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*Dedicated to Professor Yeol Je Cho on the occasion of his retirement*

# Fixed point problems concerning contractive type operators on KST-Spaces

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ABSTRACT. In this paper, using the concept of $w$-distance we prove some results on the existence of fixed points for contractive type operators, namely; $(\alpha, \mu)$-$\psi$-contractive operators. Applications are also presented. Our results improve and generalize a number of known results of fixed point theory including the recent results of Guran and Bota [ Guran, L. and Bota, M.-F., *Ulam-Hyers Stability Problems for Fixed Point Theorems concerning $\alpha$-$\psi$-Type Contractive Operators on $KST$-Spaces*, Submitted in press.] and Ansari [Ansari, A. H. and Shukla, S., *Some fixed point theorems for ordered $F$-$(\mathcal{F}, h)$-contraction and subcontractions in $\theta$-$f$-orbitally complete partial metric spaces*, J. Adv. Math. Stud., **9** (2016), No. 1, 37–53].

## 1. INTRODUCTION AND PRELIMINARIES

In 2012, Samet et al. [23] introduced the notion of $\alpha$-$\psi$-contractive type operator and proved fixed point results for such operators, generalizing a number of known fixed point results including the Banach Contraction Principle.

In 1996, Kada et al. [15] introduced the notion of $w$-distance on metric spaces. Using this notion, they improved a number of known results including the Caristi fixed point theorem [4].

The first stability problem raised by Ulam [26] during his talk at the University of Wisconsin in 1940, concerns the stability of group homomorphisms. The first affirmative partial answer to the question of Ulam was given for Banach spaces by Hyers [13] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability. Ulam-Hyers stability results in fixed point theory have been investigated by many authors, see; [13, 14, 18, 19, 21]. While, in [5, 6] Cho proved a number of interesting fixed point results for contractive type operators in various type of spaces.

In this paper we study existence, uniqueness and generalized Ulam-Hyers stability for fixed point equations concerning a new class of contractions, $(\alpha, \mu)$-$\psi$-type, on a $KST$-spaces.

First we recall some essential definitions and fundamental results.

Let $(X, d)$ be a metric space and $f : X \to X$ be a singlevalued operator. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;
$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;
$P_{cp}(X)$ - *the set of all nonempty compact subsets of* X;
$Fix(f) := \{x \in X \mid x = f(x)\}$ - *the set of fixed points of* $f$.

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The concept of $w$-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [15]) as follows.

**Definition 1.1.** Let $(X, d)$ be a metric space. Then $w : X \times X \to [0, \infty)$ is called a weak distance (briefly $w$-distance) on X if the following axioms are satisfied :

(1) $w(x, z) \leq w(x, y) + w(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X$, $w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ implies $d(x, y) \leq \varepsilon$.

The triple $(X, d, w)$ is a $KST$-space if $X$ is a nonempty set, $d : X \times X \to \mathbb{R}_+$ is a metric on $X$ and $w : X \times X \to [0, \infty)$ is a $w$-distance on $X$.

Let $(X, d, w)$ be a $KST$-space. We say that $(X, d, w)$ is a complete $KST$-space if the metric space $(X, d)$ is complete.

Some examples of $w$-distance can be found in [15].

For our main results, we need the following crucial result for $w$-distance.

**Lemma 1.1.** [25] *Let $(X, d)$ be a metric space and let $w$ be a $w$-distance on $X$. Let $(x_n)$ and $(y_n)$ be two sequences in $X$, let $(\alpha_n)$, $(\beta_n)$ be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:*

*(1) If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
*(2) If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $(y_n)$ converges to $z$.*
*(3) If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $(x_n)$ is a Cauchy sequence.*
*(4) If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $(x_n)$ is a Cauchy sequence.*

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by $\Phi$, the class of all comparison functions $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples , see e.g. [3, 20].

**Lemma 1.2.** [3, 20] *If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) *each iterate $\varphi^k$ of $\varphi$, $k \geq 1$, is also a comparison function;*
(2) *$\varphi$ is continuous at 0;*
(3) *$\varphi(t) < t$, for any $t > 0$.*

Next, we present the definitions of $\alpha$-$\psi$-contractive and $\alpha$-admissible mappings introduced by Samet et al. [23] and $\eta$-subadmissible mapping introduced by [22].

We denote with $\Psi$ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where $\psi^n$ is the $n$-th iterate of $\psi$. It is clear that if $\Psi \subset \Phi$ and hence, by Lemma 1.2 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

**Definition 1.2.** [23] Let $f : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$. We say that $f$ is an $\alpha$-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$, for every $x, y \in X$.

