamma)$ with $n \in T(m)$, then $g(n) \in T(m)$, that is T satisfies Definition 2.2. Hence Theorem 2.2 guarantees that $Coin(g,T) \cap Fix(T) \neq \emptyset$. In fact, $Coin(g,T) \cap Fix(T) = \left\{0, \frac{1}{2^2}\right\}$.

Step 3: We check that the given map T does not satisfy Definition 1.1. For $(m,n) = (\frac{1}{2}, 0)$, then $T\left(\frac{1}{2}\right) = \{\frac{1}{2}, \frac{1}{2^2}\}$ and $T(0) = \{0,1\}$. Let $a \in T(m)$. If $a = \frac{1}{2^2}$, then $(a,0) \in E(\Gamma)$ and

$$\varphi\left(d\left(\frac{1}{2}, 0\right)\right) d\left(\frac{1}{2}, 0\right) + h(g(0))\, d\left(g(0), T\left(\frac{1}{2}\right)\right) = \frac{1}{2^3} < d(a,0) = \frac{1}{2^2}.$$

Therefore, for $(m,n) = \left(\frac{1}{2}, 0\right)$ and $(a,b) = \left(\frac{1}{2^2}, 0\right)$, Theorem 1.1 of Alfuraidan et al. [3] does not apply.

In the following result, we drop the continuity of g and property $(\mathcal{P})$ of the metric space (M,d) while T is g-invariant and the mapping $m \mapsto d(m,T(m))$ is lower semi-continuous (l.s.c. for short).

Theorem 2.3. *Let (M,d) be a complete metric space endowed with a digraph Γ. Let $g : M \to M$ be a self-mapping and $T : M \to \mathcal{C}(M)$ be a weak Γ-contraction mapping with respect to g. Suppose that T is g-invariant and $E(\Gamma) \cap Graph(T) \neq \emptyset$. Then there exists a sequence (m_i) which converges to some limit $\widetilde{m} \in M$ and $m_{i+1} \in T(m_i)$, for all $i \in \mathbb{N}$. If further the function $d(m,T(m))$ is l.s.c. at $\widetilde{m}$, then $Coin(g,T) \cap Fix(T) \neq \emptyset$.*

Proof. The sequence (m_i) constructed in the proof of Theorem 1.1 converges to some limit $\widetilde{m} \in M$ and satisfies $m_{i+1} \in T(m_i)$, for all $i \in \mathbb{N}$. Moreover, we have

$$0 \leq d(m_i, T(m_i)) \leq d(m_i, m_{i+1}).$$

Passing to the limit yields $\lim\limits_{i\to\infty} d(m_i,T(m_i)) = 0$. The function $d(m,T(m))$ being lower semi-continuous at $\widetilde{m}$, we obtain that

$$d(\widetilde{m}, T(\widetilde{m})) \leq \liminf_{i\to\infty} d(m_i, T(m_i)) = 0.$$

Since $T(\widetilde{m})$ is closed, then $\widetilde{m} \in T(\widetilde{m})$. Finally using the g-invariance assumption on T, we deduce that $g(\widetilde{m}) \in T(\widetilde{m})$. Therefore $\widetilde{m} \in Coin(g,T) \cap Fix(T)$, as claimed. □

Theorem 2.2 is now extended to the class of weak (ψ,Γ)-contraction mappings:

Theorem 2.4. *Let (M,d) be a complete metric space such that (M,d,Γ) has property $(\mathcal{P})$. Let $g : M \to M$ be a continuous self-mapping and $T : M \to \mathcal{C}(M)$ be a weak (ψ,Γ)-contraction mapping with respect to g. In addition, suppose that $g(n) \in T(m)$, for all $(m,n) \in E(\Gamma)$ with $n \in T(m)$ and $E(\Gamma) \cap Graph(T) \neq \emptyset$. Then $Coin(g,T) \cap Fix(T) \neq \emptyset$.*

Proof. Since $E(\Gamma) \cap Graph(T) \neq \emptyset$, then there exists $m_0 \in M$ and $m_1 \in T(m_0)$ such that $(m_0,m_1) \in E(\Gamma)$. By assumption, $g(m_1) \in T(m_0)$. If $m_0 = m_1$, then $m_0 \in Coin(g,T) \cap Fix(T)$. Now suppose $m_0 \neq m_1$. Since T is a weak (ψ,Γ)-contraction, then there exists $m_2 \in T(m_1)$ such that $(m_1,m_2) \in E(\Gamma)$ and

$$\begin{aligned}\psi(d(m_1,m_2)) &\leq \gamma(d(m_0,m_1))\,\psi(\mathcal{M}(m_0,m_1)) + h(g(m_1))\,d(g(m_1),T(m_0))\\ &= \gamma(d(m_0,m_1))\,\psi(\mathcal{M}(m_0,m_1)).\end{aligned}$$