**Definition 1.3.** [22] Let $f : X \to X$, $\eta : X \times X \to \mathbb{R}^+$. We say that $f$ is an $\eta$-subadmissible mapping if $\eta(x, y) \leq 1$ implies $\eta(f(x), f(y)) \leq 1$, for every $x, y \in X$.

**Definition 1.4.** [16] An $\alpha$-admissible map $f : X \to X$ is called a triangular $\alpha$-admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

**Lemma 1.3.** [16] *Let $f : X \to X$ be a triangular $\alpha$-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, f(x_1)) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$. Then, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in N$ with $n < m$.*

**Definition 1.5.** [16] An $\eta$-subadmissible map $f : X \to X$ is called a triangular $\eta$-subadmissible if $\eta(x, z) \leq 1$ and $\eta(z, y) \leq 1$ imply $\eta(x, y) \leq 1$.

**Lemma 1.4.** [16] *Let $t\colon X \to X$ be a triangular $\eta$-subadmissible map. Assume that there exists $x_1 \in X$ such that $\eta(x_1, f(x_1)) \leq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$. Then, we have $\eta(x_n, x_m) \leq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

**Definition 1.6.** [23] Let $(X, d)$ be a metric space and $f : X \to X$ be a given mapping. We say that $f$ is an $\alpha$-$\psi$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(f(x), f(y)) \leq \psi(d(x, y)), \text{ for all } x, y \in X. \tag{1.1}$$

**Remark 1.1.** If $f : X \to X$ satisfies the Banach contraction principle, then $f$ is an $\alpha$-$\psi$-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. For example, see [23].

Let us recall some important results concerning $\alpha$-$\psi$-contractive mappings.

**Theorem 1.1.** [23] *Let $(X, d)$ be a complete metric space and $f : X \to X$ be an $\alpha$-$\psi$-contractive mapping satisfying the following conditions:*

(i) *$f$ is $\alpha$-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
(iii) *$f$ is continuous.*

*Then, $Fix(f) \neq \emptyset$.*

**Theorem 1.2.** [23] *Let $(X, d)$ be a complete metric space and $f : X \to X$ be an $\alpha$-$\psi$-contractive mapping satisfying the following conditions:*

(i) *$f$ is $\alpha$-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n$.*

*Then, $Fix(f) \neq \emptyset$.*

Further, the existence of fixed points for $\alpha$-$\psi$-contractive mappings with respect to $w$-distance on $KST$-spaces studied in [11, 12]. Recently, Latif et al. [17] studied the existence of fixed points for cyclic admissible generalized contractive type mappings.

Now, let us recall some new concepts.

**Definition 1.7.** [1, 2] We say that $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type $I$ if

$$x \geq 1 \Longrightarrow h(1, y) \leq h(x, y), \text{ for all } x, y \in \mathbb{R}^+.$$

**Definition 1.8.** [1, 2] Let $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say pair $(F, h)$ is a upper class of type $I$ if $F$ is a function, $h$ is a subclass of type $I$ and

$$0 \leq s \leq 1 \Longrightarrow F(s, t) \leq F(1, t),$$

$$h(1, y) \leq F(s, t) \Longrightarrow y \leq st, \text{ for all } x, y, s, t \in R^+.$$

**Definition 1.9.** Let $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say pair $(F, h)$ is a special upper class of type $I$ if $F$ is a function, $h$ is a subclass of type $I$ and

$$0 \leq s \leq 1 \Longrightarrow F(s, t) \leq F(1, t),$$

$$h(1, y) \leq F(1, t) \Longrightarrow y \leq t, \text{ for all } x, y, s, t \in R^+.$$

For explicit examples, see [1, 2].

## 2. Fixed points for $(\alpha, \mu)$-$\psi$-weakly contractive operators

First, let us give the following definition as a generalization of Definition 1.6.

**Definition 2.10.** Let $(X, d, w)$ be a $KST$-space and $F, h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be two functions such that the pair $(F, h)$ is a special upper class of type I. We say an operator $f : X \to X$ is an $(\alpha, \mu)$-$\psi$-weakly contractive of type I if there exist two functions $\alpha, \mu : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$h(\alpha(x,y), w(f(x), f(y))) \leq F(\mu(x,y), \psi(w(x,y))), \text{ for all } x, y \in X. \tag{2.2}$$

Our first main result is the following.