By induction, we obtain a sequence $\{m_i\}$ such that, for all $i \in \mathbb{N}$, $m_{i+1} \in T(m_i)$, $(m_i,m_{i+1}) \in E(\Gamma)$, and

$$\psi(d(m_i,m_{i+1})) \leq \gamma(d(m_{i-1},m_i))\,\psi(\mathcal{M}(m_{i-1},m_i)), \tag{2.1}$$

where

$$\begin{aligned}\mathcal{M}(m_{i-1},m_i) &= \max\Big\{d(m_{i-1},m_i), d(T(m_{i-1}),m_{i-1}), d(T(m_i),m_i),\\ &\qquad \frac{d(m_i,T(m_{i-1})) + d(m_{i-1},T(m_i))}{2}\Big\}\\ &= \max\left\{d(m_{i-1},m_i), d(T(m_i),m_i), \frac{d(m_{i-1},T(m_i))}{2}\right\}.\end{aligned}$$

Observe that we have assumed $m_{i-1} \neq m_i$, otherwise m_i is a fixed point of T and by assumption, $g(m_i) \in T(m_i)$ because (m_i,m_i) lies in $E(\Gamma)$ and $m_i \in T(m_{i-1}) = T(m_i)$, that is $m_i \in Coin(g,T) \cap Fix(T)$. Also $m_i \notin T(m_i)$, for all $i \in \mathbb{N}$.

Step 1. $\lim\limits_{i\to\infty} d(m_i,m_{i+1}) = 0$. We distinguish between two cases:

Case 1. $\mathcal{M}(m_{i-1},m_i)=\dfrac{d(m_{i-1},T(m_i))}{2}$, i.e., $\dfrac{d(m_{i-1},T(m_i))}{2}>d(m_{i-1},m_i)$ and $\dfrac{d(m_{i-1},T(m_i))}{2}>d(T(m_i),m_i)$. Then

$$d(m_{i-1},T(m_i))>d(m_{i-1},m_i)+d(m_i,T(m_i))\geq d(m_{i-1},T(m_i)),$$

leading to a contradiction.

Case 2. $\mathcal{M}(m_{i-1},m_i)=d(T(m_i),m_i)$. In this case, we have

$$\psi(d(m_i,m_{i+1}))\leq\gamma(d(m_{i-1},m_i))\ \psi(d(T(m_i),m_i)).$$

Since $m_{i+1}\in T(m_i)$ and the function ψ is monotone, we have

$$\psi(d(m_i,m_{i+1}))\leq\gamma(d(m_{i-1},m_i))\ \psi(d(m_{i+1},m_i))<\psi(d(m_{i+1},m_i)),$$

and again a contradiction is reached. Thus $\mathcal{M}(m_{i-1},m_i)=d(m_{i-1},m_i)$ and so inequality (2.1) becomes

$$\psi(d(m_i,m_{i+1}))\leq\gamma(d(m_{i-1},m_i))\ \psi(d(m_{i-1},m_i)), \tag{2.2}$$

for all $i\in\mathbb{N}$. Since $0<\gamma(t)<1$, for all $t>0$, then

$$\psi(d(m_i,m_{i+1}))<\psi(d(m_{i-1},m_i)).$$

Hence $(\psi(d(m_i,m_{i+1})))$ is a decreasing sequence of positive numbers. Let

$$l=\lim_{i\longrightarrow\infty}\psi(d(m_i,m_{i+1})).$$

By taking the limit in (2.2), we find

$$l\leq l\ \limsup_{i\to\infty}\gamma(d(m_{i-1},m_i))<l$$

which is a contradiction unless $l=0$. Hence $\lim_{i\longrightarrow\infty}\psi(d(m_i,m_{i+1}))=0$. As $\psi\in\Psi$, we deduce that $\lim_{i\longrightarrow\infty}d(m_i,m_{i+1})=0$, as claimed.