**Theorem 2.3.** *Let $(X, d, w)$ be a complete $KST$-space. Let $f : X \to X$ be an $(\alpha, \mu)$-$\psi$-weakly contractive operator of type I satisfying the following conditions:*

(i) *$f$ is $\alpha$-admissible and $\mu$-subadmissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1, \mu(x_0, f(x_0)) \leq 1$;*
(iii) *$f$ is continuous.*

*Then $Fix(f) \neq \emptyset$.*

*Proof.* By hypothesis (ii), let us consider a point $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ and $\mu(x_0, f(x_0)) \leq 1$. Inductively, we define a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ by $x_{n+1} = f(x_n),$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for $f$ and the proof finishes. We assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $f$ is $\alpha$-admissible and $\mu$-subadmissible, we have:

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0)) \geq 1 \Longrightarrow \alpha(f(x_0), f(x_1)) = \alpha(x_1, x_2) \geq 1 \tag{2.3}$$

$$\mu(x_0, x_1) = \mu(x_0, f(x_0)) \leq 1 \Longrightarrow \mu(f(x_0), f(x_1)) = \mu(x_1, x_2) \leq 1. \tag{2.4}$$

By induction, we get:

$$\begin{aligned} \alpha(x_n, x_{n+1}) &\geq 1, \text{ for all } n \in \mathbb{N}. \\ \mu(x_n, x_{n+1}) &\leq 1, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{2.5}$$

Applying the inequality (2.2) with $x = x_{n-1}$ and $y = x_n$, and using (2.5), we obtain:

$$\begin{aligned} h(1, w(x_n, x_{n+1})) = h(1, w(f(x_{n-1}), f(x_n))) &\leq h(\alpha(x_{n-1}, x_n), w(f(x_{n-1}), f(x_n))) \\ &\leq F(\mu(x_{n-1}, x_n), \psi(w(x_{n-1}, x_n))) \leq F(1, \psi(w(x_{n-1}, x_n))). \end{aligned} \tag{2.6}$$

Then, $w(x_n, x_{n+1}) \leq \psi(w(x_{n-1}, x_n))$. Since $\psi$ is nondecreasing, by induction we obtain a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that:

(i) $x_{n+1} = f(x_n)$, for any $n \in \mathbb{N}$;
(ii) $w(x_n, x_{n+1}) \leq \psi^n(w(x_0, x_1)),$ for all $n \in \mathbb{N}$.

Thus, for $n, m \in \mathbb{N}$, with $n < m$, we have:

$$\begin{aligned} w(x_n, x_m) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{m-1}, x_m) \\ &\leq \psi^n(w(x_0, x_1)) + \psi^{n+1}(w(x_0, x_1)) + ... + \psi^{m-1}(w(x_0, x_1)) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)). \end{aligned}$$

Since $\psi \in \Phi$, we have that $\psi^n(t) \to 0$ as $n \to \infty$. Letting $n \to \infty$ and using Lemma 1.2 we obtain:

$$\lim_{n\to\infty} w(x_n, x_m) \leq \lim_{n\to\infty} \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \tag{2.7}$$

By Lemma 1.1 (3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(X, d, w)$ is complete, there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n \overset{d}{\to} x^*$ as $n \to \infty$. From the continuity of

$f$, it follows that $x_{n+1} = f(x_n) \overset{d}{\to} f(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = f(x^*)$, that is, $x^*$ is a fixed point of $f$. □

Further, we prove another result where we replace the continuity hypothesis of $f$ with an other suitable condition.

**Theorem 2.4.** *Let $(X, d, w)$ be a complete $KST$-space. Let $f : X \to X$ be an $(\alpha, \mu)$-$\psi$-weakly contractive operator of type I satisfying the following conditions:*

(i) *$f$ is $\alpha$-admissible and $\mu$-subadmissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1, \mu(x_0, f(x_0)) \leq 1$;*
(iii) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ , $\mu(x_n, x_{n+1}) \leq 1$ for all $n$ and $x_n \overset{d}{\to} x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, $\mu(x_n, x) \leq 1$ for all $n$.*

*Then, $Fix(f) \neq \emptyset$.*

*Proof.* Following the proof of Theorem 2.3, we know that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete $KST$-space $(X, d, w)$. Then, there exists $x^* \in X$ such that $x_n \overset{d}{\to} x^*$ as $n \to \infty$. On the other hand we have the inequality $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1,$ for all $n \in \mathbb{N}$ and by $(2.5)$ and the hypothesis $(iii)$, we have:

$$\alpha(x_n, x^*) \geq 1, \mu(x_n, x^*) \leq 1 \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