Step 2. $\{m_i\}$ is a Cauchy sequence in M. Since $\limsup_{s\to t^+}\gamma(s)<1$, for all $t\in[0,+\infty)$, then there exist $\delta>0$ and $a\in(0,1)$ such that

$$\gamma(t)<a,\ \forall\, t\in(0,\delta).$$

Since $\lim_{i\longrightarrow\infty}d(m_{i-1},m_i)=0$, for $\varepsilon=\delta$, there exists $I\in\mathbb{N}$ such that $d(m_{i-1},m_i)<\delta$, for all $i\geq I$. From the inequality in (2.2), we infer that for all $i\geq I$

$$\psi(d(m_i,m_{i+1}))\leq a\psi(d(m_{i-1},m_i))\leq\ldots\leq a^{i-I}\psi(d(m_I,m_{I+1})).$$

For each $i,j\in\mathbb{N}$ with $j>i\geq I$, we have

$$\begin{aligned}\psi(d(m_i,m_j)) &\leq \psi(\sum_{k=i}^{j-1}d(m_k,m_{k+1}))\\ &\leq \sum_{k=i}^{j-1}\psi(d(m_k,m_{k+1}))\\ &\leq \sum_{k=i}^{j-1}a^{k-I}\psi(d(m_I,m_{I+1})).\end{aligned}$$

By taking the limit as $i,j\to\infty$, we find $\psi(d(m_i,m_j))\to 0$, whence $d(m_i,m_j)\to 0$, as $i,j\to\infty$. This proves that (m_i) is a Cauchy sequence which converges to $m\in M$ since M is complete.

Step 3. $m \in Coin(g,T) \cap Fix(T)$. By property $(\mathcal{P})$, there exists a subsequence $(m_{\phi(i)})$ such that $(m_{\phi(i)}, m) \in E(\Gamma)$, for each $i \in \mathbb{N}$. Since T is a weak (ψ,Γ)-contraction, then there exists $n_i \in T(m)$ such that for all $i \in \mathbb{N}$,

$$\psi(d(m_{\phi(i)+1}, n_i)) \leq \gamma(d(m_{\phi(i)}, m))\ \psi(\mathcal{M}(m_{\phi(i)}, m)) + h(g(m))\ d(g(m), T(m_{\phi(i)})).$$

Since $m_{\phi(i)+1} \in T(m_{\phi(i)})$, then $g(m_{\phi(i)+1}) \in T(m_{\phi(i)})$ which implies that

$$\psi(d(m_{\phi(i)+1}, n_i)) \leq \gamma(d(m_{\phi(i)}, m))\ \psi(\mathcal{M}(m_{\phi(i)}, m)) + h(g(m))\ d(g(m), g(m_{\phi(i)+1})),$$

where we have set

$$\begin{aligned}\mathcal{M}(m_{\phi(i)}, m) \quad &= \quad \max\Big\{d(m_{\phi(i)}, m), d(T(m_{\phi(i)}), m_{\phi(i)}), d(T(m), m),\\ &\qquad \frac{d(m, T(m_{\phi(i)})) + d(m_{\phi(i)}, T(m))}{2}\Big\}.\end{aligned}$$

Since $\lim\limits_{i\to+\infty} d(m_{\phi(i)}, m) = 0$, then there exists $I_1 \in \mathbb{N}$ such that

$$d(m_{\phi(i)}, m) \leq \frac{d(m, T(m))}{4},\ \forall\, i \geq I_1.$$

Moreover the estimate $d(T(m_{\phi(i)}), m_{\phi(i)}) \leq d(m_{\phi(i)+1}, m_{\phi(i)})$ guarantees that

$$\lim_{i\to+\infty} d(T(m_{\phi(i)}), m_{\phi(i)}) = 0.$$

Thus, we can find some $I_2 \in \mathbb{N}$ such that

$$d(T(m_{\phi(i)}), m_{\phi(i)}) \leq \frac{d(m, T(m))}{4},\ \forall\, i \geq I_2.$$

As a consequence

$$\begin{aligned}\frac{d(m, T(m_{\phi(i)})) + d(m_{\phi(i)}, T(m))}{2} \quad &\leq \quad \frac{d(m, m_{\phi(i)})}{2} + \frac{d(m_{\phi(i)}, T(m_{\phi(i)}))}{2}\\ &\qquad + \frac{d(m_{\phi(i)}, m)}{2} + \frac{d(m, T(m))}{2}.\end{aligned}$$

For $i \geq I_0 = \max\{I_1, I_2\}$, we have

$$\frac{d(m, T(m_{\phi(i)})) + d(m_{\phi(i)}, T(m))}{2} \leq d(m, T(m)).$$

We conclude that one can choose $I_0 \in \mathbb{N}$ such that

$$M(m_{\phi(i)}, m) = d(T(m), m),\ \forall\, i \geq I_0.$$

Since $n_i \in T(m)$, the properties of ψ and g lead to the estimates:

$$\begin{aligned}\psi(d(m_{\phi(i)+1}, n_i)) \quad &\leq \quad \gamma(d(m_{\phi(i)}, m))\ \psi(d(T(m), m)) + h(g(m))\ d(g(m), g(m_{\phi(i)+1}))\\ &\leq \quad \gamma(d(m_{\phi(i)}, m))\ \psi(d(n_i, m)) + h(g(m))\ d(g(m), g(m_{\phi(i)+1}))\\ &\leq \quad \gamma(d(m_{\phi(i)}, m))\ \psi(d(n_i, m_{\phi(i)+1}) + \psi(d(m_{\phi(i)+1}, m))\\ &\qquad + h(g(m))\ d(g(m), g(m_{\phi(i)+1})).\end{aligned}$$

Hence

$$(1 - \gamma(d(m_{\phi(i)}, m)))\psi(d(m_{\phi(i)+1}, n_i)) \leq \psi(d(m_{\phi(i)+1}, m)) +\ h(g(m))\ d(g(m), g(m_{\phi(i)+1})).$$

Taking into account the property of the function γ in Definition 1.1, the limit as $i \to \infty$ yields $\lim\limits_{i\longrightarrow\infty} \psi(d(m_{\phi(i)+1}, n_i)) = 0$, which implies that

$$\lim_{i\longrightarrow\infty} d(m_{\phi(i)+1}, n_i) = 0.$$

As a consequence the sequence (n_i) also converges to m. Since $T(m)$ is closed, we deduce that $m \in T(m)$, a fixed point of T. By assumption, $g(m) \in T(m)$. Therefore, $m \in Coin(g,T) \cap Fix(T)$. $\square$

To illustrate Theorem 2.4, we again discuss [3, Example 2.3] showing that T is in fact a weak (ψ,Γ)-contraction.

Example 2.2. Let $M = \left\{\frac{1}{2^i},\ i \in \mathbb{N}\right\} \cup \{0\}$ with the metric $d(m,n) = |m-n|$ for all $m,n \in M$ and let

$$E(\Gamma) = \Delta \cup \left\{\left(\frac{1}{2^{2i+1}}, 0\right),\ i \in \mathbb{N}\right\}.$$

Define $\gamma : [0,\infty) \to [0,1)$ by $\gamma(t) = \frac{1}{2}$ for all $t \in [0,\infty)$ and let $\psi : (0,\infty) \to [0,1)$, $T : M \to \mathcal{C}(M)$ be defined by $\psi(t) = \frac{t}{t+1}$ and

$$T(m) = \begin{cases} \{0,1\}, & \text{if } m = 0, \\ \left\{\frac{1}{2^{2i+1}}, \frac{1}{2^{2i+3}}, \frac{1}{2^{2i+5}}, \cdots\right\}, & \text{if } m = \frac{1}{2^{2i+1}}, \\ \left\{\frac{1}{2^{2i+2}}\right\}, & \text{if } m = \frac{1}{2^{2i}}. \end{cases}$$

Let $g : M \to M$ and $h : M \to [0,\infty)$ be defined by:

$$g(m) = \begin{cases} 1, & \text{if } m \in \{0,1\}, \\ \frac{1}{2^{i+2}}, & \text{if } m = \frac{1}{2^i},\ i \in \{2,3,\dots\} \end{cases}$$

and

$$h(m) = \begin{cases} 1, & \text{if } m \in \{0,1\}, \\ 2^{i+1}, & \text{if } m = \frac{1}{2^i}. \end{cases}$$