For $m, n \in \mathbb{N}$ with $m > n$, from the proof of Theorem 2.3 and using the triangular inequality, we have: $w(x_n, x_m) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1))$. Since $(x_n)_{n\in\mathbb{N}}$ converge to $x^*$ and $w(x_n, \cdot)$ is lower semicontinuous we have:

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \lim_{m\to\infty} \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)).$$

Since $\psi \in \Phi$, we have that $\psi^n(t) \to 0$ as $n \to \infty$. Letting $n \to \infty$ and using Lemma 1.2 for every $n \in \mathbb{N}$ we have that:

$$w(x_n, x^*) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \tag{2.9}$$

Let $f(x^*) \in X$ and $x_n = f(x_{n-1})$. Then, by the definition of $(\alpha, \mu)$-$\psi$-weakly contractive operator of type I and letting $n \to \infty$ we obtain the following:

$$\begin{aligned} h(1, w(x_n, f(x^*))) &= h(1, w(f(x_{n-1}), f(x^*))) \leq h(\alpha(x_{n-1}, x^*), w(f(x_{n-1}), f(x^*))) \\ &\leq F(\mu(x_{n-1}, x^*), \psi(w(x_{n-1}, x^*))) \leq F(1, \psi(w(x_{n-1}, x^*))) \end{aligned}$$

Then by the definition of pair $(F, h)$ upper class we obtain:

$$w(x_n, f(x^*)) \leq \psi(w(x_{n-1}, x^*)) \leq \psi(\sum_{n=k}^{\infty} \psi^k(w(x_0, x_1))) < \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \tag{2.10}$$

Then, by (2.9) and (2.10), we have that $w(x_n, x^*) \overset{d}{\to} 0$ and $w(x_n, f(x^*)) \overset{d}{\to} 0$. Using Lemma 1.1 (1) we obtain that $x^* = f(x^*)$. □

The following result assures the uniqueness of the fixed point for $KST$-spaces.

**Theorem 2.5.** *Adding to the hypothesis of Theorem 2.3 (resp. Theorem 2.4) the following condition:*
*$(H)$ : for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1, \mu(x, z) \leq 1$ and $\alpha(y, z) \geq 1, \mu(y, z) \leq 1$. we obtain uniqueness of the fixed point of $f$.*

*Proof.* Suppose that $x^*$ and $y^*$ are two fixed point of $f$. From the new condition $(H)$, there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1, \mu(x^*, z) \leq 1 \text{ and } \alpha(y^*, z) \geq 1, \mu(y^*, z) \leq 1. \tag{2.11}$$

Since $f$ is $\alpha$-admissible and $\mu$-subadmissible, from (2.11), we get:

$$\alpha(x^*, f^n(z)) \geq 1,\ \mu(x^*, f^n(z)) \leq 1 \text{ and } \alpha(y^*, f^n(z)) \geq 1,\ \mu(y^*, f^n(z)) \leq 1. \tag{2.12}$$

By the definition of $(\alpha, \mu)$-$\psi$-weakly contractive operator and using (2.2) and (2.12), we get:

$$\begin{aligned} h(1, w(x^*, f^n(z))) &= h(1, w(f(x^*), f(f^{n-1}(z)))) \leq h(\alpha(x^*, f^{n-1}(z)), w(f(x^*), f(f^{n-1}(z)))) \\ &\leq F(\mu(x^*, f^{n-1}(z)), \psi(w(x^*, f^{n-1}(z)))) \leq F(1, \psi(w(x^*, f^{n-1}(z)))). \end{aligned}$$

Then, we obtain the inequality: $w(x^*, f^n(z)) \leq \psi(w(x^*, f^{n-1}(z)))$. Thus, we have

$$w(x^*, f^n(z)) \leq \psi^{n-1}(w(x^*, z)), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get

$$w(x^*, f^n(z)) \stackrel{d}{\to} 0. \tag{2.13}$$

For $x^* = f(x^*)$, we suppose that $w(x^*, x^*) \neq 0$. Then, we have:

$$\begin{aligned} h(1, w(x^*, x^*)) &= h(1, w(f(x^*), f(x^*))) \leq h(\alpha(x^*, x^*), w(f(x^*), f(x^*))) \\ &\leq F(\mu(x^*, x^*), \psi(w(x^*, x^*))) \leq F(1, \psi(w(x^*, x^*))). \end{aligned}$$