We claim that $T : M \to \mathcal{C}(M)$ is a weak (ψ,Γ)-contraction with respect to g. For this, let $m,n \in M$ be such that $(m,n) \in E(\Gamma)$. If $m = n$, there is nothing to prove. If $(m,n) = \left(\frac{1}{2^{2i+1}}, 0\right)$, then $T\left(\frac{1}{2^{2i+1}}\right) = \left\{\frac{1}{2^{2i+1}}, \frac{1}{2^{2i+3}}, \frac{1}{2^{2i+5}} \cdots\right\}$ and $T(0) = \{0,1\}$. Let $a \in T(m)$. If $a = \frac{1}{2^{2i+s}}$ for some $s \in \{1,3,5,7,\cdots\}$, then $(a,0) \in E(\Gamma)$ and

$$\begin{aligned} \frac{1}{2}\psi\left(\mathcal{M}\left(\frac{1}{2^{2i+1}}, 0\right)\right) + h(g(0))\, d\left(g(0), T\left(\frac{1}{2^{2i+1}}\right)\right) &= \frac{1}{2} \cdot \frac{1}{1+2^{2i+1}} + 1 - \frac{1}{2^{2i+1}} \\ &\geq \psi(d(a,b)) \\ &= \frac{1}{1+2^{2i+s}}. \end{aligned}$$

Consequently $T : M \to \mathcal{C}(M)$ is a weak (ψ,Γ)-contraction with respect to g. Moreover, it is easy to check that if $(m,n) \in E(\Gamma)$ with $n \in T(m)$, then $g(n) \in T(m)$. From Theorem 2.4, we conclude that $Coin(g,T) \cap Fix(T) \neq \emptyset$. In fact, we have $Coin(g,T) \cap Fix(T) = \left\{0, \frac{1}{2^{2i+1}},\ i \in \mathbb{N}\right\}$.

The proof of the following theorem will be omitted for it is similar to those of Theorems 2.2 and 2.4.

Theorem 2.5. *Let (M,d) be a complete metric space such that (M,d,Γ) has property ($\mathcal{P}$). Let $g : M \to M$ be a continuous self-mapping and $T : M \to \mathcal{C}(M)$ be a mapping with the property that for any $m,n \in M$ such that $m \neq n$ and $(m,n) \in E(\Gamma)$, we have two conditions:*

(i) *if $a \in T(m)$, then there exists $b \in T(n)$ such that $(a,b) \in E(\Gamma)$ and*

(ii) *$\psi(d(a,b)) \leq \zeta(d(m,T(m)), d(m,n))\ \psi(d(m,n)) + h(g(n))\ d(g(n),T(m))$, where $\psi \in \Psi$ while $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi function and $h : M \to [0,+\infty)$.*

Suppose further that $g(n) \in T(m)$ for all $(m,n) \in E(\Gamma)$ with $n \in T(m)$ and $E(\Gamma) \cap Graph(T) \neq \emptyset$. Then $Coin(g,T) \cap Fix(T) \neq \emptyset$.

Remark 2.1. (a) As in Theorem 2.3, when (M,d,Γ) not satisfies property ($\mathcal{P}$) and the mapping $m \mapsto d(m,T(m))$ is lower semi-continuous, we can here again drop the continuity of T and replace $\psi(d(m,n))$ by $\psi(M(m,n))$ in condition (ii).
(b) If we take $g(m) = m$ and $h = 0$ in Theorem 2.5, then we obtain another Mizoguchi-Takahashi type fixed point theorem.

Remark 2.2. [Concluding remarks]
(1) If in Theorem 2.2, we take $\zeta(u,v) = \lambda$, $g(m) = m$, $h = 0$, and Γ with $E(\Gamma) = \{(m,n) \in M \times M :\ d(m,n) < \epsilon\}$, then we obtain an extension of Edelstein's Coincidence Point Theorem in [9].
(2) If in Theorem 2.2, we consider the digraph Γ with $E(\Gamma) = M \times M$, then we recover the main coincidence point theorem in [8].
(3) If in Theorem 2.2, we take $\zeta(u,v) = \varphi(v)$, $g(m) = m$, $h = 0$, and Γ with $E(\Gamma) = M \times M$, then we recapture Mizoguchi-Takahasi Fixed Point Theorem.
(4) If in Theorem 2.2, we take $\zeta(u,v) = \varphi(v)$, $g(m) = m$, $h = S$, and Γ with $E(\Gamma) = M \times M$, then Berinde-Berinde Theorem is recovered (see [4]).
(5) Putting $\zeta(u,v) = \gamma(v)$ in Theorem 2.2 yields Theorem 1.1.
(6) Alfuraindan-Khamsi Theorem 1.1 follows from Theorem 2.4 with $\psi(t) = 1$.

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