So, we obtain $w(x^*, x^*) \leq \psi(w(x^*, x^*)) < w(x^*, x^*)$, a contradiction. Thus, we have $w(x^*, x^*) = 0$. By (2.13) and using Lemma 1.1(1) we have $f^n(z) \stackrel{d}{\to} x^*$. Similarly, for $y^* = f(y^*)$ using (2.12) and (2.2), we get $f^n(z) \stackrel{d}{\to} y^*$ as $n \to \infty$. Hence, the uniqueness of the limit gives $x^* = y^*$. □

**Example 2.1.** Let $(X, d, w)$ be a KST-space such that $X = \mathbb{R}^+ \cup \{0\}$, $d = |x - y|$ is the usual metric.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping given by $f(x) = \begin{cases} \frac{1}{3}x^2, & \text{for } x \in [0,1]. \\ \frac{x+1}{6}, & \text{for } x \in (1, \infty). \end{cases}$ Define a w-distance on $X$ by $w(x, y) = \max\{d(f(x), y), d(f(x), f(y))\}$. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\psi(t) = \frac{1}{3}t$.

We define the mappings $\alpha, \mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$, $\mu(x, y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 4, & \text{otherwise.} \end{cases}$ and $h, F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $h(x, y) = y$ and $F(s, t) = t$, for every $x, y, s, t \in \mathbb{R}^+$.

Then, all the hypotheses of Theorem 2.3 (respectively Theorem 2.4) are satisfied and consequently, $f$ has a fixed point.

*Proof.* Clearly, $(X, d, w)$ is a complete $KST$ space and, obviously, $f$ is a continuous mapping. We show that $f$ is an $\alpha$-admissible mapping. Let $x, y \in X$, if $\alpha(x, y) \geq 1$, then $x, y \in [0, 1]$. On the other hand, for all $x \in [0, 1]$ we have $f(x) = \frac{1}{3}x^2 \leq x \leq 1$. It follows that $\alpha(f(x), f(y)) \geq 1$. Hence, the assertion holds. In reason of the above arguments, $\alpha(0, f(0)) \geq 1$.

Next, we prove that $f$ is an $\mu$-subadmissible mapping. Let $x, y \in X$, if $\mu(x, y) \leq 1$, then $x, y \in [0, 1]$. On the other hand, for all $x \in [0, 1]$, we have $f(x) \leq 1$. It follows that $\mu(f(x), f(y)) \leq 1$. Also, $\mu(0, f(0)) \leq 1$. Then $f$ is an $\mu$-subadmissible mapping.

Obviously the pair $(F, h)$ is a upper class of type I. We have to check the validity of contractive condition (2.2). By the definitions of the mappings we get:

$$h(\alpha(x,y), w(f(x), f(y))) \leq F(\mu(x,y), \psi(w(x,y))) \Longrightarrow w(f(x), f(y)) \leq \psi(w(x,y))$$

Then, for $x \in [0,1]$ and knowing that $f(x) = \dfrac{1}{3}x^2 \leq x$, we have:

$$\begin{aligned} w(f(x), f(y)) &= \max\{d(f(f(x)), f(y)), d(f(f(x)), f(f(y)))\} \\ &\leq \frac{1}{3}\max\left\{\left|\frac{1}{3}x^2 - y\right|, \left|\frac{1}{3}x^2 - \frac{1}{3}y^2\right|\right\} \\ &= \frac{1}{3}\max\{d(f(x), y), d(f(x), f(y))\} \\ &= \frac{1}{3}w(x,y) = \psi(w(x,y)). \end{aligned}$$

Then the contractive condition is satisfied for $x, y \in [0,1]$. Now, for $x, y \in (1, \infty)$ we have:

$$\begin{aligned} w(f(x), f(y)) &= \max\{d(f(f(x)), f(y)), d(f(f(x)), f(f(y)))\} \\ &= \frac{1}{6}\max\left\{\left|\frac{x+1}{6} - y\right|, \left|\frac{x+7}{6} - \frac{y+7}{6}\right|\right\} \\ &= \frac{1}{6}\max\{d(f(x), y), d(f(x), f(y))\} \leq \frac{1}{3}w(x,y) = \psi(w(x,y)). \end{aligned}$$

That is; the contractive condition is satisfied for $x, y \in (1, \infty)$. Hence, the contractive condition is satisfied for all $x \in X$. Now, if $\{x_n\}$ is a sequence on $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\mu(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$ then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies $\alpha(x_n, x) \leq 1$ and $\mu(x_n, x) \leq 1$ for all $n \in \mathbb{N}$. Therefore, Theorem (2.3) (also Theorem (2.4) guarantees the existence of a fixed point. □

